Notes and exercises from *Linear Algebra and Geometry*

John Peloquin

Introduction

This document contains notes and exercises from [1].

Chapter III

Section 1

Remark. If V, W are supplementary subspaces of E and c is the unique point in common to the varieties $a + V$ and $b + W$ (3.1.15), then $a + V = c + V$ and $b + W = c + W$ (3.1.12), so

$$-c + (a + V) = V \qquad \text{and} \qquad -c + (b + W) = W$$

In other words, taking the origin c (3.2.21), the varieties become their direction subspaces.

Exercise (4)**.** Let V, W be a pair of supplementary subspaces of E. Every subspace U containing V is the direct sum of V with $U \cap W$.

Proof. If $u \in U$, then $u = v + w$ for some $v \in V$ and $w \in W$, and $w = u - v \in U$. So $U = V + (U \cap W)$, and $V \cap U \cap W = \{0\}$. □

Section 2

Exercise (1)**.** *Projections and idempotents:* If p, q are the projections for a direct sum $E = V + W$, then $p, q \in \operatorname{End}(E)$ are such that $p^2 = p$, $q^2 = q$, and $p + q = 1$. Conversely, if $p \in \operatorname{End}(E)$ is such that $p^2 = p$, then $E = p(E) + p^{-1}(0)$ is a direct sum. Moreover, if $q = 1 - p$, then $q^2 = q$, $q(E) = p^{-1}(0)$, and $q^{-1}(0) = p(E)$.

Proof. For the forward direction, we know $p, q \in \operatorname{End}(E)$ and $p + q = 1$ (3.2.2). It follows that $p^2 = p \circ (1 - q) = p - pq = p$ and $q^2 = (1 - p)^2 = 1 - p = q$.

For the converse, $p + q = 1$, so $E = p(E) + q(E)$. Also $pq = p - p^2 = 0$, so $p(E) \cap q(E) = \{0\}$ and $q(E) \subseteq p^{-1}(0)$. If $x \in p^{-1}(0)$, then $q(x) = x$, so $x \in q(E)$. Hence $q(E) = p^{-1}(0)$ and similarly $q^{-1}(0) = p(E)$. Finally $q^2 = q$ as above. □

Exercise (2)**.** If W and W' are both supplementary to V in E, then W and W' are isomorphic.

Proof. If p is the projection of E onto W', then the restriction of p to W is an isomorphism from W to W'. □

Exercise (3)**.** If $E = V + W$ is a direct sum with inclusions $i : V \to E$ and $j : W \to E$, and $v : V \to F$ and $w : W \to F$ are linear maps, then there is a unique linear map $u : E \to F$ with $u \circ i = v$ and $u \circ j = w$.

Proof. If p, q are the projections on V, W respectively, then $u = v \circ p + w \circ q$. □

Exercise (11)**.** $\mathbf{GA}(E)/E \cong \mathbf{GL}(E)$.

Proof. Define $\varphi : \mathbf{GA}(E) \to \mathbf{GL}(E)$ by $\varphi(t_a \circ v) = v$. Note that φ is well-defined by (3.2.17), φ is a homomorphism by (3.2.19), and φ is clearly surjective. Also $\varphi(u) = 1$ if and only if u is a translation, so $\ker \varphi = T(E)$, the normal subgroup of translations. It follows that $\mathbf{GA}(E)/T(E) \cong \mathbf{GL}(E)$. Finally, the mapping $a \mapsto t_a$ is an isomorphism $E \cong T(E)$ from the additive group E. □

Exercise (13)**.** If $u : E \to F$ is affine and L is a variety in F, then $u^{-1}(L)$ is empty or a variety in E.

Proof. If $a \in u^{-1}(L)$ and L_0 is the direction of L, then $L = u(a) + L_0$ and hence $u^{-1}(L) = a + u^{-1}(L_0)$. □

Section 3

Remark. If $a, b \in E$, then the segment ab (3.3.4) consists of all points $x = \alpha a + \beta b$ where $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$—the *convex combinations* of a and b.

Exercise (3)**.** A necessary and sufficient condition for a nonempty subset V of a vector space to be a variety is that for all pairs x, y of distinct points of V, the line D_{xy} is contained in V.

Proof. The condition is necessary by (3.3.2).

If the condition holds, choose $v \in V$ and let $V_0 = -v + V$. We claim V_0 is a subspace, from which it follows that $V = v + V_0$ is a variety. First, $0 = -v + v \in V_0$. If $x \in V_0$ and $x \neq 0$, then $v + x \in V$ and $v + x \neq v$, so $D_{v,v+x} = \{ v + \xi x \mid \xi \in \mathbf{R} \} \subseteq V$. It follows that $\xi x \in V_0$ for all $\xi \in \mathbf{R}$. If also $y \in V_0$ and $y \neq x$, then $D_{v+x,v+y} \subseteq V$, so in particular $v + 2^{-1}(x+y) \in V$ and $2^{-1}(x+y) \in V_0$. By the previous result, it then follows that $x + y \in V_0$. Therefore V_0 is a subspace as claimed. □

Exercise (4)**.** *Translations and homothetic maps:*

- A necessary and sufficient condition for an affine map to be a translation or a homothetic map is that its associated linear map be homothetic.
- A necessary and sufficient condition for an affine map to preserve the direction of lines is that it be a translation or a bijective homothetic map.
- If u_1, u_2 are translations or homothetic maps, then so is $u_1 \circ u_2$.
- If u_1, u_2 and $u_1 \circ u_2$ are homothetic maps with ratios not equal to 1, their centers are collinear. If instead $u_1 \circ u_2$ is a translation, then it is either the identity or a translation in the direction of the line through the centers of u_1 and u_2.
- If $v \in \mathbf{GA}(E)$, then $v \circ h_{a,\lambda} \circ v^{-1} = h_{v(a),\lambda}$.
- The subset $H(E)$ of translations and homothetic maps in $\mathbf{GA}(E)$ forms a subgroup, and $H(E)/E \cong \mathbf{R}^*$.

Proof.

- This follows from the equations $t_a = t_a \circ h_1$ and $h_{a,\lambda} = t_{(1-\lambda)a} \circ h_\lambda$ and $t_a \circ h_\lambda = h_{(1-\lambda)^{-1}a,\lambda}$ ($\lambda \neq 1$).
- The condition is sufficient because such a map has the form $t_a \circ h_\lambda$ with $\lambda \neq 0$, which clearly preserves the direction of lines. Conversely, suppose $u = t_a \circ v$ preserves the direction of lines. If $x \neq 0$, let D be the vector line through x. Then $v(D) = D$, so $v(x) = \lambda x$ for some $\lambda \in \mathbf{R}$ with $\lambda \neq 0$, and in fact $v(y) = \lambda y$ for all $y \in D$. We claim $v = h_\lambda$, from which the result follows. If $y \notin D$, then by considering the vector line D' through y we have $v(y) = \mu y$ for some $\mu \in \mathbf{R}$. Now $v(D_{xy}) = D_{v(x)v(y)} = D_{\lambda x,\mu y}$, and since v preserves direction there is $\xi \in \mathbf{R}$ with $\mu y - \lambda x = \xi(y - x)$, or $(\mu - \xi)y = (\lambda - \xi)x$. Since $y \notin D$, this implies $\mu = \xi = \lambda$. Therefore $v = h_\lambda$ as claimed.

