Mixed up with Cayley and Hamilton (draft)

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Introduction

In [2], Bapat and Roy proved the Cayley-Hamilton theorem for mixed discriminants using graph-theoretic methods. In this paper, we provide a proof of the theorem in the simpler case of commuting linear transformations using results from mixed exterior algebra as developed by Greub and Vanstone in [3] and [4]. It is a question whether a similar approach can be used in the noncommutative case.

Throughout this paper, $\mathbb{F}$ is a field of characteristic zero, V is an n-dimensional vector space over $\mathbb{F}$, and ι is the identity transformation on V.

Determinants

We start with a generalization of the exterior powers of a linear transformation:

Definition 1. The p-ary *box product* of linear transformations $\varphi_1, \ldots, \varphi_p$ of V is the linear transformation $\varphi_1 \Box \cdots \Box \varphi_p$ of $\bigwedge^p V$ satisfying

$$\begin{aligned}(\varphi_1 \Box \cdots \Box \varphi_p)(x_1 \wedge \cdots \wedge x_p) &= \sum_{\sigma \in S_p} (-1)^\sigma \varphi_1 x_{\sigma(1)} \wedge \cdots \wedge \varphi_p x_{\sigma(p)} \\ &= \sum_{\sigma \in S_p} \varphi_{\sigma(1)} x_1 \wedge \cdots \wedge \varphi_{\sigma(p)} x_p \end{aligned} \tag{1}$$

If φ is a linear transformation of V, it follows from (1) that

$$\bigwedge^p \varphi = \frac{1}{p!} \underbrace{\varphi \Box \cdots \Box \varphi}_{p \text{ factors}} \tag{2}$$

Using the box product, we obtain a generalization of the determinant:

Definition 2. The *mixed determinant*[1] of linear transformations $\varphi_1, \ldots, \varphi_n$ of V is the scalar

$$\operatorname{Det}(\varphi_1, \ldots, \varphi_n) = \operatorname{tr}(\varphi_1 \Box \cdots \Box \varphi_n) \tag{3}$$

[1] This is also called a *mixed discriminant*, and historically was called a *cubic*, *3-dimensional*, or *3-way* determinant. It is sometimes divided by $n!$. See [1], [5], and [6].

If Δ is a determinant function on V with $\Delta(x_1,\ldots,x_n)=1$, then

$$\operatorname{Det}(\varphi_1,\ldots,\varphi_n)=\sum_{\sigma\in S_n}\Delta(\varphi_{\sigma(1)}x_1,\ldots,\varphi_{\sigma(n)}x_n) \tag{4}$$

It follows from (2) and (3) and the fact that $\det\varphi=\operatorname{tr}(\bigwedge^n\varphi)$, or from (4), that

$$\det\varphi=\frac{1}{n!}\operatorname{Det}(\varphi,\ldots,\varphi) \tag{5}$$

Adjoints

We also have a generalization of the classical adjoint:

Definition 3. The *mixed adjoint* of linear transformations $\varphi_1,\ldots,\varphi_{n-1}$ of V is the linear transformation of V given by

$$\operatorname{Adj}(\varphi_1,\ldots,\varphi_{n-1})=D(\varphi_1\,\square\cdots\square\,\varphi_{n-1}) \tag{6}$$

where D is the mixed Poincaré dual map.[2]

Again if $\Delta(x_1,\ldots,x_n)=1$, then

$$\operatorname{Adj}(\varphi_1,\ldots,\varphi_{n-1})x=\sum_{\sigma\in S_n}\Delta(\underbrace{\varphi_{\sigma(1)}x_1,\ldots,x}_{\sigma^{-1}(n)},\ldots,\varphi_{\sigma(n)}x_n)x_{\sigma^{-1}(n)} \tag{7}$$

It follows from (6) or (7) that the classical adjoint is given by

$$\operatorname{adj}\varphi=\frac{1}{(n-1)!}\operatorname{Adj}(\varphi,\ldots,\varphi) \tag{8}$$

A fundamental relationship between the mixed adjoint and the mixed determinant is found in:

