

# Notes and exercises from *Linear Algebra and Geometry*

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## Introduction

This document contains notes and exercises from [1].

## Chapter III

### Section 1

**Exercise (4).** Let  $V, W$  be a pair of supplementary subspaces of  $E$ . Every subspace  $U$  containing  $V$  is the direct sum of  $V$  with  $U \cap W$ .

*Proof.* If  $u \in U$ , then  $u = v + w$  for some  $v \in V$  and  $w \in W$ , and  $w = u - v \in U$ . So  $U = V + (U \cap W)$ , and  $V \cap U \cap W = \{0\}$ .  $\square$

### Section 2

**Exercise (1).** If  $p, q$  are the projections corresponding to a direct sum  $E = V + W$ , then  $p, q \in \text{End}(E)$  are such that  $p^2 = p$ ,  $q^2 = q$ , and  $p + q = 1$ . Conversely, if  $p \in \text{End}(E)$  is such that  $p^2 = p$ , then  $E = p(E) + p^{-1}(0)$  is a direct sum. Moreover, if  $q = 1 - p$ , then  $q^2 = q$ ,  $q(E) = p^{-1}(0)$ , and  $q^{-1}(0) = p(E)$ .

*Proof.* For the forward direction, we know  $p, q \in \text{End}(E)$  and  $p + q = 1$  (3.2.2). It follows that  $p^2 = p \circ (1 - q) = p - pq = p$  and  $q^2 = (1 - p)^2 = 1 - p = q$ .

For the converse,  $p + q = 1$ , so  $E = p(E) + q(E)$ . Also  $pq = p - p^2 = 0$ , so  $p(E) \cap q(E) = \{0\}$  and  $q(E) \subseteq p^{-1}(0)$ . If  $x \in p^{-1}(0)$ , then  $q(x) = x$ , so  $x \in q(E)$ . Hence  $q(E) = p^{-1}(0)$  and similarly  $q^{-1}(0) = p(E)$ . Finally  $q^2 = q$  as above.  $\square$

**Exercise (2).** If  $W$  and  $W'$  are both supplementary to  $V$  in  $E$ , then  $W$  and  $W'$  are isomorphic.

*Proof.* If  $p$  is the projection of  $E$  onto  $W'$ , then the restriction of  $p$  to  $W$  is an isomorphism from  $W$  to  $W'$ .  $\square$

**Exercise (3).** If  $E = V + W$  is a direct sum with inclusions  $i : V \rightarrow E$  and  $j : W \rightarrow E$ , and  $v : V \rightarrow F$  and  $w : W \rightarrow F$  are linear maps, then there is a unique linear map  $u : E \rightarrow F$  with  $u \circ i = v$  and  $u \circ j = w$ .

*Proof.* If  $p, q$  are the projections on  $V, W$  respectively, then  $u = v \circ p + w \circ q$ .  $\square$

**Exercise (11).**  $\mathbf{GA}(E)/E \cong \mathbf{GL}(E)$ .

*Proof.* Define  $\varphi : \mathbf{GA}(E) \rightarrow \mathbf{GL}(E)$  by  $\varphi(t_a \circ v) = v$ . Note that  $\varphi$  is well-defined by (3.2.17),  $\varphi$  is a homomorphism by (3.2.19), and  $\varphi$  is clearly surjective. Also  $\varphi(u) = 1$  if and only if  $u$  is a translation, so  $\ker \varphi = T(E)$ , the normal subgroup of translations. It follows that  $\mathbf{GA}(E)/T(E) \cong \mathbf{GL}(E)$ . Finally, the mapping  $a \mapsto t_a$  is an isomorphism  $E \cong T(E)$  from the additive group  $E$ .  $\square$

**Exercise (13).** If  $u : E \rightarrow F$  is affine and  $L$  is a variety in  $F$ , then  $u^{-1}(L)$  is empty or a variety in  $E$ .