- If $u_1 = t_a \circ h_\lambda$ and $u_2 = t_b \circ h_\mu$, then by (3.2.19.1), $u_1 \circ u_2 = t_{a+\lambda b} \circ h_{\lambda\mu}$.
- Write $u_1 = h_{a,\lambda}$, $u_2 = h_{b,\mu}$, and $u_1 \circ u_2 = h_{c,\nu}$ with $\lambda, \mu, \nu \neq 1$. Then for all x,

  $$(1-\nu)c + \nu x = (1-\lambda)a + \lambda(1-\mu)b + \lambda\mu x$$

  Taking $x = 0$ and $x \neq 0$ (we assume such exist!) yields $\nu = \lambda\mu$ and

  $$c = (1-\lambda\mu)^{-1}[(1-\lambda)a + \lambda(1-\mu)b] = (1-\lambda\mu)^{-1}(\lambda-1)(b-a) + b$$

  so that a, b, c are collinear. If instead $u_1 \circ u_2 = t_c$, then $\lambda\mu = 1$ and $c = (\lambda-1)(b-a)$, from which the second result follows.
- Write $v = t_b \circ w$, where $w \in \mathbf{GL}(E)$, so that $v^{-1} = w^{-1} \circ t_{-b}$. By repeated application of (3.2.19.1),

  $$\begin{aligned} v \circ h_{a,\lambda} \circ v^{-1} &= t_b \circ w \circ t_{(1-\lambda)a} \circ h_\lambda \circ w^{-1} \circ t_{-b} \\ &= t_{v((1-\lambda)a)} \circ w \circ h_\lambda \circ w^{-1} \circ t_{-b} \\ &= t_{b+(1-\lambda)w(a)} \circ h_\lambda \circ t_{-b} \\ &= t_{b+(1-\lambda)w(a)-\lambda b} \circ h_\lambda \\ &= t_{(1-\lambda)v(a)} \circ h_\lambda \\ &= h_{v(a),\lambda} \end{aligned}$$

- First, $1 \in H(E)$. If $u_1, u_2 \in H(E)$, then $u_1 \circ u_2 \in H(E)$ by a previous item. If $u_1 = t_a \circ h_\lambda$ with $\lambda \neq 0$, then $u_1^{-1} = t_{-\lambda^{-1}a} \circ h_{\lambda^{-1}} \in H(E)$. Therefore $H(E)$ is a subgroup of $\mathbf{GA}(E)$. Define $\varphi : H(E) \to \mathbf{R}^*$ by $\varphi(t_a \circ h_\lambda) = \lambda$. Note that φ is well-defined (since $E \neq \{0\}$!), φ is a homomorphism by a previous item, φ is surjective, and $\ker\varphi = T(E) \cong E$. It follows that $H(E)/E \cong \mathbf{R}^*$. □

Exercise (5)**.** A variety not parallel to a hyperplane meets the hyperplane. If H is a vector hyperplane in E and V is a subspace of E not contained in H, then $V \cap H$ is a vector hyperplane in V.

Proof. Any vector in the direction of the variety which is not in the direction of the hyperplane determines a line in the variety which meets the hyperplane (3.3.8).

If D is the vector line through a vector in $V - H$, then D is supplementary to H in E, so D is supplementary to $V \cap H$ in V (Exercise 3.1.4). □

Exercise (6)**.** *Dilations and transvections:* Let H be a vector hyperplane in E and suppose $u \in \mathrm{End}(E)$ fixes every element of H.

- There is $\gamma \in \mathbf{R}$ unique such that $u(a) \in \gamma a + H$ for all $a \in E - H$.
- If $\gamma \neq 1$, then γ is an eigenvalue of u and $E(\gamma; u)$ is a line S supplementary to $E(1; u) = H$. A subspace V satisfies $u(V) \subseteq V$ if and only if $S \subseteq V$ or $V \subseteq H$. In particular, a vector line D satisfies $u(D) \subseteq D$ if and only if $D = S$ or $D \subseteq H$.
- If $\gamma = 1$, and $g(x) = 0$ is an equation of H, there is $c \in H$ unique such that $u(x) = x + g(x)c$ for all $x \in E$. u is bijective. If $u \neq 1$ (so $c \neq 0$), then the line $T = D_{0c}$ is independent of g. The scalar 1 is the only eigenvalue of u if $H \neq \{0\}$, and $E(1; u) = H$ if $u \neq 1$. If $u \neq 1$, a subspace V satisfies $u(V) \subseteq V$ if and only if $T \subseteq V$ or $V \subseteq H$; in particular, a vector line D satisfies $u(D) = D$ if and only if $D \subseteq H$.
- The set $\Gamma(E, H)$ of automorphisms of E leaving the hyperplane H fixed pointwise is a subgroup of $\mathbf{GL}(E)$. The subset $\Theta(E, H)$ of transvections is a normal abelian subgroup of $\Gamma(E, H)$ isomorphic to H. $\Gamma(E, H)/H \cong \mathbf{R}^*$.

Proof.

- If $a \in E - H$, then $E = \mathbf{R}a + H$, so $u(a) = \gamma a + t$ for some $\gamma \in \mathbf{R}$ and $t \in H$. If $b \in E$, then $b = \beta a + h$ for some $\beta \in \mathbf{R}$ and $h \in H$, so

$$
\begin{aligned}
u(b) &= u(\beta a + h) \\
&= \beta u(a) + u(h) \\
&= \beta(\gamma a + t) + h \\
&= \gamma(\beta a + h) + (1-\gamma)h + \beta t \\
&= \gamma b + (1-\gamma)h + \beta t
\end{aligned}
\tag{1}
$$

  Therefore $u(b) \in \gamma b + H$. If also $u(b) \in \gamma' b + H$, then $(\gamma' - \gamma)b \in H$, which implies $\gamma' = \gamma$ if $b \notin H$. Therefore γ is unique for $b \notin H$.
- Let $x = a - (1-\gamma)^{-1} t$. Then $x \neq 0$ since $a \notin H$ and

$$
\begin{aligned}
u(x) &= u(a - (1-\gamma)^{-1} t) \\
&= u(a) - (1-\gamma)^{-1} u(t) \\
&= \gamma a + t - (1-\gamma)^{-1} t \\
&= \gamma[a - (1-\gamma)^{-1} t] \\
&= \gamma x
\end{aligned}
$$

Therefore x is an eigenvector of u with eigenvalue γ. Let S be the vector line through x. Then $S \subseteq E(\gamma; u)$. Conversely if $b \in E(\gamma; u)$, then $u(b) = \gamma b$, which implies $(1-\gamma)h + \beta t = 0$ in (1), so $h = -\beta(1-\gamma)^{-1} t$ and

$$b = \beta a + h = \beta[a - (1-\gamma)^{-1} t] = \beta x \in S$$

Therefore $S = E(\gamma; u)$. By hypothesis $H \subseteq E(1; u)$. Conversely if $b \in E(1; u)$, then $b = u(b) \in \gamma b + H$, so $(1-\gamma)b \in H$, so $b \in H$. Therefore $H = E(1; u)$. S is supplementary to H since $x \notin H$.

