Notes and exercises from *Linear Algebra and Geometry*

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Introduction

This document contains notes and exercises from [1].

Chapter III

Section 1

Exercise (4)**.** Let V, W be a pair of supplementary subspaces of E. Every subspace U containing V is the direct sum of V with $U \cap W$.

Proof. If $u \in U$, then $u = v + w$ for some $v \in V$ and $w \in W$, and $w = u - v \in U$. So $U = V + (U \cap W)$, and $V \cap U \cap W = \{0\}$. $\square$

Section 2

Exercise (1)**.** *Projections and idempotents:* If p, q are the projections for a direct sum $E = V + W$, then $p, q \in \operatorname{End}(E)$ are such that $p^2 = p$, $q^2 = q$, and $p + q = 1$. Conversely, if $p \in \operatorname{End}(E)$ is such that $p^2 = p$, then $E = p(E) + p^{-1}(0)$ is a direct sum. Moreover, if $q = 1 - p$, then $q^2 = q$, $q(E) = p^{-1}(0)$, and $q^{-1}(0) = p(E)$.

Proof. For the forward direction, we know $p, q \in \operatorname{End}(E)$ and $p + q = 1$ (3.2.2). It follows that $p^2 = p \circ (1 - q) = p - pq = p$ and $q^2 = (1 - p)^2 = 1 - p = q$.

For the converse, $p + q = 1$, so $E = p(E) + q(E)$. Also $pq = p - p^2 = 0$, so $p(E) \cap q(E) = \{0\}$ and $q(E) \subseteq p^{-1}(0)$. If $x \in p^{-1}(0)$, then $q(x) = x$, so $x \in q(E)$. Hence $q(E) = p^{-1}(0)$ and similarly $q^{-1}(0) = p(E)$. Finally $q^2 = q$ as above. $\square$

Exercise (2)**.** If W and W' are both supplementary to V in E, then W and W' are isomorphic.

Proof. If p is the projection of E onto W', then the restriction of p to W is an isomorphism from W to W'. □

Exercise (3)**.** If $E = V + W$ is a direct sum with inclusions $i : V \to E$ and $j : W \to E$, and $v : V \to F$ and $w : W \to F$ are linear maps, then there is a unique linear map $u : E \to F$ with $u \circ i = v$ and $u \circ j = w$.

Proof. If p, q are the projections on V, W respectively, then $u = v \circ p + w \circ q$. □

Exercise (11)**.** $\mathbf{GA}(E)/E \cong \mathbf{GL}(E)$.

Proof. Define $\varphi : \mathbf{GA}(E) \to \mathbf{GL}(E)$ by $\varphi(t_a \circ v) = v$. Note that φ is well-defined by (3.2.17), φ is a homomorphism by (3.2.19), and φ is clearly surjective. Also $\varphi(u) = 1$ if and only if u is a translation, so $\ker\varphi = T(E)$, the normal subgroup of translations. It follows that $\mathbf{GA}(E)/T(E) \cong \mathbf{GL}(E)$. Finally, the mapping $a \mapsto t_a$ is an isomorphism $E \cong T(E)$ from the additive group E. □

Exercise (13)**.** If $u : E \to F$ is affine and L is a variety in F, then $u^{-1}(L)$ is empty or a variety in E.

Proof. If $a \in u^{-1}(L)$ and L_0 is the direction of L, then $L = u(a) + L_0$ and hence $u^{-1}(L) = a + u^{-1}(L_0)$. □

Section 3

Exercise (3)**.** A necessary and sufficient condition for a nonempty subset V of a vector space to be a variety is that for all pairs x, y of distinct points of V, the line D_{xy} is contained in V.

Proof. The condition is necessary by (3.3.2).