*Proof.* If  $a \in u^{-1}(L)$  and  $L_0$  is the direction of  $L$ , then  $L = u(a) + L_0$  and hence  $u^{-1}(L) = a + u^{-1}(L_0)$ .  $\square$

### Section 3

**Exercise (3).** A necessary and sufficient condition for a nonempty subset  $V$  of a vector space to be a variety is that for all pairs  $x, y$  of distinct points of  $V$ , the line  $D_{xy}$  is contained in  $V$ .

*Proof.* The condition is necessary by (3.3.2).

If the condition holds, choose  $v \in V$  and let  $V_0 = -v + V$ . We claim  $V_0$  is a subspace, from which it follows that  $V = v + V_0$  is a variety. First,  $0 = -v + v \in V_0$ . If  $x \in V_0$  and  $x \neq 0$ , then  $v + x \in V$  and  $v + x \neq v$ , so  $D_{v, v+x} = \{v + \xi x \mid \xi \in \mathbf{R}\} \subseteq V$ . It follows that  $\xi x \in V_0$  for all  $\xi \in \mathbf{R}$ . If also  $y \in V_0$  and  $y \neq x$ , then  $D_{v+x, v+y} \subseteq V$ , so in particular  $v + 2^{-1}(x + y) \in V$  and  $2^{-1}(x + y) \in V_0$ . By the previous result, it then follows that  $x + y \in V_0$ . Therefore  $V_0$  is a subspace as claimed.  $\square$

**Exercise (4).**

- A necessary and sufficient condition for an affine map to be a translation or a homothetic map is that its associated linear map be homothetic.
- A necessary and sufficient condition for an affine map to preserve the direction of lines is that it be a translation or a bijective homothetic map.
- If  $u_1, u_2$  are translations or homothetic maps, then so is  $u_1 \circ u_2$ .
- If  $u_1, u_2$  and  $u_1 \circ u_2$  are homothetic maps with ratios not equal to 1, their centers are collinear. If instead  $u_1 \circ u_2$  is a translation, then it is either the identity or a translation in the direction of the line through the centers of  $u_1$  and  $u_2$ .
- If  $v \in \mathbf{GA}(E)$ , then  $v \circ h_{a,\lambda} \circ v^{-1} = h_{v(a),\lambda}$ .
- The subset  $H(E)$  of translations and homothetic maps in  $\mathbf{GA}(E)$  forms a subgroup, and  $H(E)/E \cong \mathbf{R}^*$ .

*Proof.*

- This follows from the equations  $t_a = t_a \circ h_1$  and  $h_{a,\lambda} = t_{(1-\lambda)a} \circ h_\lambda$  and  $t_a \circ h_\lambda = h_{(1-\lambda)^{-1}a,\lambda}$  ( $\lambda \neq 1$ ).
- The condition is sufficient because such a map has the form  $t_a \circ h_\lambda$  with  $\lambda \neq 0$ , which clearly preserves the direction of lines. Conversely, suppose  $u = t_a \circ v$  preserves the direction of lines. If  $x \neq 0$ , let  $D$  be the vector line through  $x$ . Then  $v(D) = D$ , so  $v(x) = \lambda x$  for some  $\lambda \in \mathbf{R}$  with  $\lambda \neq 0$ , and in fact  $v(y) = \lambda y$  for all  $y \in D$ . We claim  $v = h_\lambda$ , from which the result follows. If  $y \notin D$ , then by considering the vector line  $D'$  through  $y$  we have  $v(y) = \mu y$  for some  $\mu \in \mathbf{R}$ . Now  $v(D_{xy}) = D_{v(x)v(y)} = D_{\lambda x, \mu y}$ , and since  $v$  preserves direction there is  $\xi \in \mathbf{R}$  with  $\mu y - \lambda x = \xi(y - x)$ , or  $(\mu - \xi)y = (\lambda - \xi)x$ . Since  $y \notin D$ , this implies  $\mu = \xi = \lambda$ . Therefore  $v = h_\lambda$  as claimed.
- If  $u_1 = t_a \circ h_\lambda$  and  $u_2 = t_b \circ h_\mu$ , then by (3.2.19.1),  $u_1 \circ u_2 = t_{a+\lambda b} \circ h_{\lambda\mu}$ .
- Write  $u_1 = h_{a,\lambda}$ ,  $u_2 = h_{b,\mu}$ , and  $u_1 \circ u_2 = h_{c,v}$  with  $\lambda, \mu, v \neq 1$ . Then for all  $x$ ,