By hypothesis $u(V) \subseteq V$ for any subspace $V \subseteq H$. If $S \subseteq V$, then $V = S + V \cap H$ (Exercise 3.1.4), so clearly $u(V) \subseteq V$. Conversely if $u(V) \subseteq V$ and $v \in V - H$, then $v = s + h$ for some $s \in S$ with $s \neq 0$ and $h \in H$, so $u(v) = \gamma s + h$ and $v - u(v) = (1-\gamma)s \in V$, which implies $s \in V$ and $S = \mathbf{R}s \subseteq V$.

- Fix $e \in E$ with $g(e) = 1$ and let $c = u(e) - e$. Since $u(e) \in e + H$, $g(c) = 0$ and $c \in H$. Now $u(x) = x + g(x)c$ holds for $x = e$, and for $x \in H$, so by linearity it holds for all $x \in \mathbf{R}e + H = E$. Note c is unique since if $u(e) = e + g(e)c'$, then $c' = u(e) - e = c$.

  The map $x \mapsto x - g(x)c$ is clearly the inverse of u, so u is bijective.

  If $h(x) = 0$ is another equation of H, then by the above there exists $c' \in H$ such that $u(x) = x + h(x)c'$ for all $x \in E$. But $h = \lambda g$ for some $\lambda \neq 0$ (3.3.6), so $u(x) = x + g(x)(\lambda c')$ for all $x \in E$ and $c = \lambda c'$ by uniqueness of c. If $u \neq 1$, then $T = D_{0c} = D_{0c'}$ is independent of g.

  If $u(x) = x + g(x)c = \lambda x$, then $(\lambda - 1)x \in H$. If $\lambda \neq 1$, then $x \in H$, so actually $(\lambda - 1)x = 0$ and $x = 0$. If $\lambda = 1$, then $g(x)c = 0$, so either $g(x) = 0$ and $x \in H$, or $c = 0$ and $u = 1$.

  As above, $u(V) \subseteq V$ if $V \subseteq H$ or $T \subseteq V$. Conversely if $u(V) \subseteq V$ and $x \in V - H$, then $g(x) \neq 0$ and $g(x)c = u(x) - x \in V$, so $c \in V$ and $T = \mathbf{R}c \subseteq V$. $\square$

- $\Gamma(E, H)$ is obviously a subgroup of $\mathbf{GL}(E)$. Define $\varphi : \Gamma(E, H) \to \mathbf{R}^*$ by $\varphi(u) = \gamma$. Then φ is a well-defined, surjective homomorphism and $\ker \varphi = \Theta(E, H)$. It follows that $\Theta(E, H)$ is a normal subgroup of $\Gamma(E, H)$ and that $\Gamma(E, H)/\Theta(E, H) \cong \mathbf{R}^*$. Finally, the mapping $c \mapsto (x \mapsto x + g(x)c)$ is an isomorphism $H \cong \Theta(E, H)$, so in particular $\Theta(E, H)$ is abelian.

Chapter IV

Section 1

Exercise (1)**.** If $t \neq 1$ is a transvection in E, a basis for E may be chosen relative to which

$$M(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Conversely, matrices of the form

$$B_{12}(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \qquad B_{21}(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

are matrices of transvactions relative to some basis $\{a_1, a_2\}$. These transvections have the lines D_{0a_1} and D_{0a_2} respectively.

Proof. Write $t(x) = x + g(x)c$, where $g \in E^*$ with $g \neq 0$, and $c \neq 0$ with $g(c) = 0$ (Exercise 3.3.6). Let $a_1 = c$ and choose a_2 such that $g(a_2) = 1$. Then $\{a_1, a_2\}$ is the desired basis for E.

Conversely, if t is the transformation of $B_{12}(\lambda)$ relative to $\{a_1, a_2\}$, then

$$\begin{aligned} t(x) &= t(\xi_1 a_1 + \xi_2 a_2) \\ &= \xi_1 t(a_1) + \xi_2 t(a_2) \\ &= \xi_1 a_1 + \xi_2(\lambda a_1 + a_2) \\ &= \xi_1 a_1 + \xi_2 a_2 + \lambda \xi_2 a_1 \\ &= x + g(x) a_1 \end{aligned}$$

where $g = \lambda a_2^* \in E^*$. Therefore t is a transvection in the line D_{0a_1}. A similar argument applies to $B_{21}(\lambda)$. $\square$

Exercise (3)**.** If $u \in \mathrm{End}(E)$ and $\mathrm{rank}(u) = 1$, there is a basis of E relative to which

$$M(u) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} \quad (\delta \neq 0) \qquad \text{or} \qquad M(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The second case occurs if and only if u is nilpotent, in which case $u^2 = 0$.

Proof. Let $N = u^{-1}(0)$ and $R = u(E)$. Then N and R are vector lines in E (4.1.7). If $N \cap R = \{0\}$, then $E = N + R$ is a direct sum. Choose $a_1 \in R$ and $a_2 \in N$ with

$a_1, a_2 \neq 0$. Then $u(a_1) = \delta a_1$ with $\delta \neq 0$ since $u(a_1) \in R$ and $a_1 \notin N$, and also $u(a_2) = 0$. It follows that $\{a_1, a_2\}$ is a basis of E relative to which

$$M(u) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}$$

If $N \cap R \neq \{0\}$, then $N = R$ (3.3.1). Choose $a_1 \in N$, $a_1 \neq 0$ and a_2 with $u(a_2) = a_1$. Then $\{a_1, a_2\}$ is a basis of E (since $a_2 \notin N$) relative to which

$$M(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If $M(u)$ has this form, then $M(u^2) = M(u)^2 = 0$, so $u^2 = 0$ and u is nilpotent. Conversely if u is nilpotent, there is $k > 1$ least such that $u^k = 0$. Then $u^{k-1}(E) \neq 0$ but $u^{k-1}(E) \subseteq N$, so $u^{k-1}(E) = N$. If $k > 2$, then $u^{k-1}(E)$ is a proper subspace of $u^{k-2}(E)$, lest $u^{k-2}(E) = N$ and $u^{k-1} = 0$. But then $u^{k-2}(E) = E$, impossible since $\operatorname{rank}(u) = 1$. It follows that $u^2 = 0$, so $N = R$ and $M(u)$ has this form. □

Exercise (7)**.** Let $u : E \to E$ be an injective function such that E is the smallest variety containing $u(E)$, and $u(a), u(b), u(c)$ are collinear whenever $a, b, c \in E$ are collinear.

