

Notes and exercises from *Linear Algebra and Geometry*

John Peloquin

Introduction

This document contains notes and exercises from [1].

Chapter III

Section 1

Exercise (4). Let V, W be a pair of supplementary subspaces of E . Every subspace U containing V is the direct sum of V with $U \cap W$.

Proof. If $u \in U$, then $u = v + w$ for some $v \in V$ and $w \in W$, and $w = u - v \in U$. So $U = V + (U \cap W)$, and $V \cap U \cap W = \{0\}$. \square

Section 2

Exercise (1). If p, q are the projections corresponding to a direct sum $E = V + W$, then $p, q \in \text{End}(E)$ are such that $p^2 = p$, $q^2 = q$, and $p + q = 1$. Conversely, if $p \in \text{End}(E)$ is such that $p^2 = p$, then $E = p(E) + p^{-1}(0)$ is a direct sum. Moreover, if $q = 1 - p$, then $q^2 = q$, $q(E) = p^{-1}(0)$, and $q^{-1}(0) = p(E)$.

Proof. For the forward direction, we know $p, q \in \text{End}(E)$ and $p + q = 1$ (3.2.2). It follows that $p^2 = p \circ (1 - q) = p - pq = p$ and $q^2 = (1 - p)^2 = 1 - p = q$.

For the converse, $p + q = 1$, so $E = p(E) + q(E)$. Also $pq = p - p^2 = 0$, so $p(E) \cap q(E) = \{0\}$ and $q(E) \subseteq p^{-1}(0)$. If $x \in p^{-1}(0)$, then $q(x) = x$, so $x \in q(E)$. Hence $q(E) = p^{-1}(0)$ and similarly $q^{-1}(0) = p(E)$. Finally $q^2 = q$ as above. \square

Exercise (2). If W and W' are both supplementary to V in E , then W and W' are isomorphic.

Proof. If p is the projection of E onto W' , then the restriction of p to W is an isomorphism from W to W' . \square

Exercise (3). If $E = V + W$ is a direct sum with inclusions $i : V \rightarrow E$ and $j : W \rightarrow E$, and $v : V \rightarrow F$ and $w : W \rightarrow F$ are linear maps, then there is a unique linear map $u : E \rightarrow F$ with $u \circ i = v$ and $u \circ j = w$.

Proof. If p, q are the projections on V, W respectively, then $u = v \circ p + w \circ q$. \square

Exercise (11). $\mathbf{GA}(E)/E \cong \mathbf{GL}(E)$.

Proof. Define $\varphi : \mathbf{GA}(E) \rightarrow \mathbf{GL}(E)$ by $\varphi(t_a \circ v) = v$. Note that φ is well-defined by (3.2.17), φ is a homomorphism by (3.2.19), and φ is clearly surjective. Also $\varphi(u) = 1$ if and only if u is a translation, so $\ker \varphi = T(E)$, the normal subgroup of translations. It follows that $\mathbf{GA}(E)/T(E) \cong \mathbf{GL}(E)$. Finally, the mapping $a \mapsto t_a$ is an isomorphism $E \cong T(E)$ from the additive group E . \square

Exercise (13). If $u : E \rightarrow F$ is affine and L is a variety in F , then $u^{-1}(L)$ is empty or a variety in E .

Proof. If $a \in u^{-1}(L)$ and L_0 is the direction of L , then $L = u(a) + L_0$ and hence $u^{-1}(L) = a + u^{-1}(L_0)$. \square

Section 3

Exercise (3). A necessary and sufficient condition for a nonempty subset V of a vector space to be a variety is that for all pairs x, y of distinct points of V , the line D_{xy} is contained in V .

Proof. The condition is necessary by (3.3.2).

If the condition holds, choose $v \in V$ and let $V_0 = -v + V$. We claim V_0 is a subspace, from which it follows that $V = v + V_0$ is a variety. First, $0 = -v + v \in V_0$. If $x \in V_0$ and $x \neq 0$, then $v + x \in V$ and $v + x \neq v$, so $D_{v, v+x} = \{v + \xi x \mid \xi \in \mathbf{R}\} \subseteq V$. It follows that $\xi x \in V_0$ for all $\xi \in \mathbf{R}$. If also $y \in V_0$ and $y \neq x$, then $D_{v+x, v+y} \subseteq V$, so in particular $v + 2^{-1}(x + y) \in V$ and $2^{-1}(x + y) \in V_0$. By the previous result, it then follows that $x + y \in V_0$. Therefore V_0 is a subspace as claimed. \square

Exercise (4).

