

Notes and exercises from *The Cauchy-Schwarz Master Class*

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Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (1.1). For any real numbers r_1, \dots, r_n ,

$$\sum_k r_k \leq \sqrt{n} \left(\sum_k r_k^2 \right)^{1/2}$$

and

$$\sum_k r_k \leq \left(\sum_k |r_k|^{2/3} \right)^{1/2} \left(\sum_k |r_k|^{4/3} \right)^{1/2}$$

Proof. Both inequalities are obtained from Cauchy-Schwarz,¹ the first by taking $a_k = 1$ and $b_k = r_k$ for all k and the second by taking $a_k = r_k^{1/3}$ and $b_k = r_k^{2/3}$ for all k , recalling that $r^2 = |r|^2$ for any real number r . \square

Exercise (1.2). Suppose $p_k \geq 0$ for $1 \leq k \leq n$ and $p_1 + \dots + p_n = 1$. If $a_k \geq 0$, $b_k \geq 0$, and $1 \leq a_k b_k$ for $1 \leq k \leq n$, then

$$1 \leq \left(\sum_k p_k a_k \right) \left(\sum_k p_k b_k \right)$$

Proof. $1 \leq \sqrt{a_k b_k}$, so $p_k \leq p_k \sqrt{a_k b_k} = \sqrt{p_k a_k} \sqrt{p_k b_k}$. By Cauchy-Schwarz,

$$1 = \sum_k p_k \leq \sum_k (p_k a_k)^{1/2} (p_k b_k)^{1/2} \leq \left(\sum_k p_k a_k \right)^{1/2} \left(\sum_k p_k b_k \right)^{1/2}$$

The result now follows from squaring both sides of the outer inequality. \square

¹This refers to the first inequality in [1], p. 1.

Exercise (1.3). For any real sequences $(a_k), (b_k), (c_k)$ of length n ,

$$\left(\sum_k a_k b_k c_k\right)^4 \leq \left(\sum_k a_k^2\right)^2 \sum_k b_k^4 \sum_k c_k^4 \quad (\text{a})$$

and

$$\left(\sum_k a_k b_k c_k\right)^2 \leq \sum_k a_k^2 \sum_k b_k^2 \sum_k c_k^2 \quad (\text{b})$$

Proof. By Cauchy-Schwarz,

$$\left|\sum_k a_k b_k c_k\right| \leq \left(\sum_k a_k^2\right)^{1/2} \left(\sum_k b_k^2 c_k^2\right)^{1/2}$$

so

$$\left(\sum_k a_k b_k c_k\right)^2 \leq \left(\sum_k a_k^2\right) \left(\sum_k b_k^2 c_k^2\right) \quad (\text{c})$$

By Cauchy-Schwarz again,

$$\sum_k b_k^2 c_k^2 \leq \left(\sum_k b_k^4\right)^{1/2} \left(\sum_k c_k^4\right)^{1/2}$$

which together with (c) implies

$$\left(\sum_k a_k b_k c_k\right)^2 \leq \left(\sum_k a_k^2\right) \left(\sum_k b_k^4\right)^{1/2} \left(\sum_k c_k^4\right)^{1/2}$$

which implies (a). Also, clearly

$$\sum_k b_k^2 c_k^2 \leq \left(\sum_k b_k^2\right) \left(\sum_k c_k^2\right)$$

which together with (c) implies (b). □

Exercise (1.4). For all $x, y, z > 0$,

$$\left(\frac{x+y}{x+y+z}\right)^{1/2} + \left(\frac{x+z}{x+y+z}\right)^{1/2} + \left(\frac{y+z}{x+y+z}\right)^{1/2} \leq 6^{1/2} \quad (\text{a})$$

and

$$x + y + z \leq 2 \left(\frac{x^2}{y+z} + \frac{y^2}{x+z} + \frac{z^2}{x+y} \right) \quad (\text{b})$$

Proof. Both inequalities are obtained from Cauchy-Schwarz, the first by taking $a_1 = (x + y)^{1/2}$, $a_2 = (x + z)^{1/2}$, $a_3 = (y + z)^{1/2}$, and $b_1 = b_2 = b_3 = (x + y + z)^{-1/2}$, and the second by first writing

$$x + y + z = \frac{x}{\sqrt{y + z}} \sqrt{y + z} + \frac{y}{\sqrt{x + z}} \sqrt{x + z} + \frac{z}{\sqrt{x + y}} \sqrt{x + y}$$

to obtain

$$(x + y + z)^2 \leq 2 \left(\frac{x^2}{y + z} + \frac{y^2}{x + z} + \frac{z^2}{x + y} \right) (x + y + z)$$

from which the result follows. □

References

- [1] Steele, J. Michael. *The Cauchy-Schwarz Master Class*. Cambridge, 2004.