

# Notes and exercises from *Linear Algebra and Geometry*

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## Introduction

This document contains notes and exercises from [1].

## Chapter III

### Section 1

*Remark.* If  $V, W$  are supplementary subspaces of  $E$  and  $c$  is the unique point in common to the varieties  $a + V$  and  $b + W$  (3.1.15), then  $a + V = c + V$  and  $b + W = c + W$  (3.1.12), so

$$-c + (a + V) = V \quad \text{and} \quad -c + (b + W) = W$$

In other words, taking the origin  $c$  (3.2.21), the varieties become their direction subspaces.

**Exercise (4).** Let  $V, W$  be a pair of supplementary subspaces of  $E$ . Every subspace  $U$  containing  $V$  is the direct sum of  $V$  with  $U \cap W$ .

*Proof.* If  $u \in U$ , then  $u = v + w$  for some  $v \in V$  and  $w \in W$ , and  $w = u - v \in U$ . So  $U = V + (U \cap W)$ , and  $V \cap U \cap W = \{0\}$ .  $\square$

### Section 2

**Exercise (1).** *Projections and idempotents:* If  $p, q$  are the projections for a direct sum  $E = V + W$ , then  $p, q \in \text{End}(E)$  are such that  $p^2 = p$ ,  $q^2 = q$ , and  $p + q = 1$ . Conversely, if  $p \in \text{End}(E)$  is such that  $p^2 = p$ , then  $E = p(E) + p^{-1}(0)$  is a direct sum. Moreover, if  $q = 1 - p$ , then  $q^2 = q$ ,  $q(E) = p^{-1}(0)$ , and  $q^{-1}(0) = p(E)$ .

*Proof.* For the forward direction, we know  $p, q \in \text{End}(E)$  and  $p + q = 1$  (3.2.2). It follows that  $p^2 = p \circ (1 - q) = p - pq = p$  and  $q^2 = (1 - p)^2 = 1 - p = q$ .

For the converse,  $p + q = 1$ , so  $E = p(E) + q(E)$ . Also  $pq = p - p^2 = 0$ , so  $p(E) \cap q(E) = \{0\}$  and  $q(E) \subseteq p^{-1}(0)$ . If  $x \in p^{-1}(0)$ , then  $q(x) = x$ , so  $x \in q(E)$ . Hence  $q(E) = p^{-1}(0)$  and similarly  $q^{-1}(0) = p(E)$ . Finally  $q^2 = q$  as above.  $\square$

**Exercise (2).** If  $W$  and  $W'$  are both supplementary to  $V$  in  $E$ , then  $W$  and  $W'$  are isomorphic.

*Proof.* If  $p$  is the projection of  $E$  onto  $W'$ , then the restriction of  $p$  to  $W$  is an isomorphism from  $W$  to  $W'$ .  $\square$

**Exercise (3).** If  $E = V + W$  is a direct sum with inclusions  $i : V \rightarrow E$  and  $j : W \rightarrow E$ , and  $v : V \rightarrow F$  and  $w : W \rightarrow F$  are linear maps, then there is a unique linear map  $u : E \rightarrow F$  with  $u \circ i = v$  and  $u \circ j = w$ .

*Proof.* If  $p, q$  are the projections on  $V, W$  respectively, then  $u = v \circ p + w \circ q$ .  $\square$

**Exercise (11).**  $\mathbf{GA}(E)/E \cong \mathbf{GL}(E)$ .

*Proof.* Define  $\varphi : \mathbf{GA}(E) \rightarrow \mathbf{GL}(E)$  by  $\varphi(t_a \circ v) = v$ . Note that  $\varphi$  is well-defined by (3.2.17),  $\varphi$  is a homomorphism by (3.2.19), and  $\varphi$  is clearly surjective. Also  $\varphi(u) = 1$  if and only if  $u$  is a translation, so  $\ker \varphi = T(E)$ , the normal subgroup of translations. It follows that  $\mathbf{GA}(E)/T(E) \cong \mathbf{GL}(E)$ . Finally, the mapping  $a \mapsto t_a$  is an isomorphism  $E \cong T(E)$  from the additive group  $E$ .  $\square$

**Exercise (13).** If  $u : E \rightarrow F$  is affine and  $L$  is a variety in  $F$ , then  $u^{-1}(L)$  is empty or a variety in  $E$ .

*Proof.* If  $a \in u^{-1}(L)$  and  $L_0$  is the direction of  $L$ , then  $L = u(a) + L_0$  and hence  $u^{-1}(L) = a + u^{-1}(L_0)$ .  $\square$

### Section 3

*Remark.* If  $a, b \in E$ , then the segment  $ab$  (3.3.4) consists of all points  $x = \alpha a + \beta b$  where  $\alpha, \beta \geq 0$  and  $\alpha + \beta = 1$ —the *convex combinations* of  $a$  and  $b$ .