- If $a, b, c \in E$ are not collinear, then $u(a), u(b), u(c)$ are not collinear.
- For every line D, there is a unique line D' such that $u(D) \subseteq D'$. If u is bijective, then $u(D) = D'$; moreover, if D_1, D_2 are parallel (resp. distinct, not parallel), then so are $u(D_1), u(D_2)$.
- If u is bijective, there is $v \in \mathbf{GA}(E)$ such that $u_1 = v \circ u$ fixes the origin and basis vectors $a_1, a_2 \in E$. u_1 maps lines to lines and preserves parallelism (resp. distinctness, non-parallelism); in particular, u_1 preserves the lines $D_{0a_1}, D_{0a_2}, D_{a_1a_2}$ and hence the directions of any lines parallel to these.
- Given the points $\xi a_1, \eta a_1$, it is possible to construct $(\xi + \eta)a_1$ and $\xi\eta a_1$ by intersecting lines with direction vectors derived from a_1, a_2.
- If φ is defined by $u_1(\xi a_1) = \varphi(\xi) a_1$, then φ is a field automorphism of $\mathbf{R}$, so $\varphi = 1_{\mathbf{R}}$.[1] It follows that $u_1 = 1_E$, so $u = v^{-1}$ is an affine map.

Proof.

[1] For this problem, we assume there are no nontrivial field automorphisms of $\mathbf{R}$.

- Suppose towards a contradiction that $u(a), u(b), u(c)$ are on the line Δ. Let $x \in E$. If $x \in D_{ab} \cup D_{ac} \cup D_{bc}$, then $u(x) \in \Delta$. Otherwise if, say, x and c are on opposite sides of D_{ab}, then D_{xc} and D_{ab} intersect at a unique point y by (3.3.9) and (4.1.6), $u(y) \in \Delta$ since $y \in D_{ab}$, and $u(x) \in \Delta$ since $x \in D_{yc}$. Similarly $u(x) \in \Delta$ if x and b are on opposite sides of D_{ac}, or x and a are on opposite sides of D_{bc}. Finally, if none of these cases hold, then D_{xc} cannot be parallel to D_{ab}, because if x is in the direction of $a-b$ from c (that is, if $x = c + \xi(a-b)$ for $\xi > 0$) then x and b are on opposite sides of D_{ac}, and if x is in the direction of $b-a$ from c then x and a are on opposite sides of D_{bc}. Therefore D_{xc} and D_{ab} intersect at a unique point y by (4.1.6) and $u(x) \in \Delta$ as above. Since x was arbitrary, this means $u(E) \subseteq \Delta$, contradicting the hypothesis about $u(E)$.
- By hypothesis, $u(D_{ab}) \subseteq D_{u(a)u(b)}$. If u is bijective and $x \in D_{u(a)u(b)}$, then $x = u(c)$ for some $c \in D_{ab}$ by the previous item, so $D_{u(a)u(b)} \subseteq u(D_{ab})$. If D_1, D_2 are distinct and parallel, then $D_1 \cap D_2 = \emptyset$, so $u(D_1) \cap u(D_2) = \emptyset$ by injectivity of u, so $u(D_1), u(D_2)$ are distinct and parallel. If D_1, D_2 are not parallel, they intersect at a unique point a. If $b \in D_1$ and $c \in D_2$ with $b, c \neq a$, then a, b, c are not collinear, so $u(a), u(b), u(c)$ are not collinear by the previous item and $u(D_1), u(D_2)$ are not parallel.
- Let a_1, a_2 be basis vectors of E. Then $0, a_1, a_2$ are not collinear (4.1.1), so $u(0), u(a_1), u(a_2)$ are not collinear by a previous item, so $a'_1 = u(a_1) - u(0)$ and $a'_2 = u(a_2) - u(0)$ are basis vectors of E. Let $w \in \mathbf{GL}(E)$ map $a'_1 \mapsto a_1$ and $a'_2 \mapsto a_2$ (4.1.10) and let $v = w \circ t_{-u(0)} \in \mathbf{GA}(E)$ (3.2.19). Then $u_1 = v \circ u$ fixes $0, a_1, a_2$. u_1 operates as claimed on lines by (3.2.17) and the previous item, and the observation that, for example,

$$u_1(D_{a_1 a_2}) = D_{u_1(a_1) u_1(a_2)} = D_{a_1 a_2}$$

- If L_1 is the line through ηa_1 with direction vector $a_2 - a_1$, then $L_1 \cap D_{0a_2} = \{\eta a_2\}$, so ηa_2 is constructible. If L_2 is the line through ξa_1 with direction vector a_2, and L_3 is the line through ηa_2 with direction vector a_1, then $L_2 \cap L_3 = \{\xi a_1 + \eta a_2\}$, so $\xi a_1 + \eta a_2$ is constructible. If L_4 is the line through $\xi a_1 + \eta a_2$ with direction vector $a_2 - a_1$, then $L_4 \cap D_{0a_1} = \{(\xi + \eta) a_1\}$, so $(\xi + \eta) a_1$ is constructible. If L_5 is the line through ηa_2 with direction vector $a_2 - \xi a_1$, then $L_5 \cap D_{0a_1} = \{\xi \eta a_1\}$, so $\xi \eta a_1$ is constructible.
- By previous items, $u_1(L_2)$ is the line passing through $u_1(\xi a_1)$ parallel to L_2 (and to D_{0a_2}), and $u_1(L_3)$ is the line passing through $u_1(\eta a_2)$ parallel to L_3

(and to D_{0a_1}). Since $u_1(\eta a_2) \in D_{0a_2}$ and $u_1(\xi a_1) \in D_{0a_1}$, it follows that $u_1(\xi a_1) + u_1(\eta a_2)$ is the intersection of $u_1(L_2)$ and $u_1(L_3)$. But we also have $u_1(\xi a_1 + \eta a_2) \in u_1(L_2 \cap L_3) = u_1(L_2) \cap u_1(L_3)$, so

$$u_1(\xi a_1 + \eta a_2) = u_1(\xi a_1) + u_1(\eta a_2)$$

Now $u_1(L_4)$ is the line passing through this point and $u_1((\xi + \eta)a_1)$ parallel to the line $u_1(L_1)$, which in turn passes through $u_1(\eta a_1)$ and $u_1(\eta a_2)$ parallel to L_1 (and to $D_{a_1 a_2}$). Therefore

$$u_1((\xi + \eta)a_1) = u_1(\xi a_1) + u_1(\eta a_1)$$

is the intersection of $u_1(L_4)$ and D_{0a_1}. It follows that $\varphi(\xi+\eta) = \varphi(\xi)+\varphi(\eta)$.

