Notes and exercises from
Linear Algebra and Geometry

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Introduction

This document contains notes and exercises from [1].

Chapter III

Section 1

Remark. If V, W are supplementary subspaces of E and c is the unique point in common to the varieties $a+V$ and $b+W$ (3.1.15), then $a+V=c+V$ and $b+W=c+W$ (3.1.12), so

$$-c+(a+V)=V \qquad \text{and} \qquad -c+(b+W)=W$$

In other words, taking the origin c (3.2.21), the varieties become their direction subspaces.

Exercise (4)**.** Let V, W be a pair of supplementary subspaces of E. Every subspace U containing V is the direct sum of V with $U \cap W$.

Proof. If $u \in U$, then $u = v + w$ for some $v \in V$ and $w \in W$, and $w = u - v \in U$. So $U = V + (U \cap W)$, and $V \cap U \cap W = \{0\}$. □

Section 2

Exercise (1)**.** *Projections and idempotents:* If p, q are the projections for a direct sum $E = V + W$, then $p, q \in \mathrm{End}(E)$ are such that $p^2 = p$, $q^2 = q$, and $p + q = 1$. Conversely, if $p \in \mathrm{End}(E)$ is such that $p^2 = p$, then $E = p(E) + p^{-1}(0)$ is a direct sum. Moreover, if $q = 1 - p$, then $q^2 = q$, $q(E) = p^{-1}(0)$, and $q^{-1}(0) = p(E)$.

Proof. For the forward direction, we know $p, q \in \operatorname{End}(E)$ and $p + q = 1$ (3.2.2). It follows that $p^2 = p \circ (1 - q) = p - pq = p$ and $q^2 = (1-p)^2 = 1 - p = q$.

For the converse, $p + q = 1$, so $E = p(E) + q(E)$. Also $pq = p - p^2 = 0$, so $p(E) \cap q(E) = \{0\}$ and $q(E) \subseteq p^{-1}(0)$. If $x \in p^{-1}(0)$, then $q(x) = x$, so $x \in q(E)$. Hence $q(E) = p^{-1}(0)$ and similarly $q^{-1}(0) = p(E)$. Finally $q^2 = q$ as above. □

Exercise (2)**.** If W and W' are both supplementary to V in E, then W and W' are isomorphic.

Proof. If p is the projection of E onto W', then the restriction of p to W is an isomorphism from W to W'. □

Exercise (3)**.** If $E = V + W$ is a direct sum with inclusions $i : V \to E$ and $j : W \to E$, and $v : V \to F$ and $w : W \to F$ are linear maps, then there is a unique linear map $u : E \to F$ with $u \circ i = v$ and $u \circ j = w$.

Proof. If p, q are the projections on V, W respectively, then $u = v \circ p + w \circ q$. □

Exercise (11)**.** $\mathbf{GA}(E)/E \cong \mathbf{GL}(E)$.

Proof. Define $\varphi : \mathbf{GA}(E) \to \mathbf{GL}(E)$ by $\varphi(t_a \circ v) = v$. Note that φ is well-defined by (3.2.17), φ is a homomorphism by (3.2.19), and φ is clearly surjective. Also $\varphi(u) = 1$ if and only if u is a translation, so $\ker\varphi = T(E)$, the normal subgroup of translations. It follows that $\mathbf{GA}(E)/T(E) \cong \mathbf{GL}(E)$. Finally, the mapping $a \mapsto t_a$ is an isomorphism $E \cong T(E)$ from the additive group E. □

Exercise (13)**.** If $u : E \to F$ is affine and L is a variety in F, then $u^{-1}(L)$ is empty or a variety in E.

Proof. If $a \in u^{-1}(L)$ and L_0 is the direction of L, then $L = u(a) + L_0$ and hence $u^{-1}(L) = a + u^{-1}(L_0)$. □

Section 3

Remark. If $a, b \in E$, then the segment ab (3.3.4) consists of all points $x = \alpha a + \beta b$ where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$—the *convex combinations* of a and b.

Exercise (3)**.** A necessary and sufficient condition for a nonempty subset V of a vector space to be a variety is that for all pairs x, y of distinct points of V, the line D_{xy} is contained in V.

Proof. The condition is necessary by (3.3.2).

If the condition holds, choose $v \in V$ and let $V_0 = -v + V$. We claim V_0 is a subspace, from which it follows that $V = v + V_0$ is a variety. First, $0 = -v + v \in V_0$. If $x \in V_0$ and $x \neq 0$, then $v + x \in V$ and $v + x \neq v$, so $D_{v,v+x} = \{v + \xi x \mid \xi \in \mathbf{R}\} \subseteq V$. It follows that $\xi x \in V_0$ for all $\xi \in \mathbf{R}$. If also $y \in V_0$ and $y \neq x$, then $D_{v+x,v+y} \subseteq V$, so in particular $v + 2^{-1}(x + y) \in V$ and $2^{-1}(x + y) \in V_0$. By the previous result, it then follows that $x + y \in V_0$. Therefore V_0 is a subspace as claimed. □

Exercise (4)**.** *Translations and homothetic maps:*

- A necessary and sufficient condition for an affine map to be a translation or a homothetic map is that its associated linear map be homothetic.
- A necessary and sufficient condition for an affine map to preserve the direction of lines is that it be a translation or a bijective homothetic map.
- If u_1, u_2 are translations or homothetic maps, then so is $u_1 \circ u_2$.
- If u_1, u_2 and $u_1 \circ u_2$ are homothetic maps with ratios not equal to 1, their centers are collinear. If instead $u_1 \circ u_2$ is a translation, then it is either the identity or a translation in the direction of the line through the centers of u_1 and u_2.
- If $v \in \mathbf{GA}(E)$, then $v \circ h_{a,\lambda} \circ v^{-1} = h_{v(a),\lambda}$.
- The subset $H(E)$ of translations and homothetic maps in $\mathbf{GA}(E)$ forms a subgroup, and $H(E)/E \cong \mathbf{R}^*$.

Proof.

- This follows from the equations $t_a = t_a \circ h_1$ and $h_{a,\lambda} = t_{(1-\lambda)a} \circ h_\lambda$ and $t_a \circ h_\lambda = h_{(1-\lambda)^{-1}a,\lambda}$ $(\lambda \neq 1)$.
- The condition is sufficient because such a map has the form $t_a \circ h_\lambda$ with $\lambda \neq 0$, which clearly preserves the direction of lines. Conversely, suppose $u = t_a \circ v$ preserves the direction of lines. If $x \neq 0$, let D be the vector line through x. Then $v(D) = D$, so $v(x) = \lambda x$ for some $\lambda \in \mathbf{R}$ with $\lambda \neq 0$, and in fact $v(y) = \lambda y$ for all $y \in D$. We claim $v = h_\lambda$, from which the result follows. If $y \notin D$, then by considering the vector line D' through y we have $v(y) = \mu y$ for some $\mu \in \mathbf{R}$. Now $v(D_{xy}) = D_{v(x)v(y)} = D_{\lambda x,\mu y}$, and since v preserves direction there is $\xi \in \mathbf{R}$ with $\mu y - \lambda x = \xi(y - x)$, or $(\mu - \xi)y = (\lambda - \xi)x$. Since $y \notin D$, this implies $\mu = \xi = \lambda$. Therefore $v = h_\lambda$ as claimed.

- If $u_1 = t_a \circ h_\lambda$ and $u_2 = t_b \circ h_\mu$, then by (3.2.19.1), $u_1 \circ u_2 = t_{a+\lambda b} \circ h_{\lambda\mu}$.
- Write $u_1 = h_{a,\lambda}$, $u_2 = h_{b,\mu}$, and $u_1 \circ u_2 = h_{c,\nu}$ with $\lambda, \mu, \nu \neq 1$. Then for all x,

  $$(1-\nu)c + \nu x = (1-\lambda)a + \lambda(1-\mu)b + \lambda\mu x$$

  Taking $x = 0$ and $x \neq 0$ (we assume such exist!) yields $\nu = \lambda\mu$ and

  $$c = (1-\lambda\mu)^{-1}[(1-\lambda)a + \lambda(1-\mu)b] = (1-\lambda\mu)^{-1}(\lambda-1)(b-a) + b$$

  so that a, b, c are collinear. If instead $u_1 \circ u_2 = t_c$, then $\lambda\mu = 1$ and $c = (\lambda-1)(b-a)$, from which the second result follows.
- Write $v = t_b \circ w$, where $w \in \mathbf{GL}(E)$, so that $v^{-1} = w^{-1} \circ t_{-b}$. By repeated application of (3.2.19.1),

  $$\begin{aligned} v \circ h_{a,\lambda} \circ v^{-1} &= t_b \circ w \circ t_{(1-\lambda)a} \circ h_\lambda \circ w^{-1} \circ t_{-b} \\ &= t_{v((1-\lambda)a)} \circ w \circ h_\lambda \circ w^{-1} \circ t_{-b} \\ &= t_{b+(1-\lambda)w(a)} \circ h_\lambda \circ t_{-b} \\ &= t_{b+(1-\lambda)w(a)-\lambda b} \circ h_\lambda \\ &= t_{(1-\lambda)v(a)} \circ h_\lambda \\ &= h_{v(a),\lambda} \end{aligned}$$

