Notes and exercises from *Real Analysis*

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Introduction

This document contains notes and exercises from [1].

Chapter 2

Section 6

Exercise (51)**.** Let $f : [a,b] \to \mathbb{R}$. Let

$$g(x) = \sup_{\delta>0} \inf_{|y-x|<\delta} f(y)$$
$$h(x) = \inf_{\delta>0} \sup_{|y-x|<\delta} f(y)$$

be the lower and upper envelopes of f, respectively.

(a) For all $x \in [a,b]$, $g(x) \le f(x) \le h(x)$, $g(x) = f(x)$ if and only if f is lower semicontinuous at x, $f(x) = h(x)$ if and only if f is upper semicontinuous at x, and $g(x) = h(x)$ if and only if f is continuous at x.

(b) If f is bounded, then g is lower semicontinuous and h is upper semicontinuous.

(c) If $\varphi \le f$ is lower semicontinuous, then $\varphi \le g$, and if $\psi \ge f$ is upper semicontinuous, then $\psi \ge h$.

Proof.

(a) For all $\delta > 0$, $\inf_{|y-x|<\delta} f(y) \le f(x)$, so $g(x) \le f(x)$; similarly, $f(x) \le h(x)$.

If $g(x) = f(x)$, then for $\epsilon > 0$ there is $\delta > 0$ such that $|y-x| < \delta$ implies

$$f(y) \ge \inf_{|y-x|<\delta} f(y) > g(x) - \epsilon = f(x) - \epsilon$$

Therefore f is lower semicontinuous at x (Exercise 50(a)). Similarly f is upper semicontinuous at x if $f(x) = h(x)$, and hence f is continuous at x if $g(x) = h(x)$ (Exercise 50(b)).

Conversely, if f is lower semicontinuous at x, suppose that $f(x) > g(x)$. We must have $g(x) > -\infty$. Let $\epsilon = f(x) - g(x) > 0$ and choose $\delta > 0$ such that $|y-x| < \delta$ implies $f(y) \ge f(x) - \epsilon/2$. Then $\inf_{|y-x|<\delta} f(y) \ge f(x) - \epsilon/2$, so that $g(x) \ge f(x) - \epsilon/2 > g(x)$, a contradiction. Therefore $g(x) = f(x)$. Similarly $f(x) = h(x)$ if f is upper semicontinuous at x, and hence $g(x) = h(x)$ if f is continuous at x.

(b) Since f is bounded, g is bounded. As above, given $\epsilon > 0$, there is $\delta > 0$ such that $f(y) \ge g(x) - \epsilon$ for $|y-x| < \delta$. Given y with $|y-x| < \delta$, there is $\delta' > 0$ such that $|z-y| < \delta'$ implies $|z-x| < \delta$, so $g(y) \ge g(x) - \epsilon$. Therefore g is lower semicontinuous at x. Similarly h is upper semicontinuous at x.

(c) If λ is the lower envelope of φ, then $\lambda \le g$ since $\varphi \le f$, and $\lambda = \varphi$ since φ is lower semicontinuous (by part (a)). Similarly for h. □

Chapter 3

Section 2

Proposition (5)**.** *Given any $A \subseteq \mathbb{R}$ and any $\epsilon > 0$, there is an open set O such that $A \subseteq O$ and $m^* O \le m^* A + \epsilon$. There is a $G \in G_\delta$ such that $A \subseteq G$ and $m^* A = m^* G$.*

Proof. If $m^* A = \infty$, take $O = G = \mathbb{R}$. If $m^* A < \infty$, there is a countable set $\{I_n\}$ of open intervals with $A \subseteq \bigcup I_n$ and $\sum l(I_n) < m^* A + \epsilon$. Let $O = \bigcup I_n$. Then O is open (Proposition 2.7), $A \subseteq O$, and

$$m^* O \le \sum m^* I_n = \sum l(I_n) < m^* A + \epsilon$$

by countable subadditivity (Proposition 2) and the outer measure of intervals (Proposition 1).

For each $n \geq 1$, let O_n be open with $A \subseteq O_n$ and $m^* O_n \leq m^* A + 1/n$. Let $G = \bigcap O_n$. Then $G \in G_\delta$ and $A \subseteq G$. By monotonicity, $m^* A \leq m^* G$ and

$$m^* G \leq m^* O_n \leq m^* A + 1/n$$

for all $n \geq 1$, so $m^* A = m^* G$. $\square$

Exercise (5)**.** Let $A = \mathbb{Q} \cap (0,1)$, and let $\{I_n\}$ be a finite set of open intervals with $A \subseteq \bigcup I_n$. Then $\sum l(I_n) \geq 1$.

Proof. By density of $\mathbb{Q}$ in $\mathbb{R}$ (Corollary 2.4), $[0,1] = \overline{A} \subseteq \overline{\bigcup I_n} = \bigcup \overline{I_n}$, so

$$1 = l[0,1] = m^*[0,1] \leq m^* \bigcup \overline{I_n} \leq \sum m^* \overline{I_n} = \sum l(I_n) \qquad \square$$

Remark. The assumption of finiteness is needed, since $m^* A = 0$. In the infinite case, we may have $\overline{\bigcup I_n} \not\subseteq \bigcup \overline{I_n}$.