We claim $u_1(\eta a_2) = \varphi(\eta)a_2$. Indeed, $u_1(\eta a_2) = \lambda a_2$ for some λ, and also $u_1(\eta a_2) = u_1(\eta a_1) + \mu(a_2 - a_1) = \varphi(\eta)a_1 + \mu(a_2 - a_1)$ for some μ by facts about $u_1(L_1)$ above. It follows that $\lambda = \mu = \varphi(\eta)$ by linear independence of a_1, a_2, establishing the claim.

Now $u_1(L_5)$ is the line passing through $u_1(\xi\eta a_1)$ and $u_1(\eta a_2)$, and with the direction vector $u_1(\xi a_1) - a_2$. Therefore

$$u_1(\xi\eta a_1) = u_1(\eta a_2) + \varphi(\eta)[u_1(\xi a_1) - a_2] = \varphi(\eta)u_1(\xi a_1)$$

is the intersection of $u_1(L_5)$ and D_{0a_1}. It follows that $\varphi(\xi\eta) = \varphi(\xi)\varphi(\eta)$. Since $\varphi(1) = 1$, φ is a field automorphism of $\mathbf{R}$ and hence $\varphi = 1_{\mathbf{R}}$.

By the above, u_1 fixes D_{0a_1} and D_{0a_2} pointwise. If $x = \xi_1 a_1 + \xi_2 a_2$, then x is the intersection of the line through $\xi_1 a_1$ parallel to D_{0a_2} and the line through $\xi_2 a_2$ parallel to D_{0a_1}. It follows that $u_1(x)$ is the intersection of the same lines, so $u_1(x) = x$. Therefore $u_1 = 1_E$, and $u = v^{-1}$ is affine. □

Exercise (8)**.** We have

$$\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}\begin{pmatrix}\beta_1 & \beta_2\end{pmatrix} = \begin{pmatrix}\alpha_1\beta_1 & \alpha_1\beta_2\\ \alpha_2\beta_1 & \alpha_2\beta_2\end{pmatrix}$$

This can be interpreted as the composite of a linear form $E \to \mathbf{R}$ and a linear map $\mathbf{R} \to E$; it maps the plane onto the origin or onto a vector line and hence is of rank 0 or 1.

Exercise (9)**.** If $f : E \to E$ is a function which commutes with all automorphisms in $\mathbf{GL}(E)$, then $f = h_\lambda$ for some $\lambda \in \mathbf{R}$.

Proof. First, since f commutes with $h_2 \in \mathbf{GL}(E)$,

$$f(0) = f(2 \cdot 0) = 2 \cdot f(0) = f(0) + f(0)$$

and it follows that $f(0) = 0$. Now $f(\alpha x) = \alpha f(x)$ for all $x \in E$ and $\alpha \in \mathbf{R}$, using the previous result for $\alpha = 0$ and commutativity of f with $h_\alpha \in \mathbf{GL}(E)$ for $\alpha \neq 0$.

Fix $x \neq 0$ and let $u \neq 1$ be a transvection in the line $D = D_{0x}$. Then $u(f(x)) = f(u(x)) = f(x)$, so $f(x) \in D$ and hence $f(x) = \lambda x$ for some $\lambda \in \mathbf{R}$. It follows that $f(y) = \lambda y$ for all $y \in D$. Fix $y \notin D$. We may assume $u(y) = x + y$. By reasoning as above, $f(y) = \lambda_y y$ and $f(x+y) = \lambda_{x+y}(x+y)$ for some $\lambda_y, \lambda_{x+y} \in \mathbf{R}$. But also

$$f(x+y) = f(u(y)) = u(f(y)) = u(\lambda_y y) = \lambda_y u(y) = \lambda_y(x+y)$$

so $\lambda_y = \lambda_{x+y}$ (since $x + y \neq 0$). Now considering the transvection u' in the line $D' = D_{0y}$ with $u'(x) = x + y$, it follows that $\lambda = \lambda_{x+y} = \lambda_y$. Therefore $f(y) = \lambda y$ for all $y \in E$, so $f = h_\lambda$. □

Section 2

Remark. In (4.2.5), if $a \in E$, $a \neq 0$ and $f(x) = 0$ is an equation of D_{0a} with $f \in E^*$, $f \neq 0$ (3.3.6), choose $b \in E$ with $f(b) = 1$. Then $\{a, b\}$ is a basis of E. Let Ψ be the alternating bilinear form on E with $\Psi(a, b) = 1$. Then for all $x = \xi a + \eta b$,

$$\Psi(a, x) = \eta = f(x)$$

Since Ψ is essentially the determinant (4.2.9) and the determinant measures (oriented) area, this just says that x is on the vector line determined by a if and only if the parallelogram determined by x and a has zero area.

Remark. An alternating bilinear form $\Psi \neq 0$ on E captures linear independence in that $\Psi(x, y) \neq 0$ if and only if $\{x, y\}$ is independent (4.2.5). It follows that if $u \in \mathrm{End}(E)$, then $\det(u) \neq 0$ if and only if u preserves linear independence (4.2.6.1), which is true if and only if u is bijective (4.1.8). This is just (4.2.8).

Exercise (3)**.** Relative to a fixed basis of E, let u and v be defined by

$$M(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \qquad M(v) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then $uv = v$, so $\mathrm{rank}(uv) = 1$, and $vu = 0$, so $\mathrm{rank}(vu) = 0$, but $\lambda = 0$ is the only eigenvalue of uv and vu, so $\lambda^2 = 0$ is the characteristic equation of uv and vu.

Exercise (5)**.** If $u \in \operatorname{End}(E)$ has eigenvalues, there is a basis of E relative to which

$$M(u) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \qquad \text{or} \qquad M(u) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

If, relative to some basis of E,

$$M(u) = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$$

with $\alpha \neq 0$, then

$$M(u) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

relative to some basis of E, but there is no basis of E relative to which

$$M(u) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Proof. If u has two distinct eigenvalues λ, μ with eigenvectors a, b respectively, then $\{a, b\}$ is a basis of E and

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

Otherwise, u has just one eigenvalue λ. If there are two linearly independent eigenvectors a, b, then

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Note $M(u)$ has this form only if $u = h_\lambda$. If there are not two linearly independent eigenvectors, let a be an eigenvector and $\{a, x\}$ a basis of E. Then $u(x) = \alpha a + \lambda x$ with $\alpha \neq 0$ by assumption and (4.2.14), so

$$M(u, \{a, x\}) = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$$

If $b = \alpha^{-1} x$, then $\{a, b\}$ is a basis of E and

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Note $M(u)$ has this form only if $u \neq h_\lambda$. □

Chapter V

Section 1

Remark. For all $x, y \in E$,

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

Proof. Bilinearity of the inner product. □

This is called the *parallelogram law* because it shows that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

Remark. If $a, b \in E$ with $a \neq b$ and $x = \alpha a + \beta b$ with $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$ is a point on the segment ab (see remark from (3.3) above), then

$$\alpha = \frac{\|x-b\|}{\|a-b\|} \qquad \beta = \frac{\|x-a\|}{\|a-b\|}$$

In other words, α is the ratio of the lengths of the segments xb and ab, and β that of xa and ab.

