Berkeley Problems in Mathematics

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Abstract

Alternate solutions to problems from *Berkeley Problems in Mathematics*.

Chapter 6

Section 1

Problem 6.1.9. Let G be a finite group with identity e such that for all $a, b \in G$ with $a, b \neq e$, there exists an automorphism σ of G such that $\sigma(a) = b$. Then G is abelian.

Proof. Set $n = |G|$ and assume $n \neq 1$. Then $\mathrm{Aut}(G)$ acts on G and yields two orbits, the trivial orbit and an orbit of order $n-1$. Recall $\mathrm{Inn}(G) \trianglelefteq \mathrm{Aut}(G)$, so $\mathrm{Inn}(G)$ also acts on G, and the order of any $\mathrm{Inn}(G)$-orbit divides the order of an $\mathrm{Aut}(G)$-orbit.[1] This implies the order of any $\mathrm{Inn}(G)$-orbit divides $n-1$. But by the Orbit-Stabilizer Theorem, the order of any $\mathrm{Inn}(G)$-orbit also divides $|\mathrm{Inn}(G)|$. And since there is a natural surjection $G \to \mathrm{Inn}(G)$, $|\mathrm{Inn}(G)|$ in turn divides n. It follows that there are no nontrivial $\mathrm{Inn}(G)$-orbits, so any element in G is preserved under conjugation by another element, and hence G is abelian. □

References

[1] Dummit, David S. and Richard M. Foote. *Abstract Algebra, 3rd ed.* Wiley, 2003.

[1] [1], Exercise 4.1.9(a).