Exercise (7)**.** If $E \subseteq \mathbb{R}$ and $y \in \mathbb{R}$, then $m^*(E+y) = m^* E$.

Proof. First, if $E = (a,b)$ with $-\infty \leq a < b \leq \infty$, then $E + y = (a+y, b+y)$, so

$$l(E+y) = b - a = l(E)$$

Now if E is arbitrary and $\{I_n\}$ is a countable set of open intervals with $E \subseteq \bigcup I_n$, then $\{I_n + y\}$ is a countable set of open intervals with $E + y \subseteq \bigcup (I_n + y)$ and $\sum l(I_n + y) = \sum l(I_n)$ by the above. Therefore $m^*(E+y) \leq m^* E$. Conversely,

$$m^* E = m^*((E+y) - y) \leq m^*(E+y)$$

so $m^*(E+y) = m^* E$. $\square$

Exercise (8)**.** If $m^* A = 0$, then $m^*(A \cup B) = m^* B$.

Proof. We have $m^* B \leq m^*(A \cup B) \leq m^* A + m^* B = m^* B$. $\square$

Section 3

We provide an alternative proof of Lemma 7.

Lemma (7)**.** *If $E_1, E_2 \subseteq \mathbb{R}$ are measurable, so is $E_1 \cup E_2$.*

Proof. Let $A \subseteq \mathbb{R}$. We first claim[1]

$$m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap E_1 \cap E_2) \leq m^*(A \cap E_1) + m^*(A \cap E_2) \tag{1}$$

Indeed, since $A \cap (E_1 \cup E_2) = (A \cap E_1 \cap E_2) \cup (A \cap E_1 \cap \widetilde{E_2}) \cup (A \cap \widetilde{E_1} \cap E_2)$,

$$m^*(A \cap [E_1 \cup E_2]) \leq m^*(A \cap E_1 \cap E_2) + m^*(A \cap E_1 \cap \widetilde{E_2}) + m^*(A \cap \widetilde{E_1} \cap E_2)$$

by subadditivity, so

$$\begin{aligned} m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap E_1 \cap E_2) &\leq m^*([A \cap E_1] \cap E_2) + m^*([A \cap E_1] \cap \widetilde{E_2}) \\ &\quad + m^*([A \cap E_2] \cap E_1) + m^*([A \cap E_2] \cap \widetilde{E_1}) \\ &= m^*(A \cap E_1) + m^*(A \cap E_2) \end{aligned}$$

by measurability of E_1 and E_2, establishing (1).

We now claim $m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [\widetilde{E_1 \cup E_2}]) \leq m^* A$. If $m^* A = \infty$, there is nothing to prove, so we assume $m^* A < \infty$. We have

$$\begin{aligned} m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap \widetilde{E_1} \cap \widetilde{E_2}) + m^* A &\leq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap E_1 \cap E_2) \\ &\quad + m^*(A \cap [\widetilde{E_1} \cup \widetilde{E_2}]) + m^*(A \cap \widetilde{E_1} \cap \widetilde{E_2}) \\ &\leq m^*(A \cap E_1) + m^*(A \cap E_2) \\ &\quad + m^*(A \cap \widetilde{E_1}) + m^*(A \cap \widetilde{E_2}) \\ &= m^* A + m^* A \end{aligned}$$

where the first inequality follows from $A = (A \cap E_1 \cap E_2) \cup (A \cap [\widetilde{E_1} \cup \widetilde{E_2}])$ and subadditivity, the second inequality follows from (1), and the equality follows from measurability of E_1 and E_2. The claim now follows since $m^* A < \infty$. Since A was arbitrary, $E_1 \cup E_2$ is measurable. □

Proposition (15)**.** *Let $E \subseteq \mathbb{R}$. The following are equivalent:*

(i) *E is measurable.*

(ii) *Given $\epsilon > 0$, there is an open set $O \supseteq E$ with $m^*(O - E) < \epsilon$.*

(iii) *Given $\epsilon > 0$, there is a closed set $F \subseteq E$ with $m^*(E - F) < \epsilon$.*

(iv) *There is $G \in G_\delta$ with $E \subseteq G$ and $m^*(G - E) = 0$.*

(v) *There is $F \in F_\sigma$ with $F \subseteq E$ and $m^*(E - F) = 0$.*

If $m^* E < \infty$, these are all equivalent to:

(vi) *Given $\epsilon > 0$, there is a finite union U of open intervals with*[2] $m^*(U \triangle E) < \epsilon$.

[1] This is an inclusion-exclusion inequality for outer measure.

[2] Recall $U \triangle E = (U - E) \cup (E - U)$ is the symmetric difference of U and E.