- First, $1 \in H(E)$. If $u_1, u_2 \in H(E)$, then $u_1 \circ u_2 \in H(E)$ by a previous item. If $u_1 = t_a \circ h_\lambda$ with $\lambda \neq 0$, then $u_1^{-1} = t_{-\lambda^{-1}a} \circ h_{\lambda^{-1}} \in H(E)$. Therefore $H(E)$ is a subgroup of $\mathbf{GA}(E)$. Define $\varphi : H(E) \to \mathbf{R}^*$ by $\varphi(t_a \circ h_\lambda) = \lambda$. Note that φ is well-defined (since $E \neq \{0\}$!), φ is a homomorphism by a previous item, φ is surjective, and $\ker\varphi = T(E) \cong E$. It follows that $H(E)/E \cong \mathbf{R}^*$. □

Exercise (5)**.** A variety not parallel to a hyperplane meets the hyperplane. If H is a vector hyperplane in E and V is a subspace of E not contained in H, then $V \cap H$ is a vector hyperplane in V.

Proof. Any vector in the direction of the variety which is not in the direction of the hyperplane determines a line in the variety which meets the hyperplane (3.3.8).

If D is the vector line through a vector in $V - H$, then D is supplementary to H in E, so D is supplementary to $V \cap H$ in V (Exercise 3.1.4). □

Exercise (6)**.** *Dilations and transvections:* Let H be a vector hyperplane in E and suppose $u \in \mathrm{End}(E)$ fixes every element of H.

- There is $\gamma \in \mathbf{R}$ unique such that $u(a) \in \gamma a + H$ for all $a \in E - H$.
- If $\gamma \neq 1$, then γ is an eigenvalue of u and $E(\gamma; u)$ is a line S supplementary to $E(1; u) = H$. A subspace V satisfies $u(V) \subseteq V$ if and only if $S \subseteq V$ or $V \subseteq H$. In particular, a vector line D satisfies $u(D) \subseteq D$ if and only if $D = S$ or $D \subseteq H$.
- If $\gamma = 1$, and $g(x) = 0$ is an equation of H, there is $c \in H$ unique such that $u(x) = x + g(x)c$ for all $x \in E$. u is bijective. If $u \neq 1$ (so $c \neq 0$), then the line $T = D_{0c}$ is independent of g. The scalar 1 is the only eigenvalue of u if $H \neq \{0\}$, and $E(1; u) = H$ if $u \neq 1$. If $u \neq 1$, a subspace V satisfies $u(V) \subseteq V$ if and only if $T \subseteq V$ or $V \subseteq H$; in particular, a vector line D satisfies $u(D) = D$ if and only if $D \subseteq H$.
- The set $\Gamma(E, H)$ of automorphisms of E leaving the hyperplane H fixed pointwise is a subgroup of $\mathbf{GL}(E)$. The subset $\Theta(E, H)$ of transvections is a normal abelian subgroup of $\Gamma(E, H)$ isomorphic to H. $\Gamma(E, H)/H \cong \mathbf{R}^*$.

Proof.

- If $a \in E - H$, then $E = \mathbf{R}a + H$, so $u(a) = \gamma a + t$ for some $\gamma \in \mathbf{R}$ and $t \in H$. If $b \in E$, then $b = \beta a + h$ for some $\beta \in \mathbf{R}$ and $h \in H$, so

$$\begin{aligned} u(b) &= u(\beta a + h) \\ &= \beta u(a) + u(h) \\ &= \beta(\gamma a + t) + h \\ &= \gamma(\beta a + h) + (1 - \gamma)h + \beta t \\ &= \gamma b + (1 - \gamma)h + \beta t \end{aligned} \tag{1}$$

  Therefore $u(b) \in \gamma b + H$. If also $u(b) \in \gamma' b + H$, then $(\gamma' - \gamma)b \in H$, which implies $\gamma' = \gamma$ if $b \notin H$. Therefore γ is unique for $b \notin H$.
- Let $x = a - (1 - \gamma)^{-1} t$. Then $x \neq 0$ since $a \notin H$ and

$$\begin{aligned} u(x) &= u(a - (1 - \gamma)^{-1} t) \\ &= u(a) - (1 - \gamma)^{-1} u(t) \\ &= \gamma a + t - (1 - \gamma)^{-1} t \\ &= \gamma[a - (1 - \gamma)^{-1} t] \\ &= \gamma x \end{aligned}$$

Therefore x is an eigenvector of u with eigenvalue γ. Let S be the vector line through x. Then $S \subseteq E(\gamma; u)$. Conversely if $b \in E(\gamma; u)$, then $u(b) = \gamma b$, which implies $(1-\gamma)h + \beta t = 0$ in (1), so $h = -\beta(1-\gamma)^{-1}t$ and

$$b = \beta a + h = \beta[a - (1-\gamma)^{-1}t] = \beta x \in S$$

Therefore $S = E(\gamma; u)$. By hypothesis $H \subseteq E(1; u)$. Conversely if $b \in E(1; u)$, then $b = u(b) \in \gamma b + H$, so $(1-\gamma)b \in H$, so $b \in H$. Therefore $H = E(1; u)$. S is supplementary to H since $x \notin H$.

By hypothesis $u(V) \subseteq V$ for any subspace $V \subseteq H$. If $S \subseteq V$, then $V = S + V \cap H$ (Exercise 3.1.4), so clearly $u(V) \subseteq V$. Conversely if $u(V) \subseteq V$ and $v \in V - H$, then $v = s + h$ for some $s \in S$ with $s \neq 0$ and $h \in H$, so $u(v) = \gamma s + h$ and $v - u(v) = (1-\gamma)s \in V$, which implies $s \in V$ and $S = \mathbf{R}s \subseteq V$.

- Fix $e \in E$ with $g(e) = 1$ and let $c = u(e) - e$. Since $u(e) \in e + H$, $g(c) = 0$ and $c \in H$. Now $u(x) = x + g(x)c$ holds for $x = e$, and for $x \in H$, so by linearity it holds for all $x \in \mathbf{R}e + H = E$. Note c is unique since if $u(e) = e + g(e)c'$, then $c' = u(e) - e = c$.

  The map $x \mapsto x - g(x)c$ is clearly the inverse of u, so u is bijective.

  If $h(x) = 0$ is another equation of H, then by the above there exists $c' \in H$ such that $u(x) = x + h(x)c'$ for all $x \in E$. But $h = \lambda g$ for some $\lambda \neq 0$ (3.3.6), so $u(x) = x + g(x)(\lambda c')$ for all $x \in E$ and $c = \lambda c'$ by uniqueness of c. If $u \neq 1$, then $T = D_{0c} = D_{0c'}$ is independent of g.

  If $u(x) = x + g(x)c = \lambda x$, then $(\lambda - 1)x \in H$. If $\lambda \neq 1$, then $x \in H$, so actually $(\lambda - 1)x = 0$ and $x = 0$. If $\lambda = 1$, then $g(x)c = 0$, so either $g(x) = 0$ and $x \in H$, or $c = 0$ and $u = 1$.

  As above, $u(V) \subseteq V$ if $V \subseteq H$ or $T \subseteq V$. Conversely if $u(V) \subseteq V$ and $x \in V - H$, then $g(x) \neq 0$ and $g(x)c = u(x) - x \in V$, so $c \in V$ and $T = \mathbf{R}c \subseteq V$. $\square$

- $\Gamma(E, H)$ is obviously a subgroup of $\mathbf{GL}(E)$. Define $\varphi : \Gamma(E, H) \to \mathbf{R}^*$ by $\varphi(u) = \gamma$. Then φ is a well-defined, surjective homomorphism and $\ker\varphi = \Theta(E, H)$. It follows that $\Theta(E, H)$ is a normal subgroup of $\Gamma(E, H)$ and that $\Gamma(E, H)/\Theta(E, H) \cong \mathbf{R}^*$. Finally, the mapping $c \mapsto (x \mapsto x + g(x)c)$ is an isomorphism $H \cong \Theta(E, H)$, so in particular $\Theta(E, H)$ is abelian.