**Exercise (3).** A necessary and sufficient condition for a nonempty subset  $V$  of a vector space to be a variety is that for all pairs  $x, y$  of distinct points of  $V$ , the line  $D_{xy}$  is contained in  $V$ .

*Proof.* The condition is necessary by (3.3.2).

If the condition holds, choose  $v \in V$  and let  $V_0 = -v + V$ . We claim  $V_0$  is a subspace, from which it follows that  $V = v + V_0$  is a variety. First,  $0 = -v + v \in V_0$ . If  $x \in V_0$  and  $x \neq 0$ , then  $v + x \in V$  and  $v + x \neq v$ , so  $D_{v, v+x} = \{v + \xi x \mid \xi \in \mathbf{R}\} \subseteq V$ . It follows that  $\xi x \in V_0$  for all  $\xi \in \mathbf{R}$ . If also  $y \in V_0$  and  $y \neq x$ , then  $D_{v+x, v+y} \subseteq V$ , so in particular  $v + 2^{-1}(x + y) \in V$  and  $2^{-1}(x + y) \in V_0$ . By the previous result, it then follows that  $x + y \in V_0$ . Therefore  $V_0$  is a subspace as claimed.  $\square$

**Exercise (4).** *Translations and homothetic maps:*

- A necessary and sufficient condition for an affine map to be a translation or a homothetic map is that its associated linear map be homothetic.
- A necessary and sufficient condition for an affine map to preserve the direction of lines is that it be a translation or a bijective homothetic map.
- If  $u_1, u_2$  are translations or homothetic maps, then so is  $u_1 \circ u_2$ .
- If  $u_1, u_2$  and  $u_1 \circ u_2$  are homothetic maps with ratios not equal to 1, their centers are collinear. If instead  $u_1 \circ u_2$  is a translation, then it is either the identity or a translation in the direction of the line through the centers of  $u_1$  and  $u_2$ .
- If  $v \in \mathbf{GA}(E)$ , then  $v \circ h_{a, \lambda} \circ v^{-1} = h_{v(a), \lambda}$ .
- The subset  $H(E)$  of translations and homothetic maps in  $\mathbf{GA}(E)$  forms a subgroup, and  $H(E)/E \cong \mathbf{R}^*$ .

*Proof.*

- This follows from the equations  $t_a = t_a \circ h_1$  and  $h_{a, \lambda} = t_{(1-\lambda)a} \circ h_\lambda$  and  $t_a \circ h_\lambda = h_{(1-\lambda)^{-1}a, \lambda}$  ( $\lambda \neq 1$ ).
- The condition is sufficient because such a map has the form  $t_a \circ h_\lambda$  with  $\lambda \neq 0$ , which clearly preserves the direction of lines. Conversely, suppose  $u = t_a \circ v$  preserves the direction of lines. If  $x \neq 0$ , let  $D$  be the vector line through  $x$ . Then  $v(D) = D$ , so  $v(x) = \lambda x$  for some  $\lambda \in \mathbf{R}$  with  $\lambda \neq 0$ , and in fact  $v(y) = \lambda y$  for all  $y \in D$ . We claim  $v = h_\lambda$ , from which the result follows. If  $y \notin D$ , then by considering the vector line  $D'$  through  $y$  we have  $v(y) = \mu y$  for some  $\mu \in \mathbf{R}$ . Now  $v(D_{xy}) = D_{v(x)v(y)} = D_{\lambda x, \mu y}$ , and since  $v$  preserves direction there is  $\xi \in \mathbf{R}$  with  $\mu y - \lambda x = \xi(y - x)$ , or  $(\mu - \xi)y = (\lambda - \xi)x$ . Since  $y \notin D$ , this implies  $\mu = \xi = \lambda$ . Therefore  $v = h_\lambda$  as claimed.

- If  $u_1 = t_a \circ h_\lambda$  and  $u_2 = t_b \circ h_\mu$ , then by (3.2.19.1),  $u_1 \circ u_2 = t_{a+\lambda b} \circ h_{\lambda\mu}$ .
- Write  $u_1 = h_{a,\lambda}$ ,  $u_2 = h_{b,\mu}$ , and  $u_1 \circ u_2 = h_{c,\nu}$  with  $\lambda, \mu, \nu \neq 1$ . Then for all  $x$ ,

$$(1 - \nu)c + \nu x = (1 - \lambda)a + \lambda(1 - \mu)b + \lambda\mu x$$

Taking  $x = 0$  and  $x \neq 0$  (we assume such exist!) yields  $\nu = \lambda\mu$  and

$$c = (1 - \lambda\mu)^{-1}[(1 - \lambda)a + \lambda(1 - \mu)b] = (1 - \lambda\mu)^{-1}(\lambda - 1)(b - a) + b$$

so that  $a, b, c$  are collinear. If instead  $u_1 \circ u_2 = t_c$ , then  $\lambda\mu = 1$  and  $c = (\lambda - 1)(b - a)$ , from which the second result follows.