Proof. (i) $\implies$ (ii): Let $\epsilon > 0$. There is an open set $O \supseteq E$ with $m^*O \le m^*E + \epsilon/2$ (Proposition 5). Now $O = E \cup (O - E)$, so by measurability (Theorems 10,12) and additivity (Proposition 13),

$$m^*E + m^*(O - E) = m^*O \le m^*E + \epsilon/2$$

If $m^*E < \infty$, it follows that $m^*(O-E) \le \epsilon/2 < \epsilon$. If $m^*E = \infty$, let $E_n = E \cap [-n, n]$. Then E_n is measurable and $m^*E_n < \infty$, so we may choose an open set $O_n \supseteq E_n$ with $m^*(O_n - E_n) < \epsilon/2^{n+1}$ by the above. Let $O = \bigcup O_n$. Then O is open, $E \subseteq O$, and

$$O - E = \bigcup O_n - E = \bigcup (O_n - E) \subseteq \bigcup (O_n - E_n)$$

so

$$m^*(O - E) \le \sum m^*(O_n - E_n) \le \sum \epsilon/2^{n+1} = \epsilon/2 < \epsilon$$

(i) $\implies$ (iii): Let $\epsilon > 0$. Since $\widetilde{E}$ is measurable, there is an open set $O \supseteq \widetilde{E}$ with $m^*(O - \widetilde{E}) < \epsilon$ by the above. Let $F = \widetilde{O}$. Then F is closed, $F \subseteq E$, and $E - F = O - \widetilde{E}$, so $m^*(E - F) = m^*(O - \widetilde{E}) < \epsilon$.

(ii) $\implies$ (iv): For each $n \ge 1$, choose $O_n \supseteq E$ open with $m^*(O_n - E) < 1/n$. Let $G = \bigcap O_n$. Then $G \in G_\delta$, $E \subseteq G$, and $G - E \subseteq O_n - E$ for all n, so $m^*(G - E) \le m^*(O_n - E) < 1/n$ for all n, so $m^*(G - E) = 0$.

(iii) $\implies$ (v): For each $n \ge 1$, choose $F_n \subseteq E$ closed with $m^*(E - F_n) < 1/n$. Let $F = \bigcup F_n$. Then $F \in F_\sigma$, $F \subseteq E$, and $E - F \subseteq E - F_n$ for all n, so $m^*(E - F) \le m^*(E - F_n) < 1/n$ for all n, so $m^*(E - F) = 0$.

(iv) $\implies$ (i): If $G \in G_\delta$, $E \subseteq G$, and $m^*(G - E) = 0$, then G is measurable and $G - E$ is measurable, so $E = G - (G - E)$ is also measurable.

(v) $\implies$ (i): If $F \in F_\sigma$, $F \subseteq E$, and $m^*(E - F) = 0$, then F is measurable and $E - F$ is measurable, so $E = F \cup (E - F)$ is also measurable.

The implications above establish the equivalence of (i)–(v). Finally, suppose $m^*E < \infty$.

(ii) $\implies$ (vi): Let $\epsilon > 0$. Choose $O \supseteq E$ open with $m^*(O - E) < \epsilon/2$. Write $O = \bigcup I_n$, where $\{I_n\}$ is a countable set of disjoint open intervals (Proposition 2.8). By additivity,

$$\sum m^* I_n = m^*(O) < m^*(E) + \epsilon/2 < \infty$$

so there is N with $m^*O - \sum_{n=1}^N m^* I_n < \epsilon/2$. Let $U = \bigcup_{n=1}^N I_n$. Then $U - E \subseteq O - E$, so $m^*(U - E) \le m^*(O - E) < \epsilon/2$, and $E - U \subseteq O - U$, so $m^*(E - U) \le m^*(O - U) = m^*O - m^*U < \epsilon/2$. Therefore

$$m^*(U \triangle E) \le m^*(U - E) + m^*(E - U) < \epsilon/2 + \epsilon/2 = \epsilon$$

(vi) $\implies$ (ii): Let $\epsilon > 0$. Choose a finite union U of open intervals with $m^*(U \triangle E) < \epsilon/3$. Choose $V \supseteq E - U$ open with $m^* V \le m^*(E-U) + \epsilon/3$ (Proposition 5). Let $O = U \cup V$. Then O is open, $E \subseteq O$, and $O - E \subseteq (U - E) \cup V$, so

$$\begin{aligned} m^*(O-E) &\le m^*(U-E) + m^* V \\ &\le m^*(U-E) + m^*(E-U) + \epsilon/3 \\ &< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \end{aligned}$$

□

Exercise (9)**.** If $E \subseteq \mathbb{R}$ is measurable, then $E + y$ is measurable.

Proof. Let $A \subseteq \mathbb{R}$. Then[3]

$$\begin{aligned} A \cap (E+y) &= (A-y) \cap E + y \\ A \cap (\widetilde{E+y}) &= (A-y) \cap \widetilde{E} + y \end{aligned}$$

For example, if $z \in A \cap (E+y)$, then $z \in A$ and $z = x + y$ with $x \in E$, hence $z - y = x \in (A-y) \cap E$, and $z = (z-y) + y \in (A-y) \cap E + y$. Now

$$\begin{aligned} m^*(A \cap [E+y]) + m^*(A \cap [\widetilde{E+y}]) &= m^*([A-y] \cap E + y) + m^*([A-y] \cap \widetilde{E} + y) \\ &= m^*([A-y] \cap E) + m^*([A-y] \cap \widetilde{E}) \\ &= m^*(A-y) \\ &= m^* A \end{aligned}$$

where the second and fourth equalities follow from translation invariance of outer measure (Exercise 7) and the third equality follows from measurability of E. Therefore $E + y$ is measurable. □