Theorem 1. *For linear transformations $\varphi_1,\ldots,\varphi_n$ of V,*

$$\operatorname{Det}(\varphi_1,\ldots,\varphi_n)\iota=\sum_{i=1}^{n}\operatorname{Adj}(\varphi_1,\ldots,\widehat{\varphi_i},\ldots,\varphi_n)\circ\varphi_i \tag{9}$$

$$=\sum_{i=1}^{n}\varphi_i\circ\operatorname{Adj}(\varphi_1,\ldots,\widehat{\varphi_i},\ldots,\varphi_n) \tag{10}$$

where $\widehat{\varphi_i}$ denotes deletion of φ_i.

This theorem follows from the Greub-Vanstone identity in [4] and has the following familiar corollary by (5) and (8):

Corollary 1. *For a linear transformation φ of V,*

$$(\operatorname{adj}\varphi)\varphi=(\det\varphi)\iota=\varphi(\operatorname{adj}\varphi) \tag{11}$$

[2] See [3], Chapter 7.

Cayley-Hamilton

The classical Cayley-Hamilton theorem says that a linear transformation satisfies its own characteristic equation:

Definition 4. The *characteristic polynomial* of a linear transformation φ of V is the polynomial χ_φ given by[3]

$$\chi_\varphi(\lambda) = \det(\varphi - \lambda\iota) \qquad (\lambda \in \mathbb{F}) \tag{12}$$

and the *characteristic equation* of φ is

$$\chi_\varphi(\lambda) = 0 \tag{13}$$

Theorem 2 (Cayley-Hamilton)**.** *For a linear transformation φ of V,*

$$\chi_\varphi(\varphi) = 0 \tag{14}$$

This can be proved by applying (11) to $\varphi - \lambda\iota$, establishing a relationship between adjoint and characteristic coefficients, and summing up the results to obtain zero. All of these ideas can be generalized:

Definition 5. The *mixed characteristic polynomial* of linear transformations $\varphi_1, \ldots, \varphi_n$ of V is the polynomial $\chi_{\varphi_1\cdots\varphi_n}$ given by

$$\chi_{\varphi_1\cdots\varphi_n}(\lambda_1, \ldots, \lambda_n) = \mathrm{Det}(\varphi_1 - \lambda_1\iota, \ldots, \varphi_n - \lambda_n\iota) \qquad (\lambda_i \in \mathbb{F}) \tag{15}$$

and the *mixed characteristic equation* of $\varphi_1, \ldots, \varphi_n$ is

$$\chi_{\varphi_1\cdots\varphi_n}(\lambda_1, \ldots, \lambda_n) = 0 \tag{16}$$

By multilinearity of Det, $\chi_{\varphi_1\cdots\varphi_n}$ is a polynomial function in the scalars $\lambda_1, \ldots, \lambda_n$, which is equivalent to a polynomial in *commuting* indeterminates $\lambda_1, \ldots, \lambda_n$ since the field $\mathbb{F}$ is infinite. For this reason, we state a mixed Cayley-Hamilton theorem for commuting linear transformations:

Theorem 3. *For commuting linear transformations $\varphi_1, \ldots, \varphi_n$ of V,*

$$\chi_{\varphi_1\cdots\varphi_n}(\varphi_1, \ldots, \varphi_n) = 0 \tag{17}$$

Before proving this theorem, we write

$$\chi_{\varphi_1\cdots\varphi_n}(\lambda_1, \ldots, \lambda_n) = \sum_{b \in 2^n} (-1)^b C_b(\varphi_1, \ldots, \varphi_n) \lambda_1^{b_1} \cdots \lambda_n^{b_n} \tag{18}$$

where $(-1)^b = (-1)^{b_1 + \cdots + b_n}$ and

$$C_b(\varphi_1, \ldots, \varphi_n) = \mathrm{Det}(\varphi_1^{1-b_1}, \ldots, \varphi_n^{1-b_n}) \tag{19}$$

[3] In (12), χ_φ is defined as a polynomial function in the scalar λ, which is equivalent to a polynomial in an indeterminate λ since $\mathbb{F}$ is infinite.