If the condition holds, choose $v \in V$ and let $V_0 = -v + V$. We claim V_0 is a subspace, from which it follows that $V = v + V_0$ is a variety. First, $0 = -v + v \in V_0$. If $x \in V_0$ and $x \neq 0$, then $v + x \in V$ and $v + x \neq v$, so $D_{v,v+x} = \{v + \xi x \mid \xi \in \mathbf{R}\} \subseteq V$. It follows that $\xi x \in V_0$ for all $\xi \in \mathbf{R}$. If also $y \in V_0$ and $y \neq x$, then $D_{v+x,v+y} \subseteq V$, so in particular $v + 2^{-1}(x + y) \in V$ and $2^{-1}(x + y) \in V_0$. By the previous result, it then follows that $x + y \in V_0$. Therefore V_0 is a subspace as claimed. □

Exercise (4)**.** *Translations and homothetic maps:*

- A necessary and sufficient condition for an affine map to be a translation or a homothetic map is that its associated linear map be homothetic.
- A necessary and sufficient condition for an affine map to preserve the direction of lines is that it be a translation or a bijective homothetic map.
- If u_1, u_2 are translations or homothetic maps, then so is $u_1 \circ u_2$.
- If u_1, u_2 and $u_1 \circ u_2$ are homothetic maps with ratios not equal to 1, their centers are collinear. If instead $u_1 \circ u_2$ is a translation, then it is either the identity or a translation in the direction of the line through the centers of u_1 and u_2.
- If $v \in \mathbf{GA}(E)$, then $v \circ h_{a,\lambda} \circ v^{-1} = h_{v(a),\lambda}$.
- The subset $H(E)$ of translations and homothetic maps in $\mathbf{GA}(E)$ forms a subgroup, and $H(E)/E \cong \mathbf{R}^*$.

Proof.

- This follows from the equations $t_a = t_a \circ h_1$ and $h_{a,\lambda} = t_{(1-\lambda)a} \circ h_\lambda$ and $t_a \circ h_\lambda = h_{(1-\lambda)^{-1}a,\lambda}$ ($\lambda \neq 1$).
- The condition is sufficient because such a map has the form $t_a \circ h_\lambda$ with $\lambda \neq 0$, which clearly preserves the direction of lines. Conversely, suppose $u = t_a \circ v$ preserves the direction of lines. If $x \neq 0$, let D be the vector line through x. Then $v(D) = D$, so $v(x) = \lambda x$ for some $\lambda \in \mathbf{R}$ with $\lambda \neq 0$, and in fact $v(y) = \lambda y$ for all $y \in D$. We claim $v = h_\lambda$, from which the result follows. If $y \notin D$, then by considering the vector line D' through y we have $v(y) = \mu y$ for some $\mu \in \mathbf{R}$. Now $v(D_{xy}) = D_{v(x)v(y)} = D_{\lambda x, \mu y}$, and since v preserves direction there is $\xi \in \mathbf{R}$ with $\mu y - \lambda x = \xi(y - x)$, or $(\mu - \xi)y = (\lambda - \xi)x$. Since $y \notin D$, this implies $\mu = \xi = \lambda$. Therefore $v = h_\lambda$ as claimed.
- If $u_1 = t_a \circ h_\lambda$ and $u_2 = t_b \circ h_\mu$, then by (3.2.19.1), $u_1 \circ u_2 = t_{a+\lambda b} \circ h_{\lambda\mu}$.
- Write $u_1 = h_{a,\lambda}$, $u_2 = h_{b,\mu}$, and $u_1 \circ u_2 = h_{c,\nu}$ with $\lambda, \mu, \nu \neq 1$. Then for all x,

  $$(1-\nu)c + \nu x = (1-\lambda)a + \lambda(1-\mu)b + \lambda\mu x$$

  Taking $x = 0$ and $x \neq 0$ (we assume such exist!) yields $\nu = \lambda\mu$ and

  $$c = (1-\lambda\mu)^{-1}[(1-\lambda)a + \lambda(1-\mu)b] = (1-\lambda\mu)^{-1}(\lambda-1)(b-a) + b$$

  so that a, b, c are collinear. If instead $u_1 \circ u_2 = t_c$, then $\lambda\mu = 1$ and $c = (\lambda-1)(b-a)$, from which the second result follows.