Exercise (10)**.** If $E_1, E_2 \subseteq \mathbb{R}$ are measurable, then

$$m^*(E_1 \cup E_2) + m^*(E_1 \cap E_2) = m^* E_1 + m^* E_2$$

Proof. By the same argument used in the proof of Lemma 7 above (with $A = \mathbb{R}$), except appealing to additivity instead of subadditivity to get equality. □

Exercise (11)**.** For each $n \ge 1$, let $E_n = (n, \infty)$. Then $E_{n+1} \subseteq E_n$ and $\bigcap E_n = \emptyset$, but E_n is measurable with $m^* E_n = \infty$, so

$$m^* \bigcap E_n = 0 \ne \infty = \lim_{n \to \infty} m^* E_n$$

[3]We write $A \cap B + y$ for $(A \cap B) + y$.

Exercise (12)**.** If $\{E_n\}$ is a countable sequence of disjoint measurable sets and $A \subseteq \mathbb{R}$, then

$$m^*\left(A \cap \bigcup E_n\right) = \sum m^*(A \cap E_n)$$

Proof. We have $A \cap \bigcup E_n = \bigcup (A \cap E_n)$. Let $F_n = A \cap E_n$. Then $m^* \bigcup F_n \leq \sum m^* F_n$ by subadditivity. Conversely, for any $N \geq 1$,

$$\sum_{n=1}^{N} m^* F_n = m^* \bigcup_{n=1}^{N} F_n \leq m^* \bigcup F_n$$

by finite additivity (Lemma 9) and monotonicity. Letting $N \to \infty$, we have $\sum m^* F_n \leq m^* \bigcup F_n$ and hence $\sum m^* F_n = m^* \bigcup F_n$. □

Exercise (14)**.** The Cantor ternary set C has measure zero.

Proof. Write $C = \bigcap C_n$ where C_n is a union of 2^n disjoint closed intervals each of length 3^{-n} and $C_{n+1} \subseteq C_n$. Then C_n is measurable with $m^* C_n = (2/3)^n$ and C is measurable with $m^* C = \lim_{n\to\infty} (2/3)^n = 0$ (Proposition 14). □

Section 4

Exercise (15)**.** If $E \subseteq P$ is measurable, then $mE = 0$.

Proof. Let $E_i = E \overset{\circ}{+} r_i$. Then E_i is measurable and $mE_i = mE$ (Lemma 16). Also $E_i \cap E_j = \emptyset$ for $i \neq j$ since $E_i \subseteq P_i$. It follows that $\bigcup E_i$ is measurable and

$$\sum mE = \sum mE_i = m \bigcup E_i \leq m[0,1) = 1$$

Therefore $mE = 0$. □

Exercise (17)**.** Let $E_i = P + r_i$. Then $m^* E_i = m^* P > 0$ (Exercise 7, Lemma 6), so

$$m^* \bigcup E_i \leq m^*[0,2) = 2 < \infty = \sum m^* P = \sum m^* E_i$$

Section 5

Remark. We write $\{f > \alpha\}$ for $\{x \mid f(x) > \alpha\}$, $\{f = \alpha\}$ for $\{x \mid f(x) = \alpha\}$, and similarly for other sets.

Remark. A constant function (with measurable domain) is measurable.

Remark. A continuous function f (with measurable domain) is measurable. If $\alpha \in \mathbb{R}$, then $(\alpha, \infty]$ is open, so $\{f > \alpha\} = f^{-1}(\alpha, \infty]$ is open and measurable.

Remark. A step function[4] f is measurable. If $\alpha \in \mathbb{R}$, then $\{f > \alpha\}$ is a finite union of intervals and hence measurable.

Remark. The restriction of a measurable function f to a measurable subset E of its domain is measurable. If $\alpha \in \mathbb{R}$, $\{f|_E > \alpha\} = \{f > \alpha\} \cap E$ is measurable.

Exercise (18)**.** Let E be a nonmeasurable set and $f(x) = \exp(x)[\chi_E(x) - \chi_{\tilde{E}}(x)]$. Then $\{f > 0\} = E$ is nonmeasurable, but $\{f = \alpha\}$ is measurable for each $\alpha \in \overline{\mathbb{R}}$ since f assumes each value at most once.

Exercise (19)**.** If $D \subseteq \mathbb{R}$ is dense and $f : \mathbb{R} \to \overline{\mathbb{R}}$ is such that $\{f > \alpha\}$ is measurable for each $\alpha \in D$, then f is measurable.

Proof. If $\alpha \in \mathbb{R}$, choose $\{\alpha_k\} \subseteq D$ with $\alpha_k > \alpha$ and $\alpha_k \to \alpha$ by density of D. Then

$$\{f > \alpha\} = \bigcup_{k=1}^{\infty} \{f > \alpha_k\}$$

is measurable. □

Exercise (21)**.**

(a) If $D, E \subseteq \mathbb{R}$ are measurable and $f : D \cup E \to \overline{\mathbb{R}}$, then f is measurable if and only if $f|_D$ and $f|_E$ are measurable.