Proof. For example, $x - b = \alpha a + (\beta - 1)b = \alpha(a-b)$, so $\|x-b\| = \alpha\|a-b\|$. □

Remark. In (5.1.8), $H(x) = x + H_0$ and $D(x) = x + D_0$. To see that the unique point $y \in D \cap H(x)$ (3.3.8) satisfies the properties for x, let $c \in D \cap H$ and observe that $y - c$ is the orthogonal projection of $x - c$ on D_0. Indeed, $y - c \in D_0$ since $y, c \in D$ and $(x-c) - (y-c) = x - y \in H_0$ since $y \in H(x)$. It follows (5.1.7) that $x - y$ is orthogonal to $y' - y \in D_0$ for all $y' \in D$ and

$$d(x,y) = d(x-c, y-c) \leq d(x-c, d) = d(x, d+c)$$

for all $d \in D_0$, so $d(x,y) \leq d(x,y')$ for all $y' \in D$ (since $c + D_0 = D$). Moreover, y is unique in satisfying these properties for x since $y - c$ is unique in satisfying them for $x - c$ (5.1.7). A similar argument applies for the point $z \in H \cap D(x)$.

Remark. In (5.1.15), we initially suspect that u is a translation after a symmetry (5.1.13), and hence a symmetry itself. Since $u(u(0)) = u^2(0) = 0$, we suspect that u is a symmetry about a variety through $a = \frac{1}{2}u(0)$ perpendicular to a, which turns out to be the case.

By (5.1.13),

$$\tfrac{1}{2}(x + u(x)) = a + \tfrac{1}{2}(x + v(x)) \in a + V$$

and

$$\tfrac{1}{2}(x+u(x)) - x = a - \tfrac{1}{2}(x - v(x)) \in a + W = W$$

since $-v$ is the symmetry about W and $a \in W$. Therefore $\frac{1}{2}(x+u(x))$ is the unique point in the intersection of $a+V$ and $x+W$ (3.3.8).

Remark. In (5.1.16), $t_{2b} = (t_b u_1)(u_1 t_b)$ is the composite of two symmetries about varieties parallel to V differing by b. Therefore *translations are just composites of symmetries.*

Exercise (1)**.** If D is a vector line in E and H is the orthogonal hyperplane, then D is the intersection of the vector hyperplanes orthogonal to vector lines in H (that is, the vector hyperplanes *perpendicular* to H). Dually, H is the union of the vector lines orthogonal to D.

Proof. If H' is a vector hyperplane orthogonal to a vector line $D' \subseteq H$, then D is orthogonal to D' (since D is orthogonal to H), so $D \subseteq H'$ (since H' is the orthogonal subspace of D'). Conversely, if x is in the intersection and $h \in H$, $h \neq 0$, then x is orthogonal to h since x is in the vector hyperplane orthogonal to the vector line D_{0h}. The dual follows by taking orthogonal subspaces. □

Exercise (2)**.** If a and a' are diametrically opposed points on a sphere S, then a necessary and sufficient condition for a point x to be on S is that $x-a$ and $x-a'$ be orthogonal.

Proof. Without loss of generality, we may assume that S is centered at the origin with radius ρ. Then $a' = -a$, so

$$(x-a \mid x-a') = \|x\|^2 - \rho^2 = 0 \iff \|x\| = \rho \iff x \in S$$ □

Exercise (4)**.** *Powers and coorthogonal spheres:*

- Let S be a sphere centered at c with radius ρ. If $a \in E$ with $\delta = d(a,c)$ and D is a line through a meeting S at distinct points x_1, x_2, then

  $$(x_1 - a \mid x_2 - a) = \delta^2 - \rho^2$$

  If instead D meets S at the single point x, then

  $$d(a,x)^2 = \delta^2 - \rho^2$$

- If S_1, S_2 are non-concentric spheres with respective centers c_1, c_2 and radii ρ_1, ρ_2, then the set H of points whose powers with respect to S_1 and S_2 are equal is a hyperplane perpendicular to $D_{c_1 c_2}$ and containing $S_1 \cap S_2$.

- Let $\pi_k(x) = d(x, c_k)^2 - \rho_k^2$ denote the power of x with respect to S_k (for $k = 1, 2$). Then the following are equivalent:

  (a) $\pi_2(c_1) = \rho_1^2$
  (b) $\pi_1(c_2) = \rho_2^2$
  (c) $\emptyset \neq S_1 \cap S_2 \subseteq \{x \mid (x - c_1 \mid x - c_2) = 0\}$
  (d) $\pi_1(x) + \pi_2(x) = 2(x - c_1 \mid x - c_2)$

Proof.

- Without loss of generality, we may assume that $c = 0$. In the first case, let $b = \frac{1}{2}(x_1 + x_2)$ and $y = \frac{1}{2}(x_1 - x_2)$. Then

$$(b \mid y) = \tfrac{1}{4}(\|x_1\|^2 - \|x_2\|^2) = \tfrac{1}{4}(\rho^2 - \rho^2) = 0$$

  Since y is a direction vector of D, it follows that b is perpendicular to D. Now $x_1 = b + y$ and $x_2 = b - y$, so $x_1 - a = (b - a) + y$ and $x_2 - a = (b - a) - y$, hence

$$\begin{aligned}
(x_1 - a \mid x_2 - a) &= ((b - a) + y \mid (b - a) - y) \\
&= (b - a \mid b - a) - \|y\|^2 \\
&= \delta^2 - (\|b\|^2 + \|y\|^2) && \text{as } (b \mid b - a) = 0,\ (b \mid a) = \|b\|^2 \\
&= \delta^2 - \rho^2 && \text{by Pythagoras (5.1.5)}
\end{aligned}$$

  In the second case, D must be perpendicular to x. Indeed, if not, let d be a direction vector of D with $(d \mid x) \neq 0$ and consider

$$z = x - \frac{2(d \mid x)}{\|d\|^2} d$$

  Then $z \in D$, $z \neq x$, and expansion of $(z \mid z)$ shows that $\|z\|^2 = \|x\|^2$, so $\|z\| = \rho$ and $z \in S$, contradicting that D is tangential to S at x. It follows that $(x \mid a - x) = 0$, and $d(a, x)^2 = \delta^2 - \rho^2$ by Pythagoras (5.1.5).