Chapter IV

Section 1

Exercise (1)**.** If $t \neq 1$ is a transvection in E, a basis for E may be chosen relative to which

$$M(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Conversely, matrices of the form

$$B_{12}(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad B_{21}(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

are matrices of transvactions relative to some basis $\{a_1, a_2\}$. These transvections have the lines D_{0a_1} and D_{0a_2} respectively.

Proof. Write $t(x) = x + g(x)c$, where $g \in E^*$ with $g \neq 0$, and $c \neq 0$ with $g(c) = 0$ (Exercise 3.3.6). Let $a_1 = c$ and choose a_2 such that $g(a_2) = 1$. Then $\{a_1, a_2\}$ is the desired basis for E.

Conversely, if t is the transformation of $B_{12}(\lambda)$ relative to $\{a_1, a_2\}$, then

$$\begin{aligned} t(x) &= t(\xi_1 a_1 + \xi_2 a_2) \\ &= \xi_1 t(a_1) + \xi_2 t(a_2) \\ &= \xi_1 a_1 + \xi_2(\lambda a_1 + a_2) \\ &= \xi_1 a_1 + \xi_2 a_2 + \lambda \xi_2 a_1 \\ &= x + g(x) a_1 \end{aligned}$$

where $g = \lambda a_2^* \in E^*$. Therefore t is a transvection in the line D_{0a_1}. A similar argument applies to $B_{21}(\lambda)$. □

Exercise (3)**.** If $u \in \mathrm{End}(E)$ and $\mathrm{rank}(u) = 1$, there is a basis of E relative to which

$$M(u) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} \quad (\delta \neq 0) \qquad \text{or} \qquad M(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The second case occurs if and only if u is nilpotent, in which case $u^2 = 0$.

Proof. Let $N = u^{-1}(0)$ and $R = u(E)$. Then N and R are vector lines in E (4.1.7). If $N \cap R = \{0\}$, then $E = N + R$ is a direct sum. Choose $a_1 \in R$ and $a_2 \in N$ with

$a_1, a_2 \neq 0$. Then $u(a_1) = \delta a_1$ with $\delta \neq 0$ since $u(a_1) \in R$ and $a_1 \notin N$, and also $u(a_2) = 0$. It follows that $\{a_1, a_2\}$ is a basis of E relative to which

$$M(u) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}$$

If $N \cap R \neq \{0\}$, then $N = R$ (3.3.1). Choose $a_1 \in N$, $a_1 \neq 0$ and a_2 with $u(a_2) = a_1$. Then $\{a_1, a_2\}$ is a basis of E (since $a_2 \notin N$) relative to which

$$M(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If $M(u)$ has this form, then $M(u^2) = M(u)^2 = 0$, so $u^2 = 0$ and u is nilpotent. Conversely if u is nilpotent, there is $k > 1$ least such that $u^k = 0$. Then $u^{k-1}(E) \neq 0$ but $u^{k-1}(E) \subseteq N$, so $u^{k-1}(E) = N$. If $k > 2$, then $u^{k-1}(E)$ is a proper subspace of $u^{k-2}(E)$, lest $u^{k-2}(E) = N$ and $u^{k-1} = 0$. But then $u^{k-2}(E) = E$, impossible since $\operatorname{rank}(u) = 1$. It follows that $u^2 = 0$, so $N = R$ and $M(u)$ has this form. □

Exercise (7)**.** Let $u : E \to E$ be an injective function such that E is the smallest variety containing $u(E)$, and $u(a), u(b), u(c)$ are collinear whenever $a, b, c \in E$ are collinear.

- If $a, b, c \in E$ are not collinear, then $u(a), u(b), u(c)$ are not collinear.
- For every line D, there is a unique line D' such that $u(D) \subseteq D'$. If u is bijective, then $u(D) = D'$; moreover, if D_1, D_2 are parallel (resp. distinct, not parallel), then so are $u(D_1), u(D_2)$.
- If u is bijective, there is $v \in \mathbf{GA}(E)$ such that $u_1 = v \circ u$ fixes the origin and basis vectors $a_1, a_2 \in E$. u_1 maps lines to lines and preserves parallelism (resp. distinctness, non-parallelism); in particular, u_1 preserves the lines $D_{0a_1}, D_{0a_2}, D_{a_1 a_2}$ and hence the directions of any lines parallel to these.
- Given the points $\xi a_1, \eta a_1$, it is possible to construct $(\xi + \eta) a_1$ and $\xi \eta a_1$ by intersecting lines with direction vectors derived from a_1, a_2.
- If φ is defined by $u_1(\xi a_1) = \varphi(\xi) a_1$, then φ is a field automorphism of $\mathbf{R}$, so $\varphi = 1_{\mathbf{R}}$.[1] It follows that $u_1 = 1_E$, so $u = v^{-1}$ is an affine map.

Proof.

[1] For this problem, we assume there are no nontrivial field automorphisms of $\mathbf{R}$.

- Suppose towards a contradiction that $u(a), u(b), u(c)$ are on the line Δ. Let $x \in E$. If $x \in D_{ab} \cup D_{ac} \cup D_{bc}$, then $u(x) \in \Delta$. Otherwise if, say, x and c are on opposite sides of D_{ab}, then D_{xc} and D_{ab} intersect at a unique point y by (3.3.9) and (4.1.6), $u(y) \in \Delta$ since $y \in D_{ab}$, and $u(x) \in \Delta$ since $x \in D_{yc}$. Similarly $u(x) \in \Delta$ if x and b are on opposite sides of D_{ac}, or x and a are on opposite sides of D_{bc}. Finally, if none of these cases hold, then D_{xc} cannot be parallel to D_{ab}, because if x is in the direction of $a-b$ from c (that is, if $x = c + \xi(a-b)$ for $\xi > 0$) then x and b are on opposite sides of D_{ac}, and if x is in the direction of $b-a$ from c then x and a are on opposite sides of D_{bc}. Therefore D_{xc} and D_{ab} intersect at a unique point y by (4.1.6) and $u(x) \in \Delta$ as above. Since x was arbitrary, this means $u(E) \subseteq \Delta$, contradicting the hypothesis about $u(E)$.
- By hypothesis, $u(D_{ab}) \subseteq D_{u(a)u(b)}$. If u is bijective and $x \in D_{u(a)u(b)}$, then $x = u(c)$ for some $c \in D_{ab}$ by the previous item, so $D_{u(a)u(b)} \subseteq u(D_{ab})$. If D_1, D_2 are distinct and parallel, then $D_1 \cap D_2 = \emptyset$, so $u(D_1) \cap u(D_2) = \emptyset$ by injectivity of u, so $u(D_1), u(D_2)$ are distinct and parallel. If D_1, D_2 are not parallel, they intersect at a unique point a. If $b \in D_1$ and $c \in D_2$ with $b, c \neq a$, then a, b, c are not collinear, so $u(a), u(b), u(c)$ are not collinear by the previous item and $u(D_1), u(D_2)$ are not parallel.
- Let a_1, a_2 be basis vectors of E. Then $0, a_1, a_2$ are not collinear (4.1.1), so $u(0), u(a_1), u(a_2)$ are not collinear by a previous item, so $a_1' = u(a_1) - u(0)$ and $a_2' = u(a_2) - u(0)$ are basis vectors of E. Let $w \in \mathbf{GL}(E)$ map $a_1' \mapsto a_1$ and $a_2' \mapsto a_2$ (4.1.10) and let $v = w \circ t_{-u(0)} \in \mathbf{GA}(E)$ (3.2.19). Then $u_1 = v \circ u$ fixes $0, a_1, a_2$. u_1 operates as claimed on lines by (3.2.17) and the previous item, and the observation that, for example,

$$u_1(D_{a_1 a_2}) = D_{u_1(a_1) u_1(a_2)} = D_{a_1 a_2}$$

- If L_1 is the line through ηa_1 with direction vector $a_2 - a_1$, then $L_1 \cap D_{0a_2} = \{\eta a_2\}$, so ηa_2 is constructible. If L_2 is the line through ξa_1 with direction vector a_2, and L_3 is the line through ηa_2 with direction vector a_1, then $L_2 \cap L_3 = \{\xi a_1 + \eta a_2\}$, so $\xi a_1 + \eta a_2$ is constructible. If L_4 is the line through $\xi a_1 + \eta a_2$ with direction vector $a_2 - a_1$, then $L_4 \cap D_{0a_1} = \{(\xi + \eta) a_1\}$, so $(\xi+\eta) a_1$ is constructible. If L_5 is the line through ηa_2 with direction vector $a_2 - \xi a_1$, then $L_5 \cap D_{0a_1} = \{\xi\eta a_1\}$, so $\xi\eta a_1$ is constructible.
- By previous items, $u_1(L_2)$ is the line passing through $u_1(\xi a_1)$ parallel to L_2 (and to D_{0a_2}), and $u_1(L_3)$ is the line passing through $u_1(\eta a_2)$ parallel to L_3

