Notes and exercises from *Linear Algebra and Geometry*

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Introduction

This document contains notes and exercises from [1].

Chapter III

Section 1

Exercise (4)**.** Let V, W be a pair of supplementary subspaces of E. Every subspace U containing V is the direct sum of V with $U \cap W$.

Proof. If $u \in U$, then $u = v + w$ for some $v \in V$ and $w \in W$, and $w = u - v \in U$. So $U = V + (U \cap W)$, and $V \cap U \cap W = \{0\}$. □

Section 2

Exercise (1)**.** If p, q are the projections corresponding to a direct sum $E = V + W$, then $p, q \in \mathrm{End}(E)$ are such that $p^2 = p$, $q^2 = q$, and $p + q = 1$. Conversely, if $p \in \mathrm{End}(E)$ is such that $p^2 = p$, then $E = p(E) + p^{-1}(0)$ is a direct sum. Moreover, if $q = 1 - p$, then $q^2 = q$, $q(E) = p^{-1}(0)$, and $q^{-1}(0) = p(E)$.

Proof. For the forward direction, we know $p, q \in \mathrm{End}(E)$ and $p + q = 1$ (3.2.2). It follows that $p^2 = p \circ (1 - q) = p - pq = p$ and $q^2 = (1 - p)^2 = 1 - p = q$.

For the converse, $p + q = 1$, so $E = p(E) + q(E)$. Also $pq = p - p^2 = 0$, so $p(E) \cap q(E) = \{0\}$ and $q(E) \subseteq p^{-1}(0)$. If $x \in p^{-1}(0)$, then $q(x) = x$, so $x \in q(E)$. Hence $q(E) = p^{-1}(0)$ and similarly $q^{-1}(0) = p(E)$. Finally $q^2 = q$ as above. □

Exercise (2)**.** If W and W' are both supplementary to V in E, then W and W' are isomorphic.

Proof. If p is the projection of E onto W', then the restriction of p to W is an isomorphism from W to W'. $\square$

Exercise (3)**.** If $E = V + W$ is a direct sum with inclusions $i : V \to E$ and $j : W \to E$, and $v : V \to F$ and $w : W \to F$ are linear maps, then there is a unique linear map $u : E \to F$ with $u \circ i = v$ and $u \circ j = w$.

Proof. If p, q are the projections on V, W respectively, then $u = v \circ p + w \circ q$. $\square$

Exercise (11)**.** $\mathbf{GA}(E)/E \cong \mathbf{GL}(E)$.

Proof. Define $\varphi : \mathbf{GA}(E) \to \mathbf{GL}(E)$ by $\varphi(t_a \circ v) = v$. Note that φ is well-defined by (3.2.17), φ is a homomorphism by (3.2.19), and φ is clearly surjective. Also $\varphi(u) = 1$ if and only if u is a translation, so $\ker\varphi = T(E)$, the normal subgroup of translations. It follows that $\mathbf{GA}(E)/T(E) \cong \mathbf{GL}(E)$. Finally, the mapping $a \mapsto t_a$ is an isomorphism $E \cong T(E)$ from the additive group E. $\square$

Exercise (13)**.** If $u : E \to F$ is affine and L is a variety in F, then $u^{-1}(L)$ is empty or a variety in E.

Proof. If $a \in u^{-1}(L)$ and L_0 is the direction of L, then $L = u(a) + L_0$ and hence $u^{-1}(L) = a + u^{-1}(L_0)$. $\square$

Section 3

Exercise (3)**.** A necessary and sufficient condition for a nonempty subset V of a vector space to be a variety is that for all pairs x, y of distinct points of V, the line D_{xy} is contained in V.

Proof. The condition is necessary by (3.3.2).

If the condition holds, choose $v \in V$ and let $V_0 = -v + V$. We claim V_0 is a subspace, from which it follows that $V = v + V_0$ is a variety. First, $0 = -v + v \in V_0$. If $x \in V_0$ and $x \neq 0$, then $v + x \in V$ and $v + x \neq v$, so $D_{v,v+x} = \{v + \xi x \mid \xi \in \mathbf{R}\} \subseteq V$. It follows that $\xi x \in V_0$ for all $\xi \in \mathbf{R}$. If also $y \in V_0$ and $y \neq x$, then $D_{v+x,v+y} \subseteq V$, so in particular $v + 2^{-1}(x + y) \in V$ and $2^{-1}(x + y) \in V_0$. By the previous result, it then follows that $x + y \in V_0$. Therefore V_0 is a subspace as claimed. $\square$

Exercise (4)**.**

- A necessary and sufficient condition for an affine map to be a translation or a homothetic map is that its associated linear map be homothetic.
- A necessary and sufficient condition for an affine map to preserve the direction of lines is that it be a translation or a bijective homothetic map.
- If u_1, u_2 are translations or homothetic maps, then so is $u_1 \circ u_2$.
- If u_1, u_2 and $u_1 \circ u_2$ are homothetic maps with ratios not equal to 1, their centers are collinear. If instead $u_1 \circ u_2$ is a translation, then it is either the identity or a translation in the direction of the line through the centers of u_1 and u_2.
- If $v \in \mathbf{GA}(E)$, then $v \circ h_{a,\lambda} \circ v^{-1} = h_{v(a),\lambda}$.
- The subset $H(E)$ of translations and homothetic maps in $\mathbf{GA}(E)$ forms a subgroup, and $H(E)/E \cong \mathbf{R}^*$.