- A necessary and sufficient condition for an affine map to be a translation or a homothetic map is that its associated linear map be homothetic.
- A necessary and sufficient condition for an affine map to preserve the direction of lines is that it be a translation or a bijective homothetic map.
- If u_1, u_2 are translations or homothetic maps, then so is $u_1 \circ u_2$.
- If u_1, u_2 and $u_1 \circ u_2$ are homothetic maps with ratios not equal to 1, their centers are collinear. If instead $u_1 \circ u_2$ is a translation, then it is either the identity or a translation in the direction of the line through the centers of u_1 and u_2 .
- If $v \in \mathbf{GA}(E)$, then $v \circ h_{a,\lambda} \circ v^{-1} = h_{v(a),\lambda}$.
- The subset $H(E)$ of translations and homothetic maps in $\mathbf{GA}(E)$ forms a subgroup, and $H(E)/E \cong \mathbf{R}^*$.

Proof.

- This follows from the equations $t_a = t_a \circ h_1$ and $h_{a,\lambda} = t_{(1-\lambda)a} \circ h_\lambda$ and $t_a \circ h_\lambda = h_{(1-\lambda)^{-1}a,\lambda}$ ($\lambda \neq 1$).
- The condition is sufficient because such a map has the form $t_a \circ h_\lambda$ with $\lambda \neq 0$, which clearly preserves the direction of lines. Conversely, suppose $u = t_a \circ v$ preserves the direction of lines. If $x \neq 0$, let D be the vector line through x . Then $v(D) = D$, so $v(x) = \lambda x$ for some $\lambda \in \mathbf{R}$ with $\lambda \neq 0$, and in fact $v(y) = \lambda y$ for all $y \in D$. We claim $v = h_\lambda$, from which the result follows. If $y \notin D$, then by considering the vector line D' through y we have $v(y) = \mu y$ for some $\mu \in \mathbf{R}$. Now $v(D_{xy}) = D_{v(x)v(y)} = D_{\lambda x, \mu y}$, and since v preserves direction there is $\xi \in \mathbf{R}$ with $\mu y - \lambda x = \xi(y - x)$, or $(\mu - \xi)y = (\lambda - \xi)x$. Since $y \notin D$, this implies $\mu = \xi = \lambda$. Therefore $v = h_\lambda$ as claimed.
- If $u_1 = t_a \circ h_\lambda$ and $u_2 = t_b \circ h_\mu$, then by (3.2.19.1), $u_1 \circ u_2 = t_{a+\lambda b} \circ h_{\lambda\mu}$.
- Write $u_1 = h_{a,\lambda}$, $u_2 = h_{b,\mu}$, and $u_1 \circ u_2 = h_{c,v}$ with $\lambda, \mu, v \neq 1$. Then for all x ,

$$(1 - v)c + vx = (1 - \lambda)a + \lambda(1 - \mu)b + \lambda\mu x$$

Taking $x = 0$ and $x \neq 0$ (we assume such exist!) yields $v = \lambda\mu$ and

$$c = (1 - \lambda\mu)^{-1}[(1 - \lambda)a + \lambda(1 - \mu)b] = (1 - \lambda\mu)^{-1}(\lambda - 1)(b - a) + b$$

so that a, b, c are collinear. If instead $u_1 \circ u_2 = t_c$, then $\lambda\mu = 1$ and $c = (\lambda - 1)(b - a)$, from which the second result follows.

- Write $v = t_b \circ w$, where $w \in \mathbf{GL}(E)$, so that $v^{-1} = w^{-1} \circ t_{-b}$. By repeated application of (3.2.19.1),

$$\begin{aligned}
v \circ h_{a,\lambda} \circ v^{-1} &= t_b \circ w \circ t_{(1-\lambda)a} \circ h_\lambda \circ w^{-1} \circ t_{-b} \\
&= t_{v((1-\lambda)a)} \circ w \circ h_\lambda \circ w^{-1} \circ t_{-b} \\
&= t_{b+(1-\lambda)w(a)} \circ h_\lambda \circ t_{-b} \\
&= t_{b+(1-\lambda)w(a)-\lambda b} \circ h_\lambda \\
&= t_{(1-\lambda)v(a)} \circ h_\lambda \\
&= h_{v(a),\lambda}
\end{aligned}$$