(and to D_{0a_1}). Since $u_1(\eta a_2) \in D_{0a_2}$ and $u_1(\xi a_1) \in D_{0a_1}$, it follows that $u_1(\xi a_1) + u_1(\eta a_2)$ is the intersection of $u_1(L_2)$ and $u_1(L_3)$. But we also have $u_1(\xi a_1 + \eta a_2) \in u_1(L_2 \cap L_3) = u_1(L_2) \cap u_1(L_3)$, so

$$u_1(\xi a_1 + \eta a_2) = u_1(\xi a_1) + u_1(\eta a_2)$$

Now $u_1(L_4)$ is the line passing through this point and $u_1((\xi+\eta)a_1)$ parallel to the line $u_1(L_1)$, which in turn passes through $u_1(\eta a_1)$ and $u_1(\eta a_2)$ parallel to L_1 (and to $D_{a_1 a_2}$). Therefore

$$u_1((\xi+\eta)a_1) = u_1(\xi a_1) + u_1(\eta a_1)$$

is the intersection of $u_1(L_4)$ and D_{0a_1}. It follows that $\varphi(\xi+\eta) = \varphi(\xi)+\varphi(\eta)$.

We claim $u_1(\eta a_2) = \varphi(\eta)a_2$. Indeed, $u_1(\eta a_2) = \lambda a_2$ for some λ, and also $u_1(\eta a_2) = u_1(\eta a_1) + \mu(a_2 - a_1) = \varphi(\eta)a_1 + \mu(a_2 - a_1)$ for some μ by facts about $u_1(L_1)$ above. It follows that $\lambda = \mu = \varphi(\eta)$ by linear independence of a_1, a_2, establishing the claim.

Now $u_1(L_5)$ is the line passing through $u_1(\xi\eta a_1)$ and $u_1(\eta a_2)$, and with the direction vector $u_1(\xi a_1) - a_2$. Therefore

$$u_1(\xi\eta a_1) = u_1(\eta a_2) + \varphi(\eta)[u_1(\xi a_1) - a_2] = \varphi(\eta)u_1(\xi a_1)$$

is the intersection of $u_1(L_5)$ and D_{0a_1}. It follows that $\varphi(\xi\eta) = \varphi(\xi)\varphi(\eta)$. Since $\varphi(1) = 1$, φ is a field automorphism of $\mathbf{R}$ and hence $\varphi = 1_{\mathbf{R}}$.

By the above, u_1 fixes D_{0a_1} and D_{0a_2} pointwise. If $x = \xi_1 a_1 + \xi_2 a_2$, then x is the intersection of the line through $\xi_1 a_1$ parallel to D_{0a_2} and the line through $\xi_2 a_2$ parallel to D_{0a_1}. It follows that $u_1(x)$ is the intersection of the same lines, so $u_1(x) = x$. Therefore $u_1 = 1_E$, and $u = v^{-1}$ is affine. □

Exercise (8)**.** We have

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix} = \begin{pmatrix}\alpha_1\beta_1 & \alpha_1\beta_2\\ \alpha_2\beta_1 & \alpha_2\beta_2\end{pmatrix}$$

This can be interpreted as the composite of a linear form $E \to \mathbf{R}$ and a linear map $\mathbf{R} \to E$; it maps the plane onto the origin or onto a vector line and hence is of rank 0 or 1.

Exercise (9)**.** If $f : E \to E$ is a function which commutes with all automorphisms in $\mathbf{GL}(E)$, then $f = h_\lambda$ for some $\lambda \in \mathbf{R}$.

Proof. First, since f commutes with $h_2 \in \mathbf{GL}(E)$,

$$f(0) = f(2 \cdot 0) = 2 \cdot f(0) = f(0) + f(0)$$

and it follows that $f(0) = 0$. Now $f(\alpha x) = \alpha f(x)$ for all $x \in E$ and $\alpha \in \mathbf{R}$, using the previous result for $\alpha = 0$ and commutativity of f with $h_\alpha \in \mathbf{GL}(E)$ for $\alpha \neq 0$.

Fix $x \neq 0$ and let $u \neq 1$ be a transvection in the line $D = D_{0x}$. Then $u(f(x)) = f(u(x)) = f(x)$, so $f(x) \in D$ and hence $f(x) = \lambda x$ for some $\lambda \in \mathbf{R}$. It follows that $f(y) = \lambda y$ for all $y \in D$. Fix $y \notin D$. We may assume $u(y) = x + y$. By reasoning as above, $f(y) = \lambda_y y$ and $f(x+y) = \lambda_{x+y}(x+y)$ for some $\lambda_y, \lambda_{x+y} \in \mathbf{R}$. But also

$$f(x+y) = f(u(y)) = u(f(y)) = u(\lambda_y y) = \lambda_y u(y) = \lambda_y(x+y)$$

so $\lambda_y = \lambda_{x+y}$ (since $x + y \neq 0$). Now considering the transvection u' in the line $D' = D_{0y}$ with $u'(x) = x + y$, it follows that $\lambda = \lambda_{x+y} = \lambda_y$. Therefore $f(y) = \lambda y$ for all $y \in E$, so $f = h_\lambda$. □

Section 2

Remark. In (4.2.5), if $a \in E$, $a \neq 0$ and $f(x) = 0$ is an equation of D_{0a} with $f \in E^*$, $f \neq 0$ (3.3.6), choose $b \in E$ with $f(b) = 1$. Then $\{a, b\}$ is a basis of E. Let Ψ be the alternating bilinear form on E with $\Psi(a, b) = 1$. Then for all $x = \xi a + \eta b$,

$$\Psi(a, x) = \eta = f(x)$$

Since Ψ is essentially the determinant (4.2.9) and the determinant measures (oriented) area, this just says that x is on the vector line determined by a if and only if the parallelogram determined by x and a has zero area.

Remark. An alternating bilinear form $\Psi \neq 0$ on E captures linear independence in that $\Psi(x, y) \neq 0$ if and only if $\{x, y\}$ is independent (4.2.5). It follows that if $u \in \mathrm{End}(E)$, then $\det(u) \neq 0$ if and only if u preserves linear independence (4.2.6.1), which is true if and only if u is bijective (4.1.8). This is just (4.2.8).

Exercise (3)**.** Relative to a fixed basis of E, let u and v be defined by

$$M(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad M(v) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then $uv = v$, so $\mathrm{rank}(uv) = 1$, and $vu = 0$, so $\mathrm{rank}(vu) = 0$, but $\lambda = 0$ is the only eigenvalue of uv and vu, so $\lambda^2 = 0$ is the characteristic equation of uv and vu.

Exercise (5)**.** If $u \in \mathrm{End}(E)$ has eigenvalues, there is a basis of E relative to which

$$M(u) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{or} \qquad M(u) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

If, relative to some basis of E,

$$M(u) = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$$

with $\alpha \neq 0$, then

$$M(u) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

relative to some basis of E, but there is no basis of E relative to which

$$M(u) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Proof. If u has two distinct eigenvalues λ, μ with eigenvectors a, b respectively, then $\{a, b\}$ is a basis of E and

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

Otherwise, u has just one eigenvalue λ. If there are two linearly independent eigenvectors a, b, then

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Note $M(u)$ has this form only if $u = h_\lambda$. If there are not two linearly independent eigenvectors, let a be an eigenvector and $\{a, x\}$ a basis of E. Then $u(x) = \alpha a + \lambda x$ with $\alpha \neq 0$ by assumption and (4.2.14), so

$$M(u, \{a, x\}) = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$$

If $b = \alpha^{-1} x$, then $\{a, b\}$ is a basis of E and

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Note $M(u)$ has this form only if $u \neq h_\lambda$. □

Chapter V

Section 1

Remark. For all $x, y \in E$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof. Bilinearity of the inner product. □

This is called the *parallelogram law* because it shows that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

Remark. If $a, b \in E$ with $a \neq b$ and $x = \alpha a + \beta b$ with $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ is a point on the segment ab (see remark from (3.3) above), then

$$\alpha = \frac{\|x-b\|}{\|a-b\|} \qquad \beta = \frac{\|x-a\|}{\|a-b\|}$$

In other words, α is the ratio of the lengths of the segments xb and ab, and β that of xa and ab.