(b) If $D \subseteq \mathbb{R}$ is measurable and $f : D \to \overline{\mathbb{R}}$, then f is measurable if and only if

$$g(x) = \begin{cases} f(x) & \text{if } x \in D \\ 0 & \text{otherwise} \end{cases}$$

is measurable.

Proof.

(a) The forward direction follows from the remark above about restrictions. If $f|_D$ and $f|_E$ are measurable, then $D \cup E$ is measurable and

$$\{f > \alpha\} = \{f|_D > \alpha\} \cup \{f|_E > \alpha\}$$

is measurable for all $\alpha \in \mathbb{R}$, so f is measurable.

[4] Recall that a step function is just a finite linear combination of characteristic functions of intervals.

(b) By part (a), g is measurable if and only if the restrictions $g|_D = f$ and $g|_{\tilde{D}} = 0$ are measurable, which is true if and only if f is measurable. □

Exercise (22)**.**

(a) Let $D \subseteq \mathbb{R}$ be measurable and $f : D \to \overline{\mathbb{R}}$. Let $D_1 = \{f = \infty\}$ and $D_2 = \{f = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and $f|_{D-(D_1 \cup D_2)}$ is measurable.

(b) If $f, g : D \to \overline{\mathbb{R}}$ are measurable, then fg is measurable.

(c) If $f, g : D \to \overline{\mathbb{R}}$ are measurable and $\alpha \in \overline{\mathbb{R}}$, then $f + g$ is measurable if it is defined to be α when it is of form $\infty - \infty$ or $-\infty + \infty$.

(d) If $f, g : D \to \overline{\mathbb{R}}$ are measurable and finite a.e., then $f + g$ is measurable no matter how it is defined when it is of the form $\infty - \infty$ or $-\infty + \infty$.

Proof.

(a) If f is measurable, then D_1 and D_2 are measurable (Proposition 18), so $D - (D_1 \cup D_2)$ is measurable, and $f|_{D-(D_1 \cup D_2)}$ is measurable by the remark above about restrictions.

For the converse, note $D = [D - (D_1 \cup D_2)] \cup D_1 \cup D_2$ and the restrictions $f|_{D-(D_1 \cup D_2)}$, $f|_{D_1} = \infty$, and $f|_{D_2} = -\infty$ are measurable, so f is measurable by Exercise 21(a).

(b) We have

$$\begin{aligned}\{fg = \infty\} = &\big(\{f = \infty\} \cap \{g > 0\}\big) \\ &\cup \big(\{f = -\infty\} \cap \{g < 0\}\big) \\ &\cup \big(\{f > 0\} \cap \{g = \infty\}\big) \\ &\cup \big(\{f < 0\} \cap \{g = -\infty\}\big)\end{aligned}$$

which is measurable since f and g are measurable. Similarly $\{fg = -\infty\}$ is measurable.

Let $F = \{f \neq \pm\infty\} \cap \{g \neq \pm\infty\}$. Then

$$\{fg \neq \pm\infty\} = F \cup \{f = 0\} \cup \{g = 0\}$$

Now F is measurable, so $f|_F$ and $g|_F$ are measurable, and $fg|_F = f|_F \cdot g|_F$ is measurable (Proposition 19). Also $fg|_{\{f=0\}} = 0$ and $fg|_{\{g=0\}} = 0$ are measurable. Hence $fg|_{\{fg \neq \pm\infty\}}$ is measurable by Exercise 21(a).

It follows that fg is measurable by part (a).

(c) We assume $\alpha \in \mathbb{R}$ (the cases $\alpha = \pm\infty$ are similar). Then

$$\{f+g=\infty\} = \big(\{f=\infty\}\cap\{g\neq-\infty\}\big)\cup\big(\{f\neq-\infty\}\cap\{g=\infty\}\big)$$

is measurable since f and g are measurable. Similarly $\{f+g=-\infty\}$ is measurable.

Let $F = \{f \neq \pm\infty\}\cap\{g\neq\pm\infty\}$, $I_1 = \{f=\infty\}\cap\{g=-\infty\}$, and $I_2 = \{f = -\infty\}\cap\{g=\infty\}$. Then $\{f+g\neq\pm\infty\} = F\cup I_1\cup I_2$. Now F is measurable, so $f|_F$ and $g|_F$ are measurable, and $(f+g)|_F = f|_F + g|_F$ is measurable (Proposition 19). Also $f|_{I_1\cup I_2} = \alpha$ is measurable. Hence $(f+g)|_{\{f+g\neq\pm\infty\}}$ is measurable by Exercise 21(a).

It follows that $f+g$ is measurable by part (a).

(d) By part (c), $f+g$ is equal a.e. to a measurable function (with $\alpha = 0$, say), hence is measurable (Proposition 21). □

Exercise (23)**.**

(a) Let $f:[a,b]\to\overline{\mathbb{R}}$ be measurable and finite a.e. Given $\epsilon > 0$, there exists M with $|f|\le M$ except on a set of measure less than ϵ.