- Write $v = t_b \circ w$, where $w \in \mathbf{GL}(E)$, so that $v^{-1} = w^{-1} \circ t_{-b}$. By repeated application of (3.2.19.1),

$$\begin{aligned}
v \circ h_{a,\lambda} \circ v^{-1} &= t_b \circ w \circ t_{(1-\lambda)a} \circ h_\lambda \circ w^{-1} \circ t_{-b} \\
&= t_{v((1-\lambda)a)} \circ w \circ h_\lambda \circ w^{-1} \circ t_{-b} \\
&= t_{b+(1-\lambda)w(a)} \circ h_\lambda \circ t_{-b} \\
&= t_{b+(1-\lambda)w(a)-\lambda b} \circ h_\lambda \\
&= t_{(1-\lambda)v(a)} \circ h_\lambda \\
&= h_{v(a),\lambda}
\end{aligned}$$

- First, $1 \in H(E)$. If $u_1, u_2 \in H(E)$, then $u_1 \circ u_2 \in H(E)$ by a previous item. If $u_1 = t_a \circ h_\lambda$ with $\lambda \neq 0$, then $u_1^{-1} = t_{-\lambda^{-1}a} \circ h_{\lambda^{-1}} \in H(E)$. Therefore $H(E)$ is a subgroup of $\mathbf{GA}(E)$. Define $\varphi : H(E) \to \mathbf{R}^*$ by $\varphi(t_a \circ h_\lambda) = \lambda$. Note that φ is well-defined (since $E \neq \{0\}$!), φ is a homomorphism by a previous item, φ is surjective, and $\ker \varphi = T(E) \cong E$. It follows that $H(E)/E \cong \mathbf{R}^*$. □

Exercise (5)**.** A variety not parallel to a hyperplane meets the hyperplane. If H is a vector hyperplane in E and V is a subspace of E not contained in H, then $V \cap H$ is a vector hyperplane in V.

Proof. Any vector in the direction of the variety which is not in the direction of the hyperplane determines a line in the variety which meets the hyperplane (3.3.8).

If D is the vector line through a vector in $V - H$, then D is supplementary to H in E, so D is supplementary to $V \cap H$ in V (Exercise 3.1.4). □

Exercise (6)**.** *Dilations and transvections:* Let H be a vector hyperplane in E and suppose $u \in \mathrm{End}(E)$ fixes every element of H.

- There is $\gamma \in \mathbf{R}$ unique such that $u(a) \in \gamma a + H$ for all $a \in E - H$.
- If $\gamma \neq 1$, then γ is an eigenvalue of u and $E(\gamma; u)$ is a line S supplementary to $E(1; u) = H$. A subspace V satisfies $u(V) \subseteq V$ if and only if $S \subseteq V$ or $V \subseteq H$. In particular, a vector line D satisfies $u(D) \subseteq D$ if and only if $D = S$ or $D \subseteq H$.
- If $\gamma = 1$, and $g(x) = 0$ is an equation of H, there is $c \in H$ unique such that $u(x) = x + g(x)c$ for all $x \in E$. u is bijective. If $u \neq 1$ (so $c \neq 0$), then the line $T = D_{0c}$ is independent of g. The scalar 1 is the only eigenvalue of u if $H \neq \{0\}$, and $E(1; u) = H$ if $u \neq 1$. If $u \neq 1$, a subspace V satisfies

$u(V) \subseteq V$ if and only if $T \subseteq V$ or $V \subseteq H$; in particular, a vector line D satisfies $u(D) = D$ if and only if $D \subseteq H$.

- The set $\Gamma(E, H)$ of automorphisms of E leaving the hyperplane H fixed pointwise is a subgroup of $\mathbf{GL}(E)$. The subset $\Theta(E, H)$ of transvections is a normal abelian subgroup of $\Gamma(E, H)$ isomorphic to H. $\Gamma(E, H)/H \cong \mathbf{R}^*$.