Proof. For example, $x - b = \alpha a + (\beta - 1)b = \alpha(a-b)$, so $\|x-b\| = \alpha\|a-b\|$. □

Remark. In (5.1.8), $H(x) = x + H_0$ and $D(x) = x + D_0$. To see that the unique point $y \in D \cap H(x)$ (3.3.8) satisfies the properties for x, let $c \in D \cap H$ and observe that $y - c$ is the orthogonal projection of $x - c$ on D_0. Indeed, $y - c \in D_0$ since $y, c \in D$ and $(x-c) - (y-c) = x - y \in H_0$ since $y \in H(x)$. It follows (5.1.7) that $x - y$ is orthogonal to $y' - y \in D_0$ for all $y' \in D$ and

$$d(x,y) = d(x-c, y-c) \leq d(x-c, d) = d(x, d+c)$$

for all $d \in D_0$, so $d(x,y) \leq d(x,y')$ for all $y' \in D$ (since $c + D_0 = D$). Moreover, y is unique in satisfying these properties for x since $y - c$ is unique in satisfying them for $x - c$ (5.1.7). A similar argument applies for the point $z \in H \cap D(x)$.

Remark. In (5.1.15), we initially suspect that u is a translation after a symmetry (5.1.13), and hence a symmetry itself. Since $u(u(0)) = u^2(0) = 0$, we suspect that u is a symmetry about a variety through $a = \frac{1}{2}u(0)$ perpendicular to a, which turns out to be the case.

By (5.1.13),

$$\tfrac{1}{2}(x + u(x)) = a + \tfrac{1}{2}(x + v(x)) \in a + V$$

and

$$\tfrac{1}{2}(x+u(x))-x=a-\tfrac{1}{2}(x-v(x))\in a+W=W$$

since $-v$ is the symmetry about W and $a\in W$. Therefore $\frac{1}{2}(x+u(x))$ is the unique point in the intersection of $a+V$ and $x+W$ (3.3.8).

Remark. In (5.1.16), $t_{2b}=(t_b u_1)(u_1 t_b)$ is the composite of two symmetries about varieties parallel to V differing by b. Therefore *translations are just composites of symmetries.*

Exercise (1)**.** If D is a vector line in E and H is the orthogonal hyperplane, then D is the intersection of the vector hyperplanes orthogonal to vector lines in H (that is, the vector hyperplanes *perpendicular* to H). Dually, H is the union of the vector lines orthogonal to D.

Proof. If H' is a vector hyperplane orthogonal to a vector line $D'\subseteq H$, then D is orthogonal to D' (since D is orthogonal to H), so $D\subseteq H'$ (since H' is the orthogonal subspace of D'). Conversely, if x is in the intersection and $h\in H$, $h\neq 0$, then x is orthogonal to h since x is in the vector hyperplane orthogonal to the vector line D_{0h}. The dual follows by taking orthogonal subspaces. □

Exercise (2)**.** If a and a' are diametrically opposed points on a sphere S, then a necessary and sufficient condition for a point x to be on S is that $x-a$ and $x-a'$ be orthogonal.

Proof. Without loss of generality, we may assume that S is centered at the origin with radius ρ. Then $a'=-a$, so

$$(x-a\mid x-a')=\|x\|^2-\rho^2=0\iff \|x\|=\rho\iff x\in S$$ □

Exercise (4)**.** *Powers and coorthogonal spheres:*

- Let S be a sphere centered at c with radius ρ. If $a\in E$ with $\delta=d(a,c)$ and D is a line through a meeting S at distinct points x_1,x_2, then

  $$(x_1-a\mid x_2-a)=\delta^2-\rho^2$$

  If instead D meets S at the single point x, then

  $$d(a,x)^2=\delta^2-\rho^2$$

- If S_1,S_2 are non-concentric spheres with respective centers c_1,c_2 and radii ρ_1,ρ_2, then the set H of points whose powers with respect to S_1 and S_2 are equal is a hyperplane perpendicular to $D_{c_1c_2}$ and containing $S_1\cap S_2$.

- Let $\pi_k(x) = d(x, c_k)^2 - \rho_k^2$ denote the power of x with respect to S_k (for $k = 1, 2$). Then the following are equivalent:

  (a) $\pi_2(c_1) = \rho_1^2$
  (b) $\pi_1(c_2) = \rho_2^2$
  (c) $\emptyset \neq S_1 \cap S_2 \subseteq \{x \mid (x - c_1 \mid x - c_2) = 0\}$
  (d) $\pi_1(x) + \pi_2(x) = 2(x - c_1 \mid x - c_2)$

Proof.

- Without loss of generality, we may assume that $c = 0$. In the first case, let $b = \frac{1}{2}(x_1 + x_2)$ and $y = \frac{1}{2}(x_1 - x_2)$. Then

$$(b \mid y) = \tfrac{1}{4}(\|x_1\|^2 - \|x_2\|^2) = \tfrac{1}{4}(\rho^2 - \rho^2) = 0$$

  Since y is a direction vector of D, it follows that b is perpendicular to D. Now $x_1 = b + y$ and $x_2 = b - y$, so $x_1 - a = (b - a) + y$ and $x_2 - a = (b - a) - y$, hence

$$\begin{aligned}
(x_1 - a \mid x_2 - a) &= ((b - a) + y \mid (b - a) - y) \\
&= (b - a \mid b - a) - \|y\|^2 \\
&= \delta^2 - (\|b\|^2 + \|y\|^2) && \text{as } (b \mid b - a) = 0,\ (b \mid a) = \|b\|^2 \\
&= \delta^2 - \rho^2 && \text{by Pythagoras (5.1.5)}
\end{aligned}$$

  In the second case, D must be perpendicular to x. Indeed, if not, let d be a direction vector of D with $(d \mid x) \neq 0$ and consider

$$z = x - \frac{2(d \mid x)}{\|d\|^2} d$$

  Then $z \in D$, $z \neq x$, and expansion of $(z \mid z)$ shows that $\|z\|^2 = \|x\|^2$, so $\|z\| = \rho$ and $z \in S$, contradicting that D is tangential to S at x. It follows that $(x \mid a - x) = 0$, and $d(a, x)^2 = \delta^2 - \rho^2$ by Pythagoras (5.1.5).

- We first find $c \in H \cap D_{c_1 c_2}$. We must have $c = \alpha c_1 + \beta c_2$ with $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. In fact, $\alpha = \|c - c_2\| / \delta$ and $\beta = \|c - c_1\| / \delta$ where $\delta = \|c_1 - c_2\| \neq 0$, so that

$$\begin{aligned}\alpha-\beta &= (\alpha-\beta)(\alpha+\beta)\\ &= \alpha^2-\beta^2\\ &= \frac{\|c-c_2\|^2-\|c-c_1\|^2}{\delta^2}\\ &= \frac{(\|c-c_2\|^2-\rho_2^2)+\rho_2^2-(\|c-c_1\|^2-\rho_1^2)-\rho_1^2}{\delta^2}\\ &= \frac{\rho_2^2-\rho_1^2}{\delta^2}\end{aligned}$$

since $c \in H$. It follows that

$$\alpha = \frac{1}{2}+\frac{\rho_2^2-\rho_1^2}{2\delta^2} \qquad \beta = \frac{1}{2}-\frac{\rho_2^2-\rho_1^2}{2\delta^2} \qquad c = \frac{1}{2}(c_1+c_2)+\frac{\rho_2^2-\rho_1^2}{2\delta^2}(c_1-c_2)$$

Note c is the midpoint of c_1 and c_2 if and only if $\rho_1=\rho_2$. Now

$$\begin{aligned}x\in H &\iff \|x-c_1\|^2-\rho_1^2 = \|x-c_2\|^2-\rho_2^2\\ &\iff (x-c_2 \mid x-c_2)-(x-c_1\mid x-c_1) = \rho_2^2-\rho_1^2\\ &\iff (x\mid c_1-c_2) = \tfrac{1}{2}(\|c_1\|^2-\|c_2\|^2+\rho_2^2-\rho_1^2) = (c\mid c_1-c_2)\end{aligned}$$

Therefore H is the hyperplane through c perpendicular to $D_{c_1c_2}$. Finally, $S_1\cap S_2 \subseteq H$ since $S_1\cap S_2$ consists of the points whose powers with respect to S_1 and S_2 are zero.

