Notes and exercises from *The Lambda Calculus*

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Introduction

This document contains notes and exercises from [1].

Chapter 2

Exercise (2.4.1)**.** The following terms have normal forms:

(i) $(\lambda y.yyy)((\lambda ab.a)\mathbf{I}(\mathbf{SS}))$

(ii) $(\lambda yz.zy)((\lambda x.xxx)(\lambda x.xxx))(\lambda w.\mathbf{I})$

Proof.

(i) $(\lambda y.yyy)((\lambda ab.a)\mathbf{I}(\mathbf{SS})) = (\lambda y.yyy)\mathbf{I} = \mathbf{III} = \mathbf{I}$

(ii) $(\lambda yz.zy)((\lambda x.xxx)(\lambda x.xxx))(\lambda w.\mathbf{I}) = (\lambda w.\mathbf{I})((\lambda x.xxx)(\lambda x.xxx)) = \mathbf{I}$ $\square$

Exercise (2.4.2)**.** The following terms are incompatible:

(i) $\mathbf{I} \sharp \mathbf{K}$

(ii) $\mathbf{I} \sharp \mathbf{S}$

(iii) $xy \sharp xx$ $(x \not\equiv y)$

Proof. By reducing to $\mathbf{K} \sharp \mathbf{S}$ (2.1.33).

(i) If $\mathbf{I} = \mathbf{K}$, then $\mathbf{S} = \mathbf{IS} = \mathbf{IIS} = \mathbf{KKS} = \mathbf{K}$.

(ii) If $\mathbf{I} = \mathbf{S}$, then since $\mathbf{I} = \mathbf{SKK}$ (2.2.1(i)), $\mathbf{S} = \mathbf{IS} = \mathbf{SKKS} = \mathbf{IKKS} = \mathbf{KKS} = \mathbf{K}$.

(iii) If $xy = xx$, then $MN = MM$ for all $M, N \in \Lambda$ by rule ξ and combinatory completeness (2.1.24). Therefore $\mathbf{S} = \mathbf{IS} = \mathbf{II} = \mathbf{I}$, which reduces to (ii). $\square$

Exercise (2.4.4)**.** Application is not associative; in fact, $(xy)z \sharp x(yz)$ for distinct x, y, z.

Proof. If $(xy)z = x(yz)$, then $(MN)P = M(NP)$ for all $M, N, P \in \Lambda$ by rule ξ and combinatory completeness (2.1.24). In particular,

$$\mathbf{S} = \mathbf{IS} = ((\mathbf{KI})\mathbf{K})\mathbf{S} = (\mathbf{K}(\mathbf{IK}))\mathbf{S} = \mathbf{KKS} = \mathbf{K}$$

The result follows since $\mathbf{K} \sharp \mathbf{S}$ (2.1.33). $\square$

Exercise (2.4.6)**.** There is no $F \in \Lambda$ such that $F(MN) = M$ for all $M, N \in \Lambda$.

Proof. If there were such an F, then taking $M = \mathbf{I}$ and $N = \mathbf{Y}F$ (2.1.5),[1]

$$\mathbf{I} = F(\mathbf{I}(\mathbf{Y}F)) = F(\mathbf{Y}F) = \mathbf{Y}F$$

But this contradicts the fact that $\mathbf{Y}F$ has no normal form. $\square$

Exercise (2.4.7)**.** There is $M \in \Lambda$ such that $MN = MM$ for all $N \in \Lambda$.

Proof. This is true if $M = \lambda x.MM$, which is true if $M = (\lambda yx.yy)M$. So we can take $M = \mathbf{Y}(\lambda yx.yy)$ (2.1.5). $\square$

Exercise (2.4.9)**.** $(\lambda y.(\lambda x.M))N = \lambda x.((\lambda y.M)N)$ for all $M, N \in \Lambda$, provided $x \not\equiv y$ and $x \notin \mathrm{FV}(N)$.

Proof. By β-reduction,

$$(\lambda y.(\lambda x.M))N = (\lambda x.M)[y \mapsto N] = \lambda x.M[y \mapsto N] = \lambda x.((\lambda y.M)N) \qquad \square$$

Remark. The hypotheses are necessary. For example if x, y, z are distinct,

$$(\lambda x.(\lambda x.x))y = \lambda x.x \neq \lambda x.y = \lambda x.((\lambda x.x)y)$$

and

$$(\lambda y.(\lambda x.y))x = \lambda z.x \neq \lambda x.x = \lambda x.((\lambda y.y)x)$$

Exercise (2.4.13)**.** $\lambda x.Mx = M$ for all $M \in \Lambda$ starting with λ, provided $x \notin \mathrm{FV}(M)$.

[1] Here $\mathbf{Y}F$ denotes the fixed point of F in 2.1.5.

Proof. If $M \equiv \lambda y.N$, then

$$\lambda x.(\lambda y.N)x = \lambda x.N[y \mapsto x] \equiv \lambda y.N \equiv M$$

Note this is justified when $y \not\equiv x$ because $x \notin \mathrm{FV}(N)$, so the only free occurrences of x in $N[y \mapsto x]$ correspond to free occurrences of y in N. □

Remark. The hypotheses are necessary. For example, $\lambda x.yx \neq x$ and

$$\lambda x.(\lambda y.x)x = \lambda x.x \neq \lambda y.x$$

Exercise (2.4.14)**.** $M \equiv \lambda x.x(\lambda y.yy)(\lambda y.yy)$ is I-solvable.

Proof. $M(\mathbf{K}(\mathbf{KI})) = \mathbf{I}$. □

Exercise (2.4.15)**.** We can write down short λ-terms having very long normal forms using the exponentiation combinator on Church numerals (2.2.6).

References

[1] Barendregt, H. *The Lambda Calculus, its Syntax and Semantics.* 2012.