Proof.

- If $a \in E - H$, then $E = \mathbf{R}a + H$, so $u(a) = \gamma a + t$ for some $\gamma \in \mathbf{R}$ and $t \in H$. If $b \in E$, then $b = \beta a + h$ for some $\beta \in \mathbf{R}$ and $h \in H$, so

$$\begin{aligned} u(b) &= u(\beta a + h) \\ &= \beta u(a) + u(h) \\ &= \beta(\gamma a + t) + h \\ &= \gamma(\beta a + h) + (1 - \gamma)h + \beta t \\ &= \gamma b + (1 - \gamma)h + \beta t \end{aligned} \tag{1}$$

  Therefore $u(b) \in \gamma b + H$. If also $u(b) \in \gamma' b + H$, then $(\gamma' - \gamma)b \in H$, which implies $\gamma' = \gamma$ if $b \notin H$. Therefore γ is unique for $b \notin H$.

- Let $x = a - (1 - \gamma)^{-1} t$. Then $x \neq 0$ since $a \notin H$ and

$$\begin{aligned} u(x) &= u(a - (1 - \gamma)^{-1} t) \\ &= u(a) - (1 - \gamma)^{-1} u(t) \\ &= \gamma a + t - (1 - \gamma)^{-1} t \\ &= \gamma[a - (1 - \gamma)^{-1} t] \\ &= \gamma x \end{aligned}$$

  Therefore x is an eigenvector of u with eigenvalue γ. Let S be the vector line through x. Then $S \subseteq E(\gamma; u)$. Conversely if $b \in E(\gamma; u)$, then $u(b) = \gamma b$, which implies $(1 - \gamma)h + \beta t = 0$ in (1), so $h = -\beta(1 - \gamma)^{-1} t$ and

$$b = \beta a + h = \beta[a - (1 - \gamma)^{-1} t] = \beta x \in S$$

  Therefore $S = E(\gamma; u)$. By hypothesis $H \subseteq E(1; u)$. Conversely if $b \in E(1; u)$, then $b = u(b) \in \gamma b + H$, so $(1 - \gamma)b \in H$, so $b \in H$. Therefore $H = E(1; u)$. S is supplementary to H since $x \notin H$.

  By hypothesis $u(V) \subseteq V$ for any subspace $V \subseteq H$. If $S \subseteq V$, then $V = S + V \cap H$ (Exercise 3.1.4), so clearly $u(V) \subseteq V$. Conversely if $u(V) \subseteq V$ and $v \in$

$V - H$, then $v = s + h$ for some $s \in S$ with $s \neq 0$ and $h \in H$, so $u(v) = \gamma s + h$ and $v - u(v) = (1-\gamma)s \in V$, which implies $s \in V$ and $S = \mathbf{R}s \subseteq V$.

- Fix $e \in E$ with $g(e) = 1$ and let $c = u(e) - e$. Since $u(e) \in e + H$, $g(c) = 0$ and $c \in H$. Now $u(x) = x + g(x)c$ holds for $x = e$, and for $x \in H$, so by linearity it holds for all $x \in \mathbf{R}e + H = E$. Note c is unique since if $u(e) = e + g(e)c'$, then $c' = u(e) - e = c$.

  The map $x \mapsto x - g(x)c$ is clearly the inverse of u, so u is bijective.

  If $h(x) = 0$ is another equation of H, then by the above there exists $c' \in H$ such that $u(x) = x + h(x)c'$ for all $x \in E$. But $h = \lambda g$ for some $\lambda \neq 0$ (3.3.6), so $u(x) = x + g(x)(\lambda c')$ for all $x \in E$ and $c = \lambda c'$ by uniqueness of c. If $u \neq 1$, then $T = D_{0c} = D_{0c'}$ is independent of g.