(b) Let $f:[a,b]\to\overline{\mathbb{R}}$ be measurable. Given $\epsilon>0$ and M, there exists a simple function $\varphi:[a,b]\to\mathbb{R}$ with $|f-\varphi|<\epsilon$ except where $|f|\ge M$. If $m\le f\le M$, then we may take $m\le\varphi\le M$.

(c) Let $\varphi:[a,b]\to\mathbb{R}$ be simple. Given $\epsilon>0$, there exists a step function $g:[a,b]\to\mathbb{R}$ with $\varphi = g$ except on a set of measure less than ϵ. If $m\le\varphi\le M$, then we may take $m\le g\le M$.

(d) Let $g:[a,b]\to\mathbb{R}$ be a step function. Given $\epsilon>0$, there exists a continuous function $h:[a,b]\to\mathbb{R}$ with $g=h$ except on a set of measure less than ϵ. If $m\le g\le M$, then we may take $m\le h\le M$.

Finally, if $f:[a,b]\to\overline{\mathbb{R}}$ is measurable and finite a.e., then given $\epsilon>0$ there exists a step function $g:[a,b]\to\mathbb{R}$ and a continuous function $h:[a,b]\to\mathbb{R}$ such that

$$|f-g|<\epsilon \qquad\text{and}\qquad |f-h|<\epsilon$$

except on a set of measure less than ϵ. Moreover, if $m\le f\le M$, then we may take $m\le g\le M$ and $m\le h\le M$.

Proof.

(a) Let $\epsilon > 0$. Since f is measurable,

$$F_n = \{|f| > n\} = \{f < -n\} \cup \{f > n\}$$

is measurable for all $n \geq 0$, with $mF_n < \infty$ since $F_n \subseteq [a,b]$. Moreover, $F_{n+1} \subseteq F_n$ and $\bigcap F_n = \{|f| = \infty\}$. Therefore (Proposition 14)

$$\lim mF_n = m\bigcap F_n = m\{|f| = \infty\} = 0$$

since f is finite a.e. Choose M with $mF_M < \epsilon$. Then $|f| \leq M$ except on a set of measure less than ϵ.

(b) Let $\epsilon, M > 0$. Let $-M = \alpha_0 < \cdots < \alpha_n = M$ be a partition of $[-M, M]$ with mesh less than ϵ. Let $A_i = \{\alpha_{i-1} < f \leq \alpha_i\}$ for $1 \leq i \leq n$. Then each A_i is measurable since f is measurable.

Let $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$. Then φ is simple. If $|f(x)| < M$, then $x \in A_i$ for exactly one $1 \leq i \leq n$, so $\varphi(x) = \alpha_i$ and

$$|f(x) - \varphi(x)| < \alpha_i - \alpha_{i-1} < \epsilon$$

Therefore $|f - \varphi| < \epsilon$ except where $|f| \geq M$. If $m \leq f \leq M$, then by instead partitioning $[m, M]$ we obtain $m \leq \varphi \leq M$.

(c) Let $\epsilon > 0$. Write $\varphi = \sum_{i=1}^n \alpha_i \chi_{A_i}$ where the sets A_i are measurable and partition $[a,b]$. For each A_i, there exists a finite union U_i of open intervals with $m(A_i \triangle U_i) < \epsilon/n$ (Proposition 15). Let $D_i = A_i \triangle U_i$.

Let $g = \sum_{i=1}^n \alpha_i \chi_{U_i}$ on $[a,b]$. Then g is a step function. If $\varphi(x) \neq g(x)$, then $x \in A_i$ for exactly one $1 \leq i \leq n$. If $x \notin U_i$, then $x \in A_i - U_i \subseteq D_i$. If $x \in U_i$, then we must also have $x \in U_j$ for some $j \neq i$, lest $\varphi(x) = \alpha_i = g(x)$, so $x \in U_j - A_j \subseteq D_j$. Hence $\{\varphi \neq g\} \subseteq \bigcup D_i$, so $m^*\{\varphi \neq g\} \leq \sum mD_i < n(\epsilon/n) = \epsilon$. Therefore $\varphi = g$ except on a set of measure less than ϵ.

If $m \leq \varphi \leq M$, then by setting $g(x) = m$ for $x \notin \bigcup U_i$ and for $x \in U_i \cap U_j$ with $i \neq j$, we have $m \leq g \leq M$.

(d) Let $\epsilon > 0$. Write $a = x_0 < \cdots < x_n = b$ with partition mesh δ and $g(x) = \alpha_i$ for $x \in (x_{i-1}, x_i)$, $1 \leq i \leq n$. Let $\gamma = \min(\epsilon/n, \delta)/2$. Define h to be equal to g except for $1 \leq i \leq n-1$ on $[x_i - \gamma, x_i + \gamma]$, where it is defined by

$$h(x) = \frac{\alpha_i - \alpha_{i-1}}{2\gamma}[x - (x_i - \gamma)] + \alpha_{i-1}$$

(Note h connects adjacent "steps" of g using line segments defined over small subintervals.) Then h is piecewise linear and continuous, and $g = h$ except on a set of measure less than $n(\epsilon/n) = \epsilon$.

If $m \le g \le M$, clearly $m \le h \le M$.