- We first find $c \in H \cap D_{c_1 c_2}$. We must have $c = \alpha c_1 + \beta c_2$ with $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$. In fact, $\alpha = \|c - c_2\| / \delta$ and $\beta = \|c - c_1\| / \delta$ where $\delta = \|c_1 - c_2\| \neq 0$, so that

$$
\begin{aligned}
\alpha - \beta &= (\alpha - \beta)(\alpha + \beta) \\
&= \alpha^2 - \beta^2 \\
&= \frac{\|c - c_2\|^2 - \|c - c_1\|^2}{\delta^2} \\
&= \frac{(\|c - c_2\|^2 - \rho_2^2) + \rho_2^2 - (\|c - c_1\|^2 - \rho_1^2) - \rho_1^2}{\delta^2} \\
&= \frac{\rho_2^2 - \rho_1^2}{\delta^2}
\end{aligned}
$$

since $c \in H$. It follows that

$$
\alpha = \frac{1}{2} + \frac{\rho_2^2 - \rho_1^2}{2\delta^2} \qquad \beta = \frac{1}{2} - \frac{\rho_2^2 - \rho_1^2}{2\delta^2} \qquad c = \frac{1}{2}(c_1 + c_2) + \frac{\rho_2^2 - \rho_1^2}{2\delta^2}(c_1 - c_2)
$$

Note c is the midpoint of c_1 and c_2 if and only if $\rho_1 = \rho_2$. Now

$$
\begin{aligned}
x \in H &\iff \|x - c_1\|^2 - \rho_1^2 = \|x - c_2\|^2 - \rho_2^2 \\
&\iff (x - c_2 \mid x - c_2) - (x - c_1 \mid x - c_1) = \rho_2^2 - \rho_1^2 \\
&\iff (x \mid c_1 - c_2) = \tfrac{1}{2}(\|c_1\|^2 - \|c_2\|^2 + \rho_2^2 - \rho_1^2) = (c \mid c_1 - c_2)
\end{aligned}
$$

Therefore H is the hyperplane through c perpendicular to $D_{c_1 c_2}$. Finally, $S_1 \cap S_2 \subseteq H$ since $S_1 \cap S_2$ consists of the points whose powers with respect to S_1 and S_2 are zero.

- Let $\delta = \|c_1 - c_2\|$. We have

$$
\pi_2(c_1) = \delta^2 - \rho_2^2 = \rho_1^2 \iff \rho_2^2 = \delta^2 - \rho_1^2 = \pi_1(c_2)
$$

so (a) $\iff$ (b), and these are equivalent to $\delta^2 = \rho_1^2 + \rho_2^2$. If this condition holds, then $\rho_1 - \delta \le 0$, so $2\rho_1(\rho_1 - \delta) \le 0$ and

$$
\begin{gathered}
2\rho_1^2 - 2\rho_1\delta + \rho_2^2 \le \rho_2^2 \le 2\rho_1^2 + 2\rho_1\delta + \rho_2^2 \\
\rho_1^2 - 2\rho_1\delta + \delta^2 \le \rho_2^2 \le \rho_1^2 + 2\rho_1\delta + \delta^2 \\
(\rho_1 - \delta)^2 \le \rho_2^2 \le (\rho_1 + \delta)^2 \\
|\rho_1 - \delta| \le \rho_2 \le \rho_1 + \delta
\end{gathered}
$$

so $S_1 \cap S_2 \neq \emptyset$ (5.1.11.4). If $x \in S_1 \cap S_2$, then

$$\|x - c_1\|^2 + \|x - c_2\|^2 = \rho_1^2 + \rho_2^2 = \delta^2 = \|c_1 - c_2\|^2$$

so $(x - c_1 \mid x - c_2) = 0$ (5.1.1.4). Therefore (a),(b) $\implies$ (c). Conversely, if $x \in S_1 \cap S_2$ and $(x - c_1 \mid x - c_2) = 0$, then by Pythagoras (5.1.5.1),

$$\delta^2 = \|c_1 - c_2\|^2 = \|x - c_1\|^2 + \|x - c_2\|^2 = \rho_1^2 + \rho_2^2$$

so (c) $\implies$ (a),(b). Finally, by (5.1.1.4),

$$\begin{aligned}\pi_1(x) + \pi_2(x) &= \|x - c_1\|^2 + \|x - c_2\|^2 - (\rho_1^2 + \rho_2^2)\\ &= 2(x - c_1 \mid x - c_2) + \delta^2 - (\rho_1^2 + \rho_2^2)\end{aligned}$$

so clearly (a),(b) $\iff$ (d). □

Exercise (6)**.** *Characterization of similitudes:*

- If $u : E \to E$ is a bijective function such that $(u(x) \mid u(y)) = \alpha(x \mid y)$ for all $x, y \in E$ ($\alpha > 0$), then u is linear and consequently $u \in \mathbf{GO}(E)$.

- If $u : E \to E$ is a bijective function such that $d(u(x), u(y)) = \alpha d(x, y)$ for all $x, y \in E$ ($\alpha > 0$), then u is affine and consequently $u \in \mathbf{Sm}(E)$.

- If $u \in \mathbf{GL}(E)$ is such that $(x \mid y) = 0$ implies $(u(x) \mid u(y)) = 0$ (in other words, u preserves orthogonality), then $u \in \mathbf{GO}(E)$.

Proof.

- By direct computation,

$$\begin{aligned}&(u(x+y) - u(x) - u(y) \mid u(x+y) - u(x) - u(y)) =\\ &\qquad\qquad\qquad \alpha(x + y - x - y \mid x + y - x - y) = 0\end{aligned}$$

  so $u(x+y) - u(x) - u(y) = 0$ by positive definiteness of the inner product. Similarly $u(\xi x) - \xi u(x) = 0$. Therefore u is linear.

- Let $v = t_{-u(0)} u$. Then $v : E \to E$ is a bijective function and (5.1.1.5)

$$\begin{aligned}2(v(x) \mid v(y)) &= \|v(x)\|^2 + \|v(y)\|^2 - \|v(x) - v(y)\|^2\\ &= \|u(x) - u(0)\|^2 + \|u(y) - u(0)\|^2 - \|u(x) - u(y)\|^2\\ &= \alpha^2(\|x\|^2 + \|y\|^2 - \|x - y\|^2)\\ &= 2\alpha^2(x \mid y)\end{aligned}$$

  Therefore $v \in \mathbf{GO}(E)$ by the previous item and $u \in \mathbf{Sm}(E)$ (5.1.14).

- First observe that $(u(x) \mid u(y)) = 0$ implies $(x \mid y) = 0$. Indeed, if $y = 0$ this is trivial. If $y \neq 0$, then $x = \xi y + z$ with $(z \mid y) = 0$ (5.1.7) and

  $$\begin{aligned} 0 &= (u(x) \mid u(y)) \\ &= (u(\xi y + z) \mid u(y)) \\ &= (\xi u(y) + u(z) \mid u(y)) \\ &= \xi \| u(y) \|^2 && \text{since } (u(z) \mid u(y)) = 0 \end{aligned}$$

  Since $u(y) \neq 0$, this implies $\xi = 0$, so $x = z$ and $(x \mid y) = 0$.