  If $u(x) = x + g(x)c = \lambda x$, then $(\lambda - 1)x \in H$. If $\lambda \neq 1$, then $x \in H$, so actually $(\lambda - 1)x = 0$ and $x = 0$. If $\lambda = 1$, then $g(x)c = 0$, so either $g(x) = 0$ and $x \in H$, or $c = 0$ and $u = 1$.

  As above, $u(V) \subseteq V$ if $V \subseteq H$ or $T \subseteq V$. Conversely if $u(V) \subseteq V$ and $x \in V - H$, then $g(x) \neq 0$ and $g(x)c = u(x) - x \in V$, so $c \in V$ and $T = \mathbf{R}c \subseteq V$. □

- $\Gamma(E,H)$ is obviously a subgroup of $\mathbf{GL}(E)$. Define $\varphi : \Gamma(E,H) \to \mathbf{R}^*$ by $\varphi(u) = \gamma$. Then φ is a well-defined, surjective homomorphism and $\ker\varphi = \Theta(E,H)$. It follows that $\Theta(E,H)$ is a normal subgroup of $\Gamma(E,H)$ and that $\Gamma(E,H)/\Theta(E,H) \cong \mathbf{R}^*$. Finally, the mapping $c \mapsto (x \mapsto x + g(x)c)$ is an isomorphism $H \cong \Theta(E,H)$, so in particular $\Theta(E,H)$ is abelian.

Chapter IV

Section 1

Exercise (1)**.** If $t \neq 1$ is a transvection in E, a basis for E may be chosen relative to which

$$M(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Conversely, matrices of the form

$$B_{12}(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad B_{21}(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

are matrices of transvactions relative to some basis $\{a_1, a_2\}$. These transvections have the lines D_{0a_1} and D_{0a_2} respectively.

Proof. Write $t(x) = x + g(x)c$, where $g \in E^*$ with $g \neq 0$, and $c \neq 0$ with $g(c) = 0$ (Exercise 3.3.6). Let $a_1 = c$ and choose a_2 such that $g(a_2) = 1$. Then $\{a_1, a_2\}$ is the desired basis for E.

Conversely, if t is the transformation of $B_{12}(\lambda)$ relative to $\{a_1, a_2\}$, then

$$\begin{aligned} t(x) &= t(\xi_1 a_1 + \xi_2 a_2) \\ &= \xi_1 t(a_1) + \xi_2 t(a_2) \\ &= \xi_1 a_1 + \xi_2(\lambda a_1 + a_2) \\ &= \xi_1 a_1 + \xi_2 a_2 + \lambda \xi_2 a_1 \\ &= x + g(x) a_1 \end{aligned}$$

where $g = \lambda a_2^* \in E^*$. Therefore t is a transvection in the line D_{0a_1}. A similar argument applies to $B_{21}(\lambda)$. □

Exercise (3)**.** If $u \in \operatorname{End}(E)$ and $\operatorname{rank}(u) = 1$, there is a basis of E relative to which

$$M(u) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} \quad (\delta \neq 0) \qquad \text{or} \qquad M(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The second case occurs if and only if u is nilpotent, in which case $u^2 = 0$.

Proof. Let $N = u^{-1}(0)$ and $R = u(E)$. Then N and R are vector lines in E (4.1.7). If $N \cap R = \{0\}$, then $E = N + R$ is a direct sum. Choose $a_1 \in R$ and $a_2 \in N$ with $a_1, a_2 \neq 0$. Then $u(a_1) = \delta a_1$ with $\delta \neq 0$ since $u(a_1) \in R$ and $a_1 \notin N$, and also $u(a_2) = 0$. It follows that $\{a_1, a_2\}$ is a basis of E relative to which

$$M(u) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}$$

If $N \cap R \neq \{0\}$, then $N = R$ (3.3.1). Choose $a_1 \in N$, $a_1 \neq 0$ and a_2 with $u(a_2) = a_1$. Then $\{a_1, a_2\}$ is a basis of E (since $a_2 \notin N$) relative to which