Finally, suppose $f : [a, b] \to \overline{\mathbb{R}}$ is measurable and finite a.e. and let $\epsilon > 0$. By (a), we may assume $m \le f \le M$ except on a set E_1 with $m^* E_1 < \epsilon/3$. By (b), there exists a simple function φ with $m \le \varphi \le M$ and $|f - \varphi| < \epsilon$ except on E_1. By (c), there exists a step function g with $m \le g \le M$ and $\varphi = g$ except on a set E_2 with $m^* E_2 < \epsilon/3$. Finally, by (d) there exists a continuous function h with $m \le h \le M$ and $g = h$ except on a set E_3 with $m^* E_3 < \epsilon/3$. Let $E = E_1 \cup E_2 \cup E_3$, so $m^* E < \epsilon$. Then $|f - g| < \epsilon$ and $|f - h| < \epsilon$ except on E. □

Remark. This shows that measurable functions are "almost" step functions and "almost" continuous.

Exercise (24)**.** If $f : D \to \overline{\mathbb{R}}$ is measurable and B is a Borel set, then $f^{-1}[B]$ is measurable.

Proof. Let $\mathscr{C}$ be the class of sets X such that $f^{-1}[X]$ is measurable. We claim that $\mathscr{C}$ is a σ-algebra containing the open intervals, so that $\mathscr{C}$ contains the Borel sets by definition.

If $X \in \mathscr{C}$, then $f^{-1}[\widetilde{X}] = \widetilde{f^{-1}[X]}$ is measurable, so $\widetilde{X} \in \mathscr{C}$. If $X_n \in \mathscr{C}$, then $f^{-1}[\bigcup X_n] = \bigcup f^{-1}[X_n]$ is measurable, so $\bigcup X_n \in \mathscr{C}$. Therefore $\mathscr{C}$ is a σ-algebra. If (α, β) is an interval, then

$$f^{-1}(\alpha, \beta) = \{\alpha < f < \beta\} = \{f > \alpha\} \cap \{f < \beta\}$$

is measurable, so $\mathscr{C}$ contains the open intervals. □

Exercise (25)**.** If $f : D \to \mathbb{R}$ is measurable and $g : \mathbb{R} \to \mathbb{R}$ is continuous, then $g \circ f : D \to \mathbb{R}$ is measurable.

Proof. For $\alpha \in \mathbb{R}$,

$$\{g \circ f > \alpha\} = (g \circ f)^{-1}(\alpha, \infty) = f^{-1}[g^{-1}(\alpha, \infty)]$$

Note $g^{-1}(\alpha, \infty)$ is open, and hence Borel, since g is continuous. Therefore $f^{-1}[g^{-1}(\alpha, \infty)]$ is measurable since f is measurable (Exercise 24). □

Remark. This shows that a continuous function of a measurable function is measurable. In particular, $|f|$ is measurable if f is measurable.

Section 6

Remark. In the proof of Proposition 23, it is sufficient that $m\bigcap E_N = 0$, which is true if $f_n \to f$ a.e. This is just Proposition 24.

Exercise (29)**.** Let $E = (0,\infty)$, so E is measurable with $mE = \infty$. Let $f_n = \chi_{(n,\infty)}$ and $f = 0$, so f_n is measurable and $f_n \to f$. Let $\epsilon = \delta = 1$. Then for all N,

$$m\{|f_N - f| \geq \epsilon\} = m\{f_N = 1\} = m(N,\infty) = \infty > \delta$$

Exercise (30 (Egoroff))**.** Let E be measurable with $mE < \infty$. Let $f_n : E \to \overline{\mathbb{R}}$ be measurable and $f : E \to \mathbb{R}$ with $f_n \to f$ a.e. on E. Given $\delta > 0$, there exists $A \subseteq E$ measurable with $mA < \delta$ such that $f_n \to f$ uniformly on $E - A$.

Proof. Let $\epsilon_n = 1/n$ and $\delta_n = \delta/2^{n+1}$. For each n, there exists N_n and $A_n \subseteq E$ measurable with $mA_n < \delta_n$ such that $|f_m - f| < \epsilon_n$ for all $m \geq N_n$ on $E - A_n$ (Proposition 24). Let $A = \bigcup A_n$. Then A is measurable and

$$mA \leq \sum mA_n \leq \sum \delta/2^{n+1} = \delta/2 < \delta$$

Given $\epsilon > 0$, choose n with $\epsilon_n < \epsilon$ and set $N = N_n$. Then for $m \geq N$ and $x \in E - A \subseteq E - A_n$,

$$|f_m(x) - f(x)| < \epsilon_n < \epsilon$$

Therefore $f_n \to f$ uniformly on $E - A$. □

Remark. This shows that a convergent sequence of measurable functions (on a set of finite measure) is "almost" uniformly convergent.

Chapter 4

Section 2

Remark. For the *Riemann* integral of a function, we partition the x-axis and consider the y-values of the function on the subintervals of the partition. For the *Lebesgue* integral, we partition the y-axis and consider the x-values mapped into the subintervals of the partition. This is seen in the proof of Proposition 3.

Remark. Let $f : E \to \mathbb{R}$ be bounded and measurable with $0 < mE < \infty$. If $f > 0$, then $\int_E f > 0$.