- First, $1 \in H(E)$. If $u_1, u_2 \in H(E)$, then $u_1 \circ u_2 \in H(E)$ by a previous item. If $u_1 = t_a \circ h_\lambda$ with $\lambda \neq 0$, then $u_1^{-1} = t_{-\lambda^{-1}a} \circ h_{\lambda^{-1}} \in H(E)$. Therefore $H(E)$ is a subgroup of $\mathbf{GA}(E)$. Define $\varphi : H(E) \rightarrow \mathbf{R}^*$ by $\varphi(t_a \circ h_\lambda) = \lambda$. Note that φ is well-defined (since $E \neq \{0\}$), φ is a homomorphism by a previous item, φ is surjective, and $\ker \varphi = T(E) \cong E$. It follows that $H(E)/E \cong \mathbf{R}^*$. \square

Exercise (5). A variety not parallel to a hyperplane meets the hyperplane. If H is a vector hyperplane in E and V is a subspace of E not contained in H , then $V \cap H$ is a vector hyperplane in V .

Proof. Any vector in the direction of the variety which is not in the direction of the hyperplane determines a line in the variety which meets the hyperplane (3.3.8).

If D is the vector line through a vector in $V - H$, then D is supplementary to H in E , so D is supplementary to $V \cap H$ in V (Exercise 3.1.4). \square

Exercise (6). Let H be a vector hyperplane in E and suppose $u \in \text{End}(E)$ fixes every element of H .

- There is $\gamma \in \mathbf{R}$ unique such that for all $a \in E - H$, $u(a) \in \gamma a + H$.
- If $\gamma \neq 1$, then γ is an eigenvalue of u and $E(\gamma; u)$ is a line S supplementary to $E(1; u) = H$. A subspace V satisfies $u(V) \subseteq V$ if and only if $S \subseteq V$ or $V \subseteq H$. In particular, a vector line D satisfies $u(D) \subseteq D$ if and only if $D = S$ or $D \subseteq H$.

Proof.

- If $a \in E - H$, then $E = \mathbf{R}a + H$, so $u(a) = \gamma a + t$ for some $\gamma \in \mathbf{R}$ and $t \in H$. If $b \in E$, then $b = \beta a + h$ for some $\beta \in \mathbf{R}$ and $h \in H$, so

$$\begin{aligned}
u(b) &= u(\beta a + h) \\
&= \beta u(a) + u(h) \\
&= \beta(\gamma a + t) + h \\
&= \gamma(\beta a + h) + (1 - \gamma)h + \beta t \\
&= \gamma b + (1 - \gamma)h + \beta t
\end{aligned} \tag{1}$$

Therefore $u(b) \in \gamma b + H$. If also $u(b) \in \gamma' b + H$, then $(\gamma' - \gamma)b \in H$, which implies $\gamma' = \gamma$ if $b \notin H$. Therefore γ is unique for $b \notin H$.

- Let $x = a - (1 - \gamma)^{-1}t$. Then $x \neq 0$ since $a \notin H$ and

$$\begin{aligned}
u(x) &= u(a - (1 - \gamma)^{-1}t) \\
&= u(a) - (1 - \gamma)^{-1}u(t) \\
&= \gamma a + t - (1 - \gamma)^{-1}t \\
&= \gamma[a - (1 - \gamma)^{-1}t] \\
&= \gamma x
\end{aligned}$$

Therefore x is an eigenvector of u with eigenvalue γ . Let S be the vector line through x . Then $S \subseteq E(\gamma; u)$. Conversely if $b \in E(\gamma; u)$, then $u(b) = \gamma b$, which implies $(1 - \gamma)h + \beta t = 0$ in (1), so $h = -\beta(1 - \gamma)^{-1}t$ and

$$b = \beta a + h = \beta[a - (1 - \gamma)^{-1}t] = \beta x \in S$$

Therefore $S = E(\gamma; u)$. By hypothesis $H \subseteq E(1; u)$. Conversely if $b \in E(1; u)$, then $b = u(b) \in \gamma b + H$, so $(1 - \gamma)b \in H$, so $b \in H$. Therefore $H = E(1; u)$. S is supplementary to H since $x \notin H$.

By hypothesis $u(V) \subseteq V$ for any subspace $V \subseteq H$. If $S \subseteq V$, then $V = S + V \cap H$ (Exercise 3.1.4), so clearly $u(V) \subseteq V$. Conversely if $u(V) \subseteq V$ and $v \in V - H$, then $v = s + h$ for some $s \in S$ with $s \neq 0$ and $h \in H$, so $u(v) = \gamma s + h$ and $v - u(v) = (1 - \gamma)s \in V$, which implies $s \in V$ and $S = \mathbf{R}s \subseteq V$. \square

References

- [1] Dieudonné, J. *Linear Algebra and Geometry*. Hermann, 1969.