$$M(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If $M(u)$ has this form, then $M(u^2) = M(u)^2 = 0$, so $u^2 = 0$ and u is nilpotent. Conversely if u is nilpotent, there is $k > 1$ least such that $u^k = 0$. Then $u^{k-1}(E) \neq 0$ but $u^{k-1}(E) \subseteq N$, so $u^{k-1}(E) = N$. If $k > 2$, then $u^{k-1}(E)$ is a proper subspace of $u^{k-2}(E)$, lest $u^{k-2}(E) = N$ and $u^{k-1} = 0$. But then $u^{k-2}(E) = E$, impossible since $\operatorname{rank}(u) = 1$. It follows that $u^2 = 0$, so $N = R$ and $M(u)$ has this form. □

Exercise (7)**.** Let $u : E \to E$ be an injective function such that E is the smallest variety containing $u(E)$, and $u(a), u(b), u(c)$ are collinear whenever $a, b, c \in E$ are collinear.

- If $a, b, c \in E$ are not collinear, then $u(a), u(b), u(c)$ are not collinear.
- For every line D, there is a unique line D' such that $u(D) \subseteq D'$. If u is bijective, then $u(D) = D'$; moreover, if D_1, D_2 are parallel (resp. distinct, not parallel), then so are $u(D_1), u(D_2)$.
- If u is bijective, there is $v \in \mathbf{GA}(E)$ such that $u_1 = v \circ u$ fixes the origin and basis vectors $a_1, a_2 \in E$.
- Given the points $\xi a_1, \eta a_1$, it is possible to construct $(\xi + \eta) a_1$ and $\xi\eta a_1$ by intersecting lines with direction vectors derived from a_1, a_2.

Proof.

- Suppose towards a contradiction that $u(a), u(b), u(c)$ are on the line Δ. Let $x \in E$. If $x \in D_{ab} \cup D_{ac} \cup D_{bc}$, then $u(x) \in \Delta$. Otherwise if, say, x and c are on opposite sides of D_{ab}, then D_{xc} and D_{ab} intersect at a unique point y by (3.3.9) and (4.1.6), $u(y) \in \Delta$ since $y \in D_{ab}$, and $u(x) \in \Delta$ since $x \in D_{yc}$. Similarly $u(x) \in \Delta$ if x and b are on opposite sides of D_{ac}, or x and a are on opposite sides of D_{bc}. Finally, if none of these cases hold, then D_{xc} cannot be parallel to D_{ab}, because if x is in the direction of $a-b$ from c (that is, if $x = c + \xi(a - b)$ for $\xi > 0$) then x and b are on opposite sides of D_{ac}, and if x is in the direction of $b - a$ from c then x and a are on opposite sides of D_{bc}. Therefore D_{xc} and D_{ab} intersect at a unique point y by (4.1.6) and $u(x) \in \Delta$ as above. Since x was arbitrary, this means $u(E) \subseteq \Delta$, contradicting the hypothesis about $u(E)$.
- By hypothesis, $u(D_{ab}) \subseteq D_{u(a)u(b)}$. If u is bijective and $x \in D_{u(a)u(b)}$, then $x = u(c)$ for some $c \in D_{ab}$ by the previous item, so $D_{u(a)u(b)} \subseteq u(D_{ab})$. If D_1, D_2 are distinct and parallel, then $D_1 \cap D_2 = \emptyset$, so $u(D_1) \cap u(D_2) = \emptyset$ by injectivity of u, so $u(D_1), u(D_2)$ are distinct and parallel. If D_1, D_2 are not parallel, they intersect at a unique point a. If $b \in D_1$ and $c \in D_2$ with $b, c \neq a$, then a, b, c are not collinear, so $u(a), u(b), u(c)$ are not collinear by the previous item and $u(D_1), u(D_2)$ are not parallel.
- Let a_1, a_2 be basis vectors of E. Then $0, a_1, a_2$ are not collinear (4.1.1), so $u(0), u(a_1), u(a_2)$ are not collinear by a previous item, so $a_1' = u(a_1) - u(0)$ and $a_2' = u(a_2) - u(0)$ are basis vectors of E. Let $w \in \mathbf{GL}(E)$ map $a_1' \mapsto a_1$