- Let $\delta = \|c_1-c_2\|$. We have

$$\pi_2(c_1) = \delta^2-\rho_2^2 = \rho_1^2 \iff \rho_2^2 = \delta^2-\rho_1^2 = \pi_1(c_2)$$

so (a) $\iff$ (b), and these are equivalent to $\delta^2 = \rho_1^2+\rho_2^2$. If this condition holds, then $\rho_1-\delta \le 0$, so $2\rho_1(\rho_1-\delta)\le 0$ and

$$\begin{aligned}2\rho_1^2-2\rho_1\delta+\rho_2^2 \le \rho_2^2 &\le 2\rho_1^2+2\rho_1\delta+\rho_2^2\\ \rho_1^2-2\rho_1\delta+\delta^2 \le \rho_2^2 &\le \rho_1^2+2\rho_1\delta+\delta^2\\ (\rho_1-\delta)^2 \le \rho_2^2 &\le (\rho_1+\delta)^2\\ |\rho_1-\delta| \le \rho_2 &\le \rho_1+\delta\end{aligned}$$

so $S_1\cap S_2 \ne \emptyset$ (5.1.11.4). If $x\in S_1\cap S_2$, then

$$\|x-c_1\|^2+\|x-c_2\|^2 = \rho_1^2+\rho_2^2 = \delta^2 = \|c_1-c_2\|^2$$

so $(x-c_1 \mid x-c_2) = 0$ (5.1.1.4). Therefore (a),(b) $\implies$ (c). Conversely, if $x \in S_1 \cap S_2$ and $(x-c_1 \mid x-c_2) = 0$, then by Pythagoras (5.1.5.1),

$$\delta^2 = \|c_1 - c_2\|^2 = \|x-c_1\|^2 + \|x-c_2\|^2 = \rho_1^2 + \rho_2^2$$

so (c) $\implies$ (a),(b). Finally, by (5.1.1.4),

$$\begin{aligned}\pi_1(x) + \pi_2(x) &= \|x-c_1\|^2 + \|x-c_2\|^2 - (\rho_1^2 + \rho_2^2) \\ &= 2(x-c_1 \mid x-c_2) + \delta^2 - (\rho_1^2 + \rho_2^2)\end{aligned}$$

so clearly (a),(b) $\iff$ (d). □

Exercise (6)**.** *Characterization of similitudes:*

- If $u : E \to E$ is a bijective function such that $(u(x) \mid u(y)) = \alpha(x \mid y)$ for all $x, y \in E$ $(\alpha > 0)$, then u is linear and consequently $u \in \mathbf{GO}(E)$.
- If $u : E \to E$ is a bijective function such that $d(u(x), u(y)) = \alpha d(x, y)$ for all $x, y \in E$ $(\alpha > 0)$, then u is affine and consequently $u \in \mathbf{Sm}(E)$.
- If $u \in \mathbf{GL}(E)$ is such that $(x \mid y) = 0$ implies $(u(x) \mid u(y)) = 0$ (in other words, u preserves orthogonality), then $u \in \mathbf{GO}(E)$.

Proof.

- By direct computation,

$$\begin{aligned}&(u(x+y) - u(x) - u(y) \mid u(x+y) - u(x) - u(y)) = \\ &\qquad\qquad \alpha(x+y-x-y \mid x+y-x-y) = 0\end{aligned}$$

  so $u(x+y) - u(x) - u(y) = 0$ by positive definiteness of the inner product. Similarly $u(\xi x) - \xi u(x) = 0$. Therefore u is linear.

- Let $v = t_{-u(0)} u$. Then $v : E \to E$ is a bijective function and (5.1.1.5)

$$\begin{aligned}2(v(x) \mid v(y)) &= \|v(x)\|^2 + \|v(y)\|^2 - \|v(x) - v(y)\|^2 \\ &= \|u(x) - u(0)\|^2 + \|u(y) - u(0)\|^2 - \|u(x) - u(y)\|^2 \\ &= \alpha^2(\|x\|^2 + \|y\|^2 - \|x-y\|^2) \\ &= 2\alpha^2(x \mid y)\end{aligned}$$

  Therefore $v \in \mathbf{GO}(E)$ by the previous item and $u \in \mathbf{Sm}(E)$ (5.1.14).

- First observe that $(u(x) \mid u(y)) = 0$ implies $(x \mid y) = 0$. Indeed, if $y = 0$ this is trivial. If $y \neq 0$, then $x = \xi y + z$ with $(z \mid y) = 0$ (5.1.7) and

  $$\begin{aligned} 0 &= (u(x) \mid u(y)) \\ &= (u(\xi y + z) \mid u(y)) \\ &= (\xi u(y) + u(z) \mid u(y)) \\ &= \xi \|u(y)\|^2 && \text{since } (u(z) \mid u(y)) = 0 \end{aligned}$$

  Since $u(y) \neq 0$, this implies $\xi = 0$, so $x = z$ and $(x \mid y) = 0$.

  Now if $y \neq 0$, it follows that $(u(x) \mid u(y)) = 0$ is another equation of the hyperplane $(x \mid y) = 0$ orthogonal to y, so (3.3.6) there is $\mu_y > 0$ with

  $$(u(x) \mid u(y)) = \mu_y (x \mid y)$$

  for all $x \in E$. We claim that μ_y is independent of y. Indeed, if $y' = 0$ then trivially $(u(x) \mid u(y')) = \mu_y (x \mid y')$ for all $x \in E$. If $(y \mid y') \neq 0$, then

  $$\mu_y = \frac{(u(y) \mid u(y'))}{(y \mid y')} = \mu_{y'}$$

  Finally, if $y' \neq 0$ and $(y \mid y') = 0$, then $(y \mid y + y') \neq 0$ and $(y' \mid y + y') \neq 0$, so

  $$\mu_y = \mu_{y+y'} = \mu_{y'}$$

  by the previous case. □

Exercise (7)**.** In $\mathbf{GL}(E)$, the normalizer of $\mathbf{O}(E)$ is $\mathbf{GO}(E)$.

Proof. The normalizer contains $\mathbf{GO}(E)$ since if $v \in \mathbf{GO}(E)$ and $u \in \mathbf{O}(E)$, then $\mu(vuv^{-1}) = 1$ (5.1.12.3), so $vuv^{-1} \in \mathbf{O}(E)$.

Conversely, suppose $v \in \mathbf{GL}(E)$ normalizes $\mathbf{O}(E)$. If $(x \mid y) = 0$ and $x = 0$, then trivially $(v(x) \mid v(y)) = 0$. If $x \neq 0$, let $u \in \mathbf{O}(E)$ be the symmetry about the hyperplane orthogonal to x, so $u(x) = -x$ and $u(y) = y$. Then $vuv^{-1} \in \mathbf{O}(E)$ by hypothesis and $vuv^{-1}v = vu$, so

$$(v(x) \mid v(y)) = (v(u(x)) \mid v(u(y))) = -(v(x) \mid v(y))$$

and hence $(v(x) \mid v(y)) = 0$. Since x, y were arbitrary, it follows that v preserves orthogonality, and therefore $v \in \mathbf{GO}(E)$ (Exercise 6). □

Section 2

Remark. In the plane, two lines are parallel if and only if there is a line to which they are both perpendicular; in this case, they are perpendicular to the same lines.

Proof. Parallel lines have the same direction, hence are perpendicular to the same lines, in particular the vector line orthogonal to that direction. Conversely, if two lines are both perpendicular to a third, then their direction is the vector line orthogonal to the direction of the third, hence they are parallel (5.2.2). □

Remark. In the plane, a line is tangential to a circle at a point if and only if the line intersects the circle at the point and is perpendicular to the line passing through the point and the center of the circle.

Proof. The forward direction is proved in Exercise 5.1.4. The reverse direction follows from Pythagoras (5.1.5). □

Remark. In (5.2.3), the mapping $f \mapsto a$ is actually a vector space *isomorphism* from the dual space E^* to E. In fact, if $\{a_1, a_2\}$ is an orthonormal basis of E, then the isomorphism is just that determined by $a_i^* \mapsto a_i$, where $\{a_1^*, a_2^*\}$ is the dual basis (4.1.15).

Remark. In (5.2.4), we have vector space isomorphisms

$$\mathscr{B}(E, E; \mathbf{R}) \cong \mathrm{Hom}(E, E^*) \cong \mathrm{End}(E)$$

Remark. In the complex plane $\mathbf{C}$, a number $z \in \mathbf{C}$ induces an endomorphism through multiplication, and its conjugate $\overline{z}$ induces the adjoint endomorphism. Note $\overline{\overline{z}} = z$ as in (5.2.5.2), $\overline{wz} = \overline{w}\,\overline{z}$ as in (5.2.5.3), $\overline{z}z = |z|^2$ is the "multiplicator" of z as in (5.2.5.5), and z is "orthogonal" if it is on the unit circle. If $z \neq 0$, then $z = \rho u$ with $\rho = |z| > 0$ and $u = z/|z|$ on the unit circle uniquely determined, as in the polar decomposition (5.1.12).