$$(1 - v)c + vx = (1 - \lambda)a + \lambda(1 - \mu)b + \lambda\mu x$$

Taking  $x = 0$  and  $x \neq 0$  (we assume such exist!) yields  $v = \lambda\mu$  and

$$c = (1 - \lambda\mu)^{-1}[(1 - \lambda)a + \lambda(1 - \mu)b] = (1 - \lambda\mu)^{-1}(\lambda - 1)(b - a) + b$$

so that  $a, b, c$  are collinear. If instead  $u_1 \circ u_2 = t_c$ , then  $\lambda\mu = 1$  and  $c = (\lambda - 1)(b - a)$ , from which the second result follows.

- Write  $v = t_b \circ w$ , where  $w \in \mathbf{GL}(E)$ , so that  $v^{-1} = w^{-1} \circ t_{-b}$ . By repeated application of (3.2.19.1),

$$\begin{aligned}
v \circ h_{a,\lambda} \circ v^{-1} &= t_b \circ w \circ t_{(1-\lambda)a} \circ h_\lambda \circ w^{-1} \circ t_{-b} \\
&= t_{v((1-\lambda)a)} \circ w \circ h_\lambda \circ w^{-1} \circ t_{-b} \\
&= t_{b+(1-\lambda)w(a)} \circ h_\lambda \circ t_{-b} \\
&= t_{b+(1-\lambda)w(a)-\lambda b} \circ h_\lambda \\
&= t_{(1-\lambda)v(a)} \circ h_\lambda \\
&= h_{v(a),\lambda}
\end{aligned}$$

- First,  $1 \in H(E)$ . If  $u_1, u_2 \in H(E)$ , then  $u_1 \circ u_2 \in H(E)$  by a previous item. If  $u_1 = t_a \circ h_\lambda$  with  $\lambda \neq 0$ , then  $u_1^{-1} = t_{-\lambda^{-1}a} \circ h_{\lambda^{-1}} \in H(E)$ . Therefore  $H(E)$  is a subgroup of  $\mathbf{GA}(E)$ . Define  $\varphi : H(E) \rightarrow \mathbf{R}^*$  by  $\varphi(t_a \circ h_\lambda) = \lambda$ . Note that  $\varphi$  is well-defined (since  $E \neq \{0\}$ ),  $\varphi$  is a homomorphism by a previous item,  $\varphi$  is surjective, and  $\ker \varphi = T(E) \cong E$ . It follows that  $H(E)/E \cong \mathbf{R}^*$ .  $\square$

**Exercise (5).** A variety not parallel to a hyperplane meets the hyperplane. If  $H$  is a vector hyperplane in  $E$  and  $V$  is a subspace of  $E$  not contained in  $H$ , then  $V \cap H$  is a vector hyperplane in  $V$ .

*Proof.* Any vector in the direction of the variety which is not in the direction of the hyperplane determines a line in the variety which meets the hyperplane (3.3.8).

If  $D$  is the vector line through a vector in  $V - H$ , then  $D$  is supplementary to  $H$  in  $E$ , so  $D$  is supplementary to  $V \cap H$  in  $V$  (Exercise 3.1.4).  $\square$

## References

- [1] Dieudonné, J. *Linear Algebra and Geometry*. Hermann, 1969.