and $a_2' \mapsto a_2$ (4.1.10) and let $v = w \circ t_{-u(0)} \in \mathbf{GA}(E)$ (3.2.19). Then $u_1 = v \circ u$ is as desired.

- If L_1 is the line through ηa_1 with direction vector $a_2 - a_1$, then $L_1 \cap D_{0a_2} = \{\eta a_2\}$, so ηa_2 is constructible. If L_2 is the line through ξa_1 with direction vector a_2, and L_3 is the line through ηa_2 with direction vector a_1, then $L_2 \cap L_3 = \{\xi a_1 + \eta a_2\}$, so $\xi a_1 + \eta a_2$ is constructible. If L_4 is the line through $\xi a_1 + \eta a_2$ with direction vector $a_2 - a_1$, then $L_4 \cap D_{0a_1} = \{(\xi + \eta) a_1\}$, so $(\xi + \eta) a_1$ is constructible. If L_5 is the line through ηa_2 with direction vector $a_2 - \xi a_1$, then $L_5 \cap D_{0a_1} = \{\xi\eta a_1\}$, so $\xi\eta a_1$ is constructible.

□

Exercise (8)**.** We have

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1\beta_1 & \alpha_1\beta_2 \\ \alpha_2\beta_1 & \alpha_2\beta_2 \end{pmatrix}$$

This can be interpreted as the composite of a linear form $E \to \mathbf{R}$ and a linear map $\mathbf{R} \to E$; it maps the plane onto the origin or onto a vector line and hence is of rank 0 or 1.

Exercise (9)**.** If $f : E \to E$ is a function which commutes with all automorphisms in $\mathbf{GL}(E)$, then $f = h_\lambda$ for some $\lambda \in \mathbf{R}$.

Proof. First, since f commutes with $h_2 \in \mathbf{GL}(E)$,

$$f(0) = f(2 \cdot 0) = 2 \cdot f(0) = f(0) + f(0)$$

and it follows that $f(0) = 0$. Now $f(\alpha x) = \alpha f(x)$ for all $x \in E$ and $\alpha \in \mathbf{R}$, using the previous result for $\alpha = 0$ and commutativity of f with $h_\alpha \in \mathbf{GL}(E)$ for $\alpha \neq 0$.

Fix $x \neq 0$ and let $u \neq 1$ be a transvection in the line $D = D_{0x}$. Then $u(f(x)) = f(u(x)) = f(x)$, so $f(x) \in D$ and hence $f(x) = \lambda x$ for some $\lambda \in \mathbf{R}$. It follows that $f(y) = \lambda y$ for all $y \in D$. Fix $y \notin D$. We may assume $u(y) = x + y$. By reasoning as above, $f(y) = \lambda_y y$ and $f(x+y) = \lambda_{x+y}(x+y)$ for some $\lambda_y, \lambda_{x+y} \in \mathbf{R}$. But also

$$f(x+y) = f(u(y)) = u(f(y)) = u(\lambda_y y) = \lambda_y u(y) = \lambda_y (x+y)$$

so $\lambda_y = \lambda_{x+y}$ (since $x + y \neq 0$). Now considering the transvection u' in the line $D' = D_{0y}$ with $u'(x) = x + y$, it follows that $\lambda = \lambda_{x+y} = \lambda_y$. Therefore $f(y) = \lambda y$ for all $y \in E$, so $f = h_\lambda$. □

References

[1] Dieudonné, J. *Linear Algebra and Geometry.* Hermann, 1969.