Endomorphisms of a plane can therefore be viewed as generalized complex numbers, with adjoints as generalized complex conjugates. This idea is made precise in (5.5).

Remark. In (5.2.8), $M(w) = (\alpha_{ij})$ with $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{12} = -\alpha_{21}$ relative to an orthonormal basis (5.2.6), so $\det(w) = \alpha_{12}^2$ (4.2.9.2). Since $w \neq 0$, $\det(w) > 0$ and w is invertible (4.2.8).

Exercise (1)**.** If $u \in \mathbf{GL}(E)$ is an involution, then $u \in \mathbf{O}(E)$ if and only if u is self-adjoint. If $p \in \operatorname{End}(E)$ is idempotent, then p is an orthogonal projection (5.1.6) if and only if p is self-adjoint.

Proof. By (5.2.5.6), $u \in \mathbf{O}(E)$ if and only if $u^* = u^{-1} = u$. By Exercise 3.2.1, p is the projection onto $p(E)$ in the direction of $p^{-1}(0)$. If $p(E)$ is orthogonal to $p^{-1}(0)$, then

$$\begin{aligned}
(p(x) \mid y) &= (p(x) \mid p(y) + (y - p(y))) \\
&= (p(x) \mid p(y)) && \text{since } (p(x) \mid y - p(y)) = 0 \\
&= (p(x) + (x - p(x)) \mid p(y)) && \text{since } (x - p(x) \mid p(y)) = 0 \\
&= (x \mid p(y))
\end{aligned}$$

for all $x, y \in E$. Therefore $p^* = p$. Conversely if $p^* = p$, then for all $x \in E$ and $y \in p^{-1}(0)$,

$$(p(x) \mid y) = (x \mid p(y)) = (x \mid 0) = 0$$

so $p(E)$ is orthogonal to $p^{-1}(0)$. □

Exercise (8)**.** Suppose $u \in \operatorname{End}(E)$ has eigenvalues $\lambda_1 \neq \lambda_2$ with corresponding eigenvectors a_1, a_2. If b_1 (resp. b_2) is nonzero and orthogonal to a_1 (resp. a_2), then b_1 (resp. b_2) is an eigenvector of u^* corresponding to λ_2 (resp. λ_1).

Proof. Since $\lambda_1 \neq \lambda_2$, $\{a_1, a_2\}$ must be linearly independent and hence a basis. If $(a_1 \mid x) = (a_1 \mid y)$ and $(a_2 \mid x) = (a_2 \mid y)$, then for any $z = \alpha_1 a_1 + \alpha_2 a_2$,

$$(z \mid x) = \alpha_1(a_1 \mid x) + \alpha_2(a_2 \mid x) = \alpha_1(a_1 \mid y) + \alpha_2(a_2 \mid y) = (z \mid y)$$

and it follows that $x = y$.

Now observe that for $z = a_1, a_2$,

$$(z \mid u^*(b_1)) = (u(z) \mid b_1) = \lambda_2(z \mid b_1) = (z \mid \lambda_2 b_1)$$

so $u^*(b_1) = \lambda_2 b_1$. Similarly $u^*(b_2) = \lambda_1 b_2$. □

Section 3

Remark. In (5.3.1), if we fix an *orientation* of the plane (4.3.1), then a similitude preserves orientation if it is direct and reverses orientation if it is inverse (4.3.2). In other words, $\mathbf{GO}^+(E) = \mathbf{GO}(E) \cap \mathbf{GL}^+(E)$. If $u \in \mathbf{GO}(E)$, then the following are equivalent:

- u is a rotation
- u preserves length and orientation
- u preserves length and oriented area[2]
- u preserves orientation and oriented area

Remark. By the polar decomposition (5.1.12) and (5.3.2), an affine similitude of the plane is the composite of a translation after a linear scaling after a linear rotation or reflection. If the scaling is trivial, then the similitude is an affine isometry; otherwise, it is a linear similitude about another origin (5.3.6).

Remark. We consider products of affine isometries $u, v \in \mathbf{Is}(E)$ with $u, v \neq 1$:

- If u is a translation and v is a rotation, then uv is a rotation about another origin (5.3.7).
- If u is a translation and v is a reflection, then uv is a glide reflection (5.3.7).
- If u and v are rotations, then uv is a translation or a rotation (5.3.7).
- If u is a rotation and v is a reflection, then uv is a glide reflection (5.3.7).
- If u and v are reflections in parallel lines, then uv is a translation in a direction orthogonal to that of the lines (5.1.16).
- If u and v are reflections in non-parallel lines, then uv is a rotation about the point of intersection of the lines. This follows from (5.3.2) after taking the origin at the point of intersection.

Exercise (2)**.** If $u \in \mathrm{End}(E)$ is normal, then vuv^{-1} is normal for all $v \in \mathbf{GO}(E)$. Also, u is self-adjoint or a direct similitude.

Proof. By (5.2.5.5), $v^* v = \alpha \cdot 1$ for some $\alpha > 0$, so $v^* = \alpha v^{-1}$, $(v^{-1})^* = \alpha^{-1} v$, and

$$\begin{aligned}
(vuv^{-1})^* vuv^{-1} &= (v^{-1})^* u^* v^* vuv^{-1} \\
&= vu^* v^{-1} vuv^{-1} \\
&= vu^* uv^{-1} \\
&= vuu^* v^{-1} && \text{since } u \text{ is normal} \\
&= vuv^{-1} vu^* v^{-1} \\
&= vuv^{-1} (vuv^{-1})^*
\end{aligned}$$

[2] By definition, u preserves oriented area if $u \in \mathbf{SL}(E)$.

Therefore vuv^{-1} is normal.

If $M(u) = (\alpha_{ij})$ relative to some fixed orthonormal basis, then $M(u^*) = (\alpha_{ji})$ (5.2.6.3) and

$$M(u^*)M(u) = M(u^* u) = M(uu^*) = M(u)M(u^*)$$

implies $\alpha_{12} = \pm\alpha_{21}$ and $\alpha_{11}(\alpha_{12}-\alpha_{21}) = \alpha_{22}(\alpha_{12}-\alpha_{21})$. If $\alpha_{12} = \alpha_{21}$, then $M(u)$ is symmetric and u is self-adjoint. If $\alpha_{12} \neq \alpha_{21}$, then $\alpha_{12} = -\alpha_{21}$ and $\alpha_{11} = \alpha_{22}$, so

$$M(u^* u) = (\alpha_{11}^2 + \alpha_{12}^2) \cdot 1 = \det(u) \cdot 1$$

Therefore $u^* u = \det(u) \cdot 1$, so u is a direct similitude. □

Exercise (5)**.** Every affine isometry of the plane is the product of at most three reflections in lines. The product of two affine rotations is a translation if and only if the associated linear rotations are mutually inverse.

Proof. If $u \in \mathbf{Is}(E)$, then by cases (5.3.7):

- If u is the identity, then u is trivially the product of zero reflections.
- If u is a nonzero translation, then u is the product of two reflections about lines perpendicular to the direction of translation (5.1.16).
- If u is a rotation, then u is the product of two reflections about lines intersecting at the center of the rotation (5.3.2).
- If u is a glide reflection, then u is a product of one or three reflections depending as the translation is zero or nonzero.

If $u_1, u_2 \in \mathbf{Is}(E)$ are affine rotations, then $u_1 = t_1 v_1$ and $u_2 = t_2 v_2$ where t_1, t_2 are translations and $v_1, v_2 \in \mathbf{O}^+(E)$. By (3.2.19.1), $u_1 u_2 = t v_1 v_2$, where t is a translation, so $u_1 u_2$ is a translation if and only if $v_1 v_2$ is, which is true if and only if $v_1 v_2 = 1$. □

Section 4

Remark. In (5.4.3), recall that $\Psi(x, y)$ is the *oriented area* of the parallelogram determined by x and y relative to the fixed orientation of the plane (see the remark on (4.2.5) above). Therefore if u is a rotation and x is a unit vector, then $\Psi(x, u(x))$ is the signed distance from $u(x)$ to the line determined by x—that is, the sine of the angle $\widehat{(x, u(x))}$ of rotation.

Remark. In (5.4.10), a line D' passing through $-a_1$ is tangential to $\mathbf{U}$ if and only if it is parallel to D, by remarks from (5.2) above. Therefore D' meets $\mathbf{U}$ at a unique point x distinct from $-a_1$ if and only if it meets D at a unique point y.