Proof.

- This follows from the equations $t_a = t_a \circ h_1$ and $h_{a,\lambda} = t_{(1-\lambda)a} \circ h_\lambda$ and $t_a \circ h_\lambda = h_{(1-\lambda)^{-1}a,\lambda}$ $(\lambda \neq 1)$.
- The condition is sufficient because such a map has the form $t_a \circ h_\lambda$ with $\lambda \neq 0$, which clearly preserves the direction of lines. Conversely, suppose $u = t_a \circ v$ preserves the direction of lines. If $x \neq 0$, let D be the vector line through x. Then $v(D) = D$, so $v(x) = \lambda x$ for some $\lambda \in \mathbf{R}$ with $\lambda \neq 0$, and in fact $v(y) = \lambda y$ for all $y \in D$. We claim $v = h_\lambda$, from which the result follows. If $y \notin D$, then by considering the vector line D' through y we have $v(y) = \mu y$ for some $\mu \in \mathbf{R}$. Now $v(D_{xy}) = D_{v(x)v(y)} = D_{\lambda x,\mu y}$, and since v preserves direction there is $\xi \in \mathbf{R}$ with $\mu y - \lambda x = \xi(y-x)$, or $(\mu-\xi)y = (\lambda-\xi)x$. Since $y \notin D$, this implies $\mu = \xi = \lambda$. Therefore $v = h_\lambda$ as claimed.
- If $u_1 = t_a \circ h_\lambda$ and $u_2 = t_b \circ h_\mu$, then by (3.2.19.1), $u_1 \circ u_2 = t_{a+\lambda b} \circ h_{\lambda\mu}$.
- Write $u_1 = h_{a,\lambda}$, $u_2 = h_{b,\mu}$, and $u_1 \circ u_2 = h_{c,\nu}$ with $\lambda, \mu, \nu \neq 1$. Then for all x,

  $$(1-\nu)c + \nu x = (1-\lambda)a + \lambda(1-\mu)b + \lambda\mu x$$

  Taking $x = 0$ and $x \neq 0$ (we assume such exist!) yields $\nu = \lambda\mu$ and

  $$c = (1-\lambda\mu)^{-1}[(1-\lambda)a + \lambda(1-\mu)b] = (1-\lambda\mu)^{-1}(\lambda-1)(b-a) + b$$

  so that a, b, c are collinear. If instead $u_1 \circ u_2 = t_c$, then $\lambda\mu = 1$ and $c = (\lambda-1)(b-a)$, from which the second result follows.

- Write $v = t_b \circ w$, where $w \in \mathbf{GL}(E)$, so that $v^{-1} = w^{-1} \circ t_{-b}$. By repeated application of (3.2.19.1),

$$\begin{aligned} v \circ h_{a,\lambda} \circ v^{-1} &= t_b \circ w \circ t_{(1-\lambda)a} \circ h_\lambda \circ w^{-1} \circ t_{-b} \\ &= t_{v((1-\lambda)a)} \circ w \circ h_\lambda \circ w^{-1} \circ t_{-b} \\ &= t_{b+(1-\lambda)w(a)} \circ h_\lambda \circ t_{-b} \\ &= t_{b+(1-\lambda)w(a)-\lambda b} \circ h_\lambda \\ &= t_{(1-\lambda)v(a)} \circ h_\lambda \\ &= h_{v(a),\lambda} \end{aligned}$$

- First, $1 \in H(E)$. If $u_1, u_2 \in H(E)$, then $u_1 \circ u_2 \in H(E)$ by a previous item. If $u_1 = t_a \circ h_\lambda$ with $\lambda \neq 0$, then $u_1^{-1} = t_{-\lambda^{-1}a} \circ h_{\lambda^{-1}} \in H(E)$. Therefore $H(E)$ is a subgroup of $\mathbf{GA}(E)$. Define $\varphi : H(E) \to \mathbf{R}^*$ by $\varphi(t_a \circ h_\lambda) = \lambda$. Note that φ is well-defined (since $E \neq \{0\}$!), φ is a homomorphism by a previous item, φ is surjective, and $\ker\varphi = T(E) \cong E$. It follows that $H(E)/E \cong \mathbf{R}^*$. □

References

[1] Dieudonné, J. *Linear Algebra and Geometry.* Hermann, 1969.