  Now if $y \neq 0$, it follows that $(u(x) \mid u(y)) = 0$ is another equation of the hyperplane $(x \mid y) = 0$ orthogonal to y, so (3.3.6) there is $\mu_y > 0$ with

  $$(u(x) \mid u(y)) = \mu_y (x \mid y)$$

  for all $x \in E$. We claim that μ_y is independent of y. Indeed, if $y' = 0$ then trivially $(u(x) \mid u(y')) = \mu_y (x \mid y')$ for all $x \in E$. If $(y \mid y') \neq 0$, then

  $$\mu_y = \frac{(u(y) \mid u(y'))}{(y \mid y')} = \mu_{y'}$$

  Finally, if $y' \neq 0$ and $(y \mid y') = 0$, then $(y \mid y + y') \neq 0$ and $(y' \mid y + y') \neq 0$, so

  $$\mu_y = \mu_{y+y'} = \mu_{y'}$$

  by the previous case. □

Exercise (7)**.** In $\mathbf{GL}(E)$, the normalizer of $\mathbf{O}(E)$ is $\mathbf{GO}(E)$.

Proof. The normalizer contains $\mathbf{GO}(E)$ since if $v \in \mathbf{GO}(E)$ and $u \in \mathbf{O}(E)$, then $\mu(vuv^{-1}) = 1$ (5.1.12.3), so $vuv^{-1} \in \mathbf{O}(E)$.

Conversely, suppose $v \in \mathbf{GL}(E)$ normalizes $\mathbf{O}(E)$. If $(x \mid y) = 0$ and $x = 0$, then trivially $(v(x) \mid v(y)) = 0$. If $x \neq 0$, let $u \in \mathbf{O}(E)$ be the symmetry about the hyperplane orthogonal to x, so $u(x) = -x$ and $u(y) = y$. Then $vuv^{-1} \in \mathbf{O}(E)$ by hypothesis and $vuv^{-1}v = vu$, so

$$(v(x) \mid v(y)) = (v(u(x)) \mid v(u(y))) = -(v(x) \mid v(y))$$

and hence $(v(x) \mid v(y)) = 0$. Since x, y were arbitrary, it follows that v preserves orthogonality, and therefore $v \in \mathbf{GO}(E)$ (Exercise 6). □

Section 2

Remark. In (5.2.3), the mapping $f \mapsto a$ is actually a vector space *isomorphism* from the dual space E^* to E. In fact, if $\{a_1, a_2\}$ is an orthonormal basis of E, then the isomorphism is just that determined by $a_i^* \mapsto a_i$, where $\{a_1^*, a_2^*\}$ is the dual basis (4.1.15).

Remark. In (5.2.4), we have vector space isomorphisms

$$\mathscr{B}(E, E; \mathbf{R}) \cong \mathrm{Hom}(E, E^*) \cong \mathrm{End}(E)$$

Remark. In the complex plane $\mathbf{C}$, a number $z \in \mathbf{C}$ induces an endomorphism through multiplication, and its conjugate $\overline{z}$ induces the adjoint endomorphism. Note $\overline{\overline{z}} = z$ as in (5.2.5.2), $\overline{wz} = \overline{w}\,\overline{z}$ as in (5.2.5.3), $\overline{z}z = |z|^2$ is the "multiplicator" of z as in (5.2.5.5), and z is "orthogonal" if it is on the unit circle. If $z \neq 0$, then $z = \rho u$ with $\rho = |z| > 0$ and $u = z/|z|$ on the unit circle uniquely determined, as in (5.1.12).

Endomorphisms of a plane can therefore be viewed as generalized complex numbers, with adjoints as generalized complex conjugates. This idea is made precise in (5.5).

Remark. In (5.2.8), $M(w) = (\alpha_{ij})$ with $\alpha_{11} = \alpha_{22} = 0$ and $\alpha_{12} = -\alpha_{21}$ relative to an orthonormal basis (5.2.6), so $\det(w) = \alpha_{12}^2$ (4.2.9.2). Since $w \neq 0$, $\det(w) > 0$ and w is invertible (4.2.8).

Exercise (1)**.** If $u \in \mathbf{GL}(E)$ is an involution, then $u \in \mathbf{O}(E)$ if and only if u is self-adjoint. If $p \in \mathrm{End}(E)$ is idempotent, then p is an orthogonal projection (5.1.6) if and only if p is self-adjoint.

Proof. By (5.2.5.6), $u \in \mathbf{O}(E)$ if and only if $u^* = u^{-1} = u$. By Exercise 3.2.1, p is the projection onto $p(E)$ in the direction of $p^{-1}(0)$. If $p(E)$ is orthogonal to $p^{-1}(0)$, then

$$\begin{aligned}
(p(x) \mid y) &= (p(x) \mid p(y) + (y - p(y))) && \\
&= (p(x) \mid p(y)) && \text{since } (p(x) \mid y - p(y)) = 0 \\
&= (p(x) + (x - p(x)) \mid p(y)) && \text{since } (x - p(x) \mid p(y)) = 0 \\
&= (x \mid p(y))
\end{aligned}$$

for all $x, y \in E$. Therefore $p^* = p$. Conversely if $p^* = p$, then for all $x \in E$ and $y \in p^{-1}(0)$,

$$(p(x) \mid y) = (x \mid p(y)) = (x \mid 0) = 0$$

so $p(E)$ is orthogonal to $p^{-1}(0)$. □

Exercise (8)**.** Suppose $u \in \operatorname{End}(E)$ has eigenvalues $\lambda_1 \neq \lambda_2$ with corresponding eigenvectors a_1, a_2. If b_1 (resp. b_2) is nonzero and orthogonal to a_1 (resp. a_2), then b_1 (resp. b_2) is an eigenvector of u^* corresponding to λ_2 (resp. λ_1).

Proof. Since $\lambda_1 \neq \lambda_2$, $\{a_1, a_2\}$ must be linearly independent and hence a basis. If $(a_1 \mid x) = (a_1 \mid y)$ and $(a_2 \mid x) = (a_2 \mid y)$, then for any $z = \alpha_1 a_1 + \alpha_2 a_2$,

$$(z \mid x) = \alpha_1 (a_1 \mid x) + \alpha_2 (a_2 \mid x) = \alpha_1 (a_1 \mid y) + \alpha_2 (a_2 \mid y) = (z \mid y)$$

and it follows that $x = y$.

Now observe that for $z = a_1, a_2$,

$$(z \mid u^*(b_1)) = (u(z) \mid b_1) = \lambda_2 (z \mid b_1) = (z \mid \lambda_2 b_1)$$

so $u^*(b_1) = \lambda_2 b_1$. Similarly $u^*(b_2) = \lambda_1 b_2$. □

References

[1] Dieudonné, J. *Linear Algebra and Geometry.* Hermann, 1969.