For $x \in \mathbf{U}'$, $(x \mid x + a_1) = (a_1 \mid x + a_1)$, so it follows from (5.1.7) and (5.4.8) that the direction of $D_{-a_1,x}$ bisects Δ_{0a_1} and Δ_{0x} and $\widehat{(a_1, x)} = 2\widehat{(a_1, x + a_1)}$. It is then immediate that $x = \cos 2\theta \cdot a_1 + \sin 2\theta \cdot a_2$ and $y = \tan\theta \cdot a_2$, where $\theta = \widehat{(a_1, x + a_1)}$.

Remark. In (5.4.13), where θ', θ'' correspond to Δ', Δ'', we have $\theta' = \widehat{(-\Delta_0, \Delta')}$ and $\theta'' = \widehat{(-\Delta_0, \Delta'')}$ (5.4.12), so

$$\theta'' - \theta' = -\theta' + \theta'' = \widehat{(\Delta', -\Delta_0)} + \widehat{(-\Delta_0, \Delta'')} = \widehat{(\Delta', \Delta'')}$$

Let x', x'' be unit vectors in Δ', Δ''. Then by (4.3.5) and (5.4.9), $\mathrm{S}^\circ(\Delta', \Delta'')$ is minor if and only if

$$\sin(\theta'' - \theta') = \sin\widehat{(\Delta', \Delta'')} = \sin\widehat{(x', x'')} = \Psi(x', x'') > 0$$

which is equivalent to $\theta'' - \theta' > 0$ (5.4.12). Similarly, $\mathrm{S}^\circ(\Delta', \Delta'')$ is major if and only if $\theta'' - \theta' < 0$.

Exercise (7)**.** In the set of pairs of vector half-lines (resp. vector lines), the orbits under the action of $\mathbf{O}^+(E)$ are the sets of pairs with the same angle; under $\mathbf{O}(E)$, they are the sets of pairs with the same or opposite angle.

Proof. By (5.4.7) and (5.3.2). □

Section 5

Exercise (3)**.** *Cayley's parametric representation:* If $z \in \mathbf{C}$ with $|z| = 1$ and $z \neq -1$, then there exists $\xi \in \mathbf{R}$ unique such that

$$z = (1 + \xi i)(1 - \xi i)^{-1}$$

Proof. Let $\theta = \arg z$, $\xi = \tan(\theta/2)$, and $w = 1 + \xi i$. By (5.4.5.3),

$$\begin{aligned} w\overline{w}^{-1} &= \frac{w^2}{|w|^2} \\ &= \frac{(1+\xi i)^2}{1+\xi^2} \\ &= \frac{1-\xi^2}{1+\xi^2} + \frac{2\xi}{1+\xi^2} i \\ &= \cos\theta + i \sin\theta = z \end{aligned}$$

If $z = (1 + \xi' i)(1 - \xi' i)^{-1}$, direct computation shows $\xi' = \xi$. □

Chapter VI

Section 2

Remark. If $\Phi \neq 0$ is an alternating *trilinear* form on E and Ψ is an alternating *bilinear* form on E, then there is a unique vector $v \in E$ with

$$\Psi(x, y) = \Phi(x, y, v) \tag{1}$$

for all vectors $x, y \in E$.

Proof. Define $\varphi : E^3 \to E$ by

$$\varphi(x, y, z) = \Psi(y, z)x - \Psi(x, z)y + \Psi(x, y)z \tag{2}$$

Then φ is alternating and trilinear, so there is a vector v with $\varphi = \Phi \cdot v$ (6.2.2.1). If x, y are linearly dependent, then both sides of (1) are zero. Otherwise there is z with $\Phi(x, y, z) = 1$ and $\varphi(x, y, z) = v$. Now by (2),

$$\begin{aligned}\Phi(x, y, v) &= \Phi(x, y, \varphi(x, y, z)) \\ &= \Psi(y, z)\Phi(x, y, x) - \Psi(x, z)\Phi(x, y, y) + \Psi(x, y)\Phi(x, y, z) \\ &= \Psi(x, y)\end{aligned}$$

The vector v is uniquely determined since $\Phi \neq 0$. □

This shows that any function measuring oriented area in planes in E actually measures oriented volume relative to a fixed vector.

Exercise (11)**.** Let $\Psi \neq 0$ be an alternating bilinear form on E and

$$N = \{x \in E \mid \Psi(x, y) = 0 \text{ for all } y \in E\}$$

Then $N \neq 0$ and there is a basis of E relative to which

$$M(\Psi) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Proof. Let $\Phi \neq 0$ be an alternating trilinear form on E. By the remark above, there is a unique vector c with

$$\Psi(x, y) = \Phi(x, y, c)$$

for all vectors x, y. Clearly $N = D_{0c} \neq 0$, and if a, b are chosen so $\Phi(a, b, c) = -1$, then $M(\Psi)$ has the desired form relative to the basis a, b, c. □

Chapter VII

Section 1

Exercise (1)**.** Let a_1, a_2, a_3 be an orthonormal basis of E, x_1, x_2, x_3 vectors in E, and u the endomorphism of E with $u(a_i) = x_i$. Define

$$G(x_1, x_2, x_3) = \begin{vmatrix} (x_1 \mid x_1) & (x_1 \mid x_2) & (x_1 \mid x_3) \\ (x_2 \mid x_1) & (x_2 \mid x_2) & (x_2 \mid x_3) \\ (x_3 \mid x_1) & (x_3 \mid x_2) & (x_3 \mid x_3) \end{vmatrix}$$

Then

$$G(x_1, x_2, x_3) = \det(u)^2 = \Psi(x_1, x_2, x_3)^2 \geq 0 \tag{1}$$

where Ψ is as in (7.1.7).

Similarly define

$$G(x_1, x_2) = \begin{vmatrix} (x_1 \mid x_1) & (x_1 \mid x_2) \\ (x_2 \mid x_1) & (x_2 \mid x_2) \end{vmatrix}$$

Then

$$G(x_1, x_2) = \|x_1 \times x_2\|^2 \tag{2}$$

Proof. Since

$$(x_i \mid x_j) = (u(a_i) \mid u(a_j)) = (a_i \mid u^* u(a_j))$$

it follows from (7.1.1.4) that

$$G(x_1, x_2, x_3) = \det(u^* u) = \det(u^*) \det(u) = \det(u)^2$$

On the other hand

$$\det(u) = \Psi(u(a_1), u(a_2), u(a_3)) = \Psi(x_1, x_2, x_3)$$

which establishes (1).

If x_1, x_2 are linearly dependent, then both sides of (2) are zero. If x_1, x_2 are linearly independent, let

$$x_3 = \frac{x_1 \times x_2}{\|x_1 \times x_2\|}$$

Then

$$G(x_1, x_2) = G(x_1, x_2, x_3) = \Psi(x_1, x_2, x_3)^2 = \|x_1 \times x_2\|^2 \qquad \square$$

Exercise (3)**.** If $u \in \mathbf{GL}(E)$, then

$$u^*(x \times y) = \det(u) \cdot u^{-1}(x) \times u^{-1}(y) \tag{1}$$

or equivalently

$$u^{-1}(x \times y) = (\det(u))^{-1} \cdot u^*(x) \times u^*(y) \tag{2}$$

Moreover

$$u^*(u(x) \times u(y)) = \det(u) \cdot x \times y = u(u^*(x) \times u^*(y)) \tag{3}$$

Proof. Let Ψ be as in (7.1.7). For any vector z,

$$\begin{aligned}
(u^*(x \times y) \mid z) &= (x \times y \mid u(z)) \\
&= \Psi(x, y, u(z)) \\
&= \Psi(u(u^{-1}(x)), u(u^{-1}(y)), u(z)) \\
&= \det(u) \cdot \Psi(u^{-1}(x), u^{-1}(y), z) \\
&= \det(u) \cdot (u^{-1}(x) \times u^{-1}(y) \mid z) \\
&= (\det(u) \cdot u^{-1}(x) \times u^{-1}(y) \mid z)
\end{aligned}$$

Now (1) follows from definiteness of the inner product.

Substituting u^* for u in (1) and noting that $(u^*)^* = u$ and $\det(u^*) = \det(u)$, we obtain

$$u(x \times y) = \det(u) \cdot (u^*)^{-1}(x) \times (u^*)^{-1}(y) \tag{4}$$

Now substituting u^{-1} for u in (4) and noting that $(u^{-1})^* = (u^*)^{-1}$, we obtain (2). By reversing these steps, we obtain (1) from (2).

Finally, (3) follows by substituting $u(x)$ for x and $u(y)$ for y in (1), and by substituting $u^*(x)$ for x and $u^*(y)$ for y in (4). Note that (3) is *self-dual*, meaning it is invariant under substitution of u^* for u. □

References

[1] Dieudonné, J. *Linear Algebra and Geometry.* Hermann, 1969.