Notes and exercises from *The Cauchy-Schwarz Master Class*

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Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (1.1)**.** For any real numbers $r_1, \ldots, r_n$,

$$\sum_k r_k \le \sqrt{n}\Big(\sum_k r_k^2\Big)^{1/2} \tag{a}$$

and

$$\sum_k r_k \le \Big(\sum_k |r_k|^{2/3}\Big)^{1/2}\Big(\sum_k |r_k|^{4/3}\Big)^{1/2} \tag{b}$$

Proof. Both inequalities are obtained from Cauchy-Schwarz,[1] the first by taking $a_k = 1$ and $b_k = r_k$ for all k and the second by taking $a_k = r_k^{1/3}$ and $b_k = r_k^{2/3}$ for all k, recalling that $r^2 = |r|^2$ for any real number r. □

Exercise (1.2)**.** Suppose $p_k \ge 0$ for $1 \le k \le n$ and $p_1 + \cdots + p_n = 1$. If $a_k \ge 0$, $b_k \ge 0$, and $1 \le a_k b_k$ for $1 \le k \le n$, then

$$1 \le \Big(\sum_k p_k a_k\Big)\Big(\sum_k p_k b_k\Big)$$

Proof. $1 \le \sqrt{a_k b_k}$, so $p_k \le p_k\sqrt{a_k b_k} = \sqrt{p_k a_k}\sqrt{p_k b_k}$. By Cauchy-Schwarz,

$$1 = \sum_k p_k \le \sum_k (p_k a_k)^{1/2}(p_k b_k)^{1/2} \le \Big(\sum_k p_k a_k\Big)^{1/2}\Big(\sum_k p_k b_k\Big)^{1/2}$$

The result now follows from squaring both sides of the outer inequality. □

[1] This refers to the first inequality in [1], p. 1.

Exercise (1.3)**.** For any real sequences $(a_k),(b_k),(c_k)$ of length n,

$$\Big(\sum_k a_k b_k c_k\Big)^4 \le \Big(\sum_k a_k^2\Big)^2 \sum_k b_k^4 \sum_k c_k^4 \tag{a}$$

and

$$\Big(\sum_k a_k b_k c_k\Big)^2 \le \sum_k a_k^2 \sum_k b_k^2 \sum_k c_k^2 \tag{b}$$

Proof. By Cauchy-Schwarz,

$$\Big|\sum_k a_k b_k c_k\Big| \le \Big(\sum_k a_k^2\Big)^{1/2}\Big(\sum_k b_k^2 c_k^2\Big)^{1/2}$$

so

$$\Big(\sum_k a_k b_k c_k\Big)^2 \le \Big(\sum_k a_k^2\Big)\Big(\sum_k b_k^2 c_k^2\Big) \tag{c}$$

By Cauchy-Schwarz again,

$$\sum_k b_k^2 c_k^2 \le \Big(\sum_k b_k^4\Big)^{1/2}\Big(\sum_k c_k^4\Big)^{1/2}$$

which together with (c) implies

$$\Big(\sum_k a_k b_k c_k\Big)^2 \le \Big(\sum_k a_k^2\Big)\Big(\sum_k b_k^4\Big)^{1/2}\Big(\sum_k c_k^4\Big)^{1/2}$$

which implies (a). Also, clearly

$$\sum_k b_k^2 c_k^2 \le \Big(\sum_k b_k^2\Big)\Big(\sum_k c_k^2\Big)$$

which together with (c) implies (b). □

Exercise (1.4)**.** For all $x,y,z>0$,

$$\left(\frac{x+y}{x+y+z}\right)^{1/2} + \left(\frac{x+z}{x+y+z}\right)^{1/2} + \left(\frac{y+z}{x+y+z}\right)^{1/2} \le 6^{1/2} \tag{a}$$

and

$$x+y+z \le 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right) \tag{b}$$

Proof. Both inequalities are obtained from Cauchy-Schwarz, the first by taking $a_1 = (x+y)^{1/2}$, $a_2 = (x+z)^{1/2}$, $a_3 = (y+z)^{1/2}$, and $b_1 = b_2 = b_3 = (x+y+z)^{-1/2}$, and the second by first writing

$$x+y+z = \frac{x}{\sqrt{y+z}}\sqrt{y+z} + \frac{y}{\sqrt{x+z}}\sqrt{x+z} + \frac{z}{\sqrt{x+y}}\sqrt{x+y}$$

to obtain

$$(x+y+z)^2 \le 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right)(x+y+z)$$

from which the result follows. □

Exercise (1.5)**.** If $p_k \ge 0$ for $1 \le k \le n$ and $p_1 + \cdots + p_n = 1$, then

$$g(x) = \sum_k p_k \cos(\beta_k x) \qquad \text{satisfies} \qquad g^2(x) \le \frac{1}{2}\big(1 + g(2x)\big)$$