We call the scalars (19) *mixed characteristic coefficients.*

We also write

$$\operatorname{Adj}(\varphi_1-\lambda_1\iota,\ldots,\widehat{\varphi_i-\lambda_i\iota},\ldots,\varphi_n-\lambda_n\iota) = \sum_{\widehat{b}\in 2^{n-1}} (-1)^{\widehat{b}} A_{\widehat{b}}(\varphi_1,\ldots,\widehat{\varphi_i},\ldots,\varphi_n)\lambda_1^{b_1}\cdots\widehat{\lambda_i^{b_i}}\cdots\lambda_n^{b_n} \tag{20}$$

where $\widehat{b}=(b_1,\ldots,\widehat{b_i},\ldots,b_n)$ and

$$A_{\widehat{b}}(\varphi_1,\ldots,\widehat{\varphi_i},\ldots,\varphi_n) = \operatorname{Adj}(\varphi_1^{1-b_1},\ldots,\widehat{\varphi_i^{1-b_i}},\ldots,\varphi_n^{1-b_n}) \tag{21}$$

We call the linear transformations (21) *mixed adjoint coefficients.* To prove the mixed Cayley-Hamilton theorem, we establish a relationship between these and the mixed characteristic coefficients:

Proof. Substituting $\varphi_1,\ldots,\varphi_n$ for $\lambda_1,\ldots,\lambda_n$ in (18), we obtain

$$\chi_{\varphi_1\cdots\varphi_n}(\varphi_1,\ldots,\varphi_n) = \sum_{b\in 2^n} (-1)^b C_b(\varphi_1,\ldots,\varphi_n)\iota\circ\varphi_1^{b_1}\cdots\varphi_n^{b_n} \tag{22}$$

On the other hand, it follows from (9), (19), and (21) that

$$C_b(\varphi_1,\ldots,\varphi_n)\iota = \sum_{i=1}^{n} A_{\widehat{b}}(\varphi_1,\ldots,\widehat{\varphi_i},\ldots,\varphi_n)\circ\varphi_i^{1-b_i} \tag{23}$$

Now it follows from (22) and (23) that

$$\begin{aligned}
\chi_{\varphi_1\cdots\varphi_n}(\varphi_1,\ldots,\varphi_n) &= \sum_{i=1}^{n}\sum_{b\in 2^n} (-1)^b A_{\widehat{b}}(\varphi_1,\ldots,\widehat{\varphi_i},\ldots,\varphi_n)\circ\varphi_1^{b_1}\cdots\varphi_i\cdots\varphi_n^{b_n} \\
&= \sum_{i=1}^{n}\sum_{\widehat{b}\in 2^{n-1}} (-1)^{\widehat{b}}\Big(A_{\widehat{b}}(\varphi_1,\ldots,\widehat{\varphi_i},\ldots,\varphi_n)\circ\varphi_1^{b_1}\cdots\varphi_i\cdots\varphi_n^{b_n} \\
&= \qquad\qquad -A_{\widehat{b}}(\varphi_1,\ldots,\widehat{\varphi_i},\ldots,\varphi_n)\circ\varphi_1^{b_1}\cdots\varphi_i\cdots\varphi_n^{b_n}\Big) \\
&= 0
\end{aligned} \tag{24}$$

which completes the proof. □

By (5), (12), and (15) we have

$$\chi_\varphi(\lambda) = \frac{1}{n!}\chi_{\varphi\cdots\varphi}(\lambda,\ldots,\lambda) \tag{25}$$

so (14) follows from (17) and the classical Cayley-Hamilton theorem is a special case of the mixed Cayley-Hamilton theorem.

Conclusion

We conclude with a question:

Question 1. Can a similar approach be used to prove the mixed Cayley-Hamilton theorem for possibly noncommuting linear transformations?

Greub and Vanstone only developed mixed exterior algebra over an arbitrary field of characteristic zero, and commutativity of the scalars in a field forced us to use commuting indeterminates in the mixed characteristic polynomial (15), which in turn limited us to commuting transformations in (17). But perhaps there is some way to extend these ideas to the noncommutative case.

References

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