Proof. Let $E_n = \{f > 1/n\}$. Then $E = \bigcup E_n$, and since $mE > 0$ there must be n with $mE_n > 0$ by countable subadditivity. Therefore (Proposition 5(iv),(v))

$$\int_E f \geq \int_{E_n} f \geq (1/n)mE_n > 0 \qquad \square$$

Remark. Let $f, g : E \to \mathbb{R}$ be bounded and measurable with $mE < \infty$ and $f \leq g$. Then $\int_E f = \int_E g$ if and only if $f = g$ a.e.

Proof. Let $F = \{f \neq g\} = \{g - f > 0\}$. If $mF > 0$, then (Proposition 5(i),(iv),(v) and the previous remark)

$$\int_E g - \int_E f = \int_E (g - f) \geq \int_F (g - f) > 0$$

The other direction is just Proposition 5(ii). $\square$

Remark. Let $f_n, f : E \to \mathbb{R}$ be bounded and measurable with $mE < \infty$ and $f_n \leq f_{n+1} \leq f$ for all n. If $\int f_n \to \int f$, then $f_n \to f$ a.e.

Proof. Let $f^* = \lim f_n = \sup f_n \leq f$. Then f^* is measurable and

$$0 \leq \int f - \int f^* \leq \int f - \int f_n \to 0$$

It follows that $\int f^* = \int f$, so $f^* = f$ a.e. by the previous remark. $\square$

Remark. This result shows that convergence in integral implies convergence almost everywhere. This is also seen in the proof of Proposition 3.

Proposition (7 - Lebesgue's criterion for Riemann integrability)**.** *A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable if and only if it is continuous a.e.*

Proof. Let g be the lower envelope for f. We claim that $R\underline{\int}_a^b f = \int_a^b g$. Indeed, there is a sequence of step functions $\varphi_n \leq g \leq f$ with $\varphi_n \uparrow g$ (Exercises 2.51(b), 2.50(g)). Therefore (Propositions 6, 5(iii), 4)

$$\int_a^b g = \lim_n \int_a^b \varphi_n = \sup_n \int_a^b \varphi_n \leq R\underline{\int}_a^b f$$

Conversely if $\varphi \leq f$ is a step function, then φ is lower semicontinuous except at a finite number of points (Exercise 2.50(f)), so $\varphi \leq g$ except at a finite number of points (Exercise 2.51(c)), so $\int_a^b \varphi \leq \int_a^b g$. Since φ was arbitrary, $R\underline{\int}_a^b f \leq \int_a^b g$, establishing the claim. By similar argument, $R\overline{\int}_a^b f = \int_a^b h$ where h is the upper envelope of f.

Now f is Riemann integrable if and only if

$$\int_a^b g = R\underline{\int}_a^b f = R\overline{\int}_a^b f = \int_a^b h$$

which is true if and only if $g = h$ a.e. by the above remark, which is true if and only if f is continuous a.e. (Exercise 2.51(a)). $\square$

Section 3

Remark. In this section, we consider *extended* real-valued functions.

Remark. In the proof of Proposition 8(ii), $h = \min(f, l)$ is the part of l less than f, and $k = l - h$ is the remainder, which must be less than g.

Remark. In Proposition 13, $f - g$ is never of form $\infty - \infty$ since $g < f$, so $f - g$ is measurable (Exercise 3.22(d)). The conclusion still holds if $g \leq f$. Indeed, $f = \infty$ on a set of measure zero since f is integrable, hence the same is true of g. Therefore $f - g = \infty - \infty$ on a set of measure zero, so $f - g$ is measurable.

This result shows that a nonnegative measurable function bounded above by an integrable function is itself integrable.

Remark. In the proof of Proposition 14, the technique of defining "truncated" versions of a function which converge to that function and then appealing to a convergence theorem for the integral is extremely important. However, it is not necessary in this proof. By definition, $\int_E f = \sup_{h \leq f} \int_E h$ where h is a bounded measurable function vanishing outside a set of finite measure. So given $\epsilon > 0$, we can choose such an h with $\int_E f - \int_E h < \epsilon/2$, then choose $\delta > 0$ such that for all $A \subseteq E$ with $mA < \delta$ we have $\int_A h < \epsilon/2$, so

$$\int_A f < \int_A h + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon$$

Section 4

Remark. In the proof of Proposition 15(ii), the following lemma is established: *if f_1 and f_2 are nonnegative integrable functions, then $f = f_1 - f_2$ is integrable and $\int f = \int f_1 - \int f_2$*. This is an extension of Proposition 13.

In the proof of the lemma, we may assume f is defined everywhere since f_1 and f_2 are integrable. Then f^+ is nonnegative and measurable, and $f^+ \leq f_1$

since f_1 and f_2 are nonnegative, so f^+ is integrable by a remark above. Similarly f^- is integrable, so f is integrable. The rest of the proof uses Descartes' trick of rewriting an equation of form $a - b = c - d$ in the form $a + d = b + c$ to eliminate negative quantities.

References

[1] Royden, H. L. *Real Analysis*, 3rd ed. Prentice Hall, 1988.