Proof. By Cauchy-Schwarz and the identity $\cos^2(\alpha) = (1+\cos(2\alpha))/2$,

$$\begin{aligned}
g^2(x) &= \Big(\sum_k \sqrt{p_k}\sqrt{p_k}\cos(\beta_k x)\Big)^2 \\
&\le \Big(\sum_k p_k\Big)\Big(\sum_k p_k \cos^2(\beta_k x)\Big) \\
&= \sum_k p_k \cos^2(\beta_k x) \\
&= \frac{1}{2}\sum_k p_k \big(1 + \cos(2\beta_k x)\big) \\
&= \frac{1}{2}\big(1 + g(2x)\big)
\end{aligned}$$

□

Exercise (1.6)**.** If $p_k > 0$ for $1 \le k \le n$ and $p_1 + \cdots + p_n = 1$, then

$$\sum_k \left(p_k + \frac{1}{p_k}\right)^2 \ge n^3 + 2n + \frac{1}{n}$$

Moreover, equality holds if and only if $p_k = 1/n$ for all k.

Proof. By Cauchy-Schwarz,

$$n^2 = \left(\sum_k \sqrt{p_k}\,\frac{1}{\sqrt{p_k}}\right)^2 \le \Big(\sum_k p_k\Big)\Big(\sum_k \frac{1}{p_k}\Big) = \sum_k \frac{1}{p_k} \tag{a}$$

so

$$n^2+1 \le \sum_k \frac{1}{p_k} + \sum_k p_k = \sum_k \left(p_k + \frac{1}{p_k}\right)$$

and by applying Exercise 1.1(a) to the last sum,

$$\frac{(n^2+1)^2}{n} \le \sum_k \left(p_k + \frac{1}{p_k}\right)^2 \tag{b}$$

which is just the desired inequality. If equality holds in (b), then equality holds in (a), so $p_k = 1/n$ for all k by (1.11); the converse is trivial. □

Exercise (1.7)**.** For any real numbers α, β, x, y,

$$(5\alpha x + \alpha y + \beta x + 3\beta y)^2 \le (5\alpha^2 + 2\alpha\beta + 3\beta^2)(5x^2 + 2xy + 3y^2)$$

Proof. In $\mathbb{R}^2$, the product

$$\langle (x_1, y_1), (x_2, y_2)\rangle = 5x_1x_2 + x_1y_2 + x_2y_1 + 3y_1y_2$$

is clearly symmetric and bilinear. Moreover,

$$\|(x,y)\|^2 = \langle (x,y),(x,y)\rangle = 5x^2 + 2xy + 3y^2$$

Fixing y and considering the quadratic polynomial $p(x) = 5x^2 + (2y)x + 3y^2$, note $p(x) \ge 0$ since $5 > 0$ and $(2y)^2 - 4\cdot 5\cdot 3y^2 = -56y^2 \le 0$, and in fact $p(x) = 0$ if and only if $x = y = 0$. Therefore the product is positive definite and so an inner product. The desired inequality is now just

$$\langle (\alpha,\beta),(x,y)\rangle^2 \le \|(\alpha,\beta)\|^2 \|(x,y)\|^2$$

which is Cauchy-Schwarz (1.16). □

Exercise (1.8)**.** The following inequalities hold:

$$\sum_{k=0}^{\infty} a_k x^k \le \frac{1}{\sqrt{1-x^2}} \Big(\sum_{k=0}^{\infty} a_k^2\Big)^{1/2} \qquad (0 \le x < 1) \tag{a}$$

$$\sum_{k=1}^{n} \frac{a_k}{k} < \sqrt{2} \Big(\sum_{k=0}^{n} a_k^2\Big)^{1/2} \tag{b}$$

$$\sum_{k=1}^{n} \frac{a_k}{\sqrt{n+k}} < (\log 2)^{1/2} \Big(\sum_{k=0}^{n} a_k^2\Big)^{1/2} \tag{c}$$

$$\sum_{k=0}^{n} \binom{n}{k} a_k \le \binom{2n}{n}^{1/2} \Big(\sum_{k=0}^{n} a_k^2\Big)^{1/2} \tag{d}$$

Proof. The inequalities all follow from Cauchy-Schwarz, together with suitable bounds for $\sum_k b_k^2$. For (a), note $0 \le x^2 < 1$, so $\sum_k (x^k)^2 = \sum_k (x^2)^k = (1-x^2)^{-1}$ by convergence of the geometric series. For (b), recall

$$\sum_{k=1}^{n} \frac{1}{k^2} < \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} < 2$$

For (c), note

$$\sum_{k=1}^{n} \frac{1}{n+k} < \int_{n}^{2n} \frac{1}{x}\, dx = \log(2n) - \log(n) = \log(2n/n) = \log(2)$$

For (d), recall for any $0 \le m \le n$,

$$\binom{n}{k} = \sum_{j=0}^{k} \binom{m}{j}\binom{n-m}{k-j}$$

so

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2 \qquad \square$$

References

[1] Steele, J. Michael. *The Cauchy-Schwarz Master Class.* Cambridge, 2004.