Notes and exercises from *The Cauchy-Schwarz Master Class*

John Peloquin

Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (1.1)**.** For any real numbers $r_1, \ldots, r_n$,

$$\sum_k r_k \leq \sqrt{n}\Big(\sum_k r_k^2\Big)^{1/2} \tag{a}$$

and

$$\sum_k r_k \leq \Big(\sum_k |r_k|^{2/3}\Big)^{1/2}\Big(\sum_k |r_k|^{4/3}\Big)^{1/2} \tag{b}$$

Proof. Both inequalities are obtained from Cauchy-Schwarz,[1] the first by taking $a_k = 1$ and $b_k = r_k$ for all k and the second by taking $a_k = r_k^{1/3}$ and $b_k = r_k^{2/3}$ for all k, recalling that $r^2 = |r|^2$ for any real number r. □

Exercise (1.2)**.** Suppose $p_k \geq 0$ for $1 \leq k \leq n$ and $p_1 + \cdots + p_n = 1$. If $a_k \geq 0$, $b_k \geq 0$, and $1 \leq a_k b_k$ for $1 \leq k \leq n$, then

$$1 \leq \Big(\sum_k p_k a_k\Big)\Big(\sum_k p_k b_k\Big)$$

Proof. $1 \leq \sqrt{a_k b_k}$, so $p_k \leq p_k\sqrt{a_k b_k} = \sqrt{p_k a_k}\sqrt{p_k b_k}$. By Cauchy-Schwarz,

$$1 = \sum_k p_k \leq \sum_k (p_k a_k)^{1/2}(p_k b_k)^{1/2} \leq \Big(\sum_k p_k a_k\Big)^{1/2}\Big(\sum_k p_k b_k\Big)^{1/2}$$

The result now follows from squaring both sides of the outer inequality. □

[1] This refers to the first inequality in [1], p. 1.

Exercise (1.3)**.** For any real sequences $(a_k), (b_k), (c_k)$ of length n,

$$\Big(\sum_k a_k b_k c_k\Big)^4 \le \Big(\sum_k a_k^2\Big)^2 \sum_k b_k^4 \sum_k c_k^4 \tag{a}$$

and

$$\Big(\sum_k a_k b_k c_k\Big)^2 \le \sum_k a_k^2 \sum_k b_k^2 \sum_k c_k^2 \tag{b}$$

Proof. By Cauchy-Schwarz,

$$\Big|\sum_k a_k b_k c_k\Big| \le \Big(\sum_k a_k^2\Big)^{1/2} \Big(\sum_k b_k^2 c_k^2\Big)^{1/2}$$

so

$$\Big(\sum_k a_k b_k c_k\Big)^2 \le \Big(\sum_k a_k^2\Big)\Big(\sum_k b_k^2 c_k^2\Big) \tag{c}$$

By Cauchy-Schwarz again,

$$\sum_k b_k^2 c_k^2 \le \Big(\sum_k b_k^4\Big)^{1/2} \Big(\sum_k c_k^4\Big)^{1/2}$$

which together with (c) implies

$$\Big(\sum_k a_k b_k c_k\Big)^2 \le \Big(\sum_k a_k^2\Big)\Big(\sum_k b_k^4\Big)^{1/2} \Big(\sum_k c_k^4\Big)^{1/2}$$

which implies (a). Also, clearly

$$\sum_k b_k^2 c_k^2 \le \Big(\sum_k b_k^2\Big)\Big(\sum_k c_k^2\Big)$$

which together with (c) implies (b). □

Exercise (1.4)**.** For all $x, y, z > 0$,

$$\left(\frac{x+y}{x+y+z}\right)^{1/2} + \left(\frac{x+z}{x+y+z}\right)^{1/2} + \left(\frac{y+z}{x+y+z}\right)^{1/2} \le 6^{1/2} \tag{a}$$

and

$$x + y + z \le 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right) \tag{b}$$

Proof. Both inequalities are obtained from Cauchy-Schwarz, the first by taking $a_1 = (x+y)^{1/2}$, $a_2 = (x+z)^{1/2}$, $a_3 = (y+z)^{1/2}$, and $b_1 = b_2 = b_3 = (x+y+z)^{-1/2}$, and the second by first writing

$$x+y+z = \frac{x}{\sqrt{y+z}}\sqrt{y+z} + \frac{y}{\sqrt{x+z}}\sqrt{x+z} + \frac{z}{\sqrt{x+y}}\sqrt{x+y}$$

to obtain

$$(x+y+z)^2 \le 2\left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y}\right)(x+y+z)$$

from which the result follows. □

Exercise (1.5)**.** If $p_k \ge 0$ for $1 \le k \le n$ and $p_1 + \cdots + p_n = 1$, then

$$g(x) = \sum_k p_k \cos(\beta_k x) \qquad \text{satisfies} \qquad g^2(x) \le \frac{1}{2}\big(1 + g(2x)\big)$$

Proof. By Cauchy-Schwarz and the identity $\cos^2(\alpha) = (1+\cos(2\alpha))/2$,

$$\begin{aligned}
g^2(x) &= \Big(\sum_k \sqrt{p_k}\sqrt{p_k}\cos(\beta_k x)\Big)^2 \\
&\le \Big(\sum_k p_k\Big)\Big(\sum_k p_k \cos^2(\beta_k x)\Big) \\
&= \sum_k p_k \cos^2(\beta_k x) \\
&= \frac{1}{2}\sum_k p_k\big(1+\cos(2\beta_k x)\big) \\
&= \frac{1}{2}\big(1+g(2x)\big)
\end{aligned}$$

□

Exercise (1.6)**.** If $p_k > 0$ for $1 \le k \le n$ and $p_1 + \cdots + p_n = 1$, then

$$\sum_k \left(p_k + \frac{1}{p_k}\right)^2 \ge n^3 + 2n + \frac{1}{n}$$

Moreover, equality holds if and only if $p_k = 1/n$ for all k.

Proof. By Cauchy-Schwarz,

$$n^2 = \left(\sum_k \sqrt{p_k}\,\frac{1}{\sqrt{p_k}}\right)^2 \le \Big(\sum_k p_k\Big)\Big(\sum_k \frac{1}{p_k}\Big) = \sum_k \frac{1}{p_k} \tag{a}$$

so

$$n^2 + 1 \le \sum_k \frac{1}{p_k} + \sum_k p_k = \sum_k \left(p_k + \frac{1}{p_k} \right)$$

and by applying Exercise 1.1(a) to the last sum,

$$\frac{(n^2+1)^2}{n} \le \sum_k \left(p_k + \frac{1}{p_k} \right)^2 \tag{b}$$

which is just the desired inequality. If equality holds in (b), then equality holds in (a), so $p_k = 1/n$ for all k by (1.11); the converse is trivial. □

References

[1] Steele, J. Michael. *The Cauchy-Schwarz Master Class.* Cambridge, 2004.