- Write  $v = t_b \circ w$ , where  $w \in \mathbf{GL}(E)$ , so that  $v^{-1} = w^{-1} \circ t_{-b}$ . By repeated application of (3.2.19.1),

$$\begin{aligned} v \circ h_{a,\lambda} \circ v^{-1} &= t_b \circ w \circ t_{(1-\lambda)a} \circ h_\lambda \circ w^{-1} \circ t_{-b} \\ &= t_{v((1-\lambda)a)} \circ w \circ h_\lambda \circ w^{-1} \circ t_{-b} \\ &= t_{b+(1-\lambda)w(a)} \circ h_\lambda \circ t_{-b} \\ &= t_{b+(1-\lambda)w(a)-\lambda b} \circ h_\lambda \\ &= t_{(1-\lambda)v(a)} \circ h_\lambda \\ &= h_{v(a),\lambda} \end{aligned}$$

- First,  $1 \in H(E)$ . If  $u_1, u_2 \in H(E)$ , then  $u_1 \circ u_2 \in H(E)$  by a previous item. If  $u_1 = t_a \circ h_\lambda$  with  $\lambda \neq 0$ , then  $u_1^{-1} = t_{-\lambda^{-1}a} \circ h_{\lambda^{-1}} \in H(E)$ . Therefore  $H(E)$  is a subgroup of  $\mathbf{GA}(E)$ . Define  $\varphi : H(E) \rightarrow \mathbf{R}^*$  by  $\varphi(t_a \circ h_\lambda) = \lambda$ . Note that  $\varphi$  is well-defined (since  $E \neq \{0\}$ !),  $\varphi$  is a homomorphism by a previous item,  $\varphi$  is surjective, and  $\ker \varphi = T(E) \cong E$ . It follows that  $H(E)/E \cong \mathbf{R}^*$ .  $\square$

**Exercise (5).** A variety not parallel to a hyperplane meets the hyperplane. If  $H$  is a vector hyperplane in  $E$  and  $V$  is a subspace of  $E$  not contained in  $H$ , then  $V \cap H$  is a vector hyperplane in  $V$ .

*Proof.* Any vector in the direction of the variety which is not in the direction of the hyperplane determines a line in the variety which meets the hyperplane (3.3.8).

If  $D$  is the vector line through a vector in  $V - H$ , then  $D$  is supplementary to  $H$  in  $E$ , so  $D$  is supplementary to  $V \cap H$  in  $V$  (Exercise 3.1.4).  $\square$

**Exercise (6).** *Dilations and transvections:* Let  $H$  be a vector hyperplane in  $E$  and suppose  $u \in \text{End}(E)$  fixes every element of  $H$ .

- There is  $\gamma \in \mathbf{R}$  unique such that  $u(a) \in \gamma a + H$  for all  $a \in E - H$ .
- If  $\gamma \neq 1$ , then  $\gamma$  is an eigenvalue of  $u$  and  $E(\gamma; u)$  is a line  $S$  supplementary to  $E(1; u) = H$ . A subspace  $V$  satisfies  $u(V) \subseteq V$  if and only if  $S \subseteq V$  or  $V \subseteq H$ . In particular, a vector line  $D$  satisfies  $u(D) \subseteq D$  if and only if  $D = S$  or  $D \subseteq H$ .
- If  $\gamma = 1$ , and  $g(x) = 0$  is an equation of  $H$ , there is  $c \in H$  unique such that  $u(x) = x + g(x)c$  for all  $x \in E$ .  $u$  is bijective. If  $u \neq 1$  (so  $c \neq 0$ ), then the line  $T = D_{0c}$  is independent of  $g$ . The scalar 1 is the only eigenvalue of  $u$  if  $H \neq \{0\}$ , and  $E(1; u) = H$  if  $u \neq 1$ . If  $u \neq 1$ , a subspace  $V$  satisfies  $u(V) \subseteq V$  if and only if  $T \subseteq V$  or  $V \subseteq H$ ; in particular, a vector line  $D$  satisfies  $u(D) = D$  if and only if  $D \subseteq H$ .
- The set  $\Gamma(E, H)$  of automorphisms of  $E$  leaving the hyperplane  $H$  fixed pointwise is a subgroup of  $\mathbf{GL}(E)$ . The subset  $\Theta(E, H)$  of transvections is a normal abelian subgroup of  $\Gamma(E, H)$  isomorphic to  $H$ .  $\Gamma(E, H)/H \cong \mathbf{R}^*$ .

*Proof.*

- If  $a \in E - H$ , then  $E = \mathbf{R}a + H$ , so  $u(a) = \gamma a + t$  for some  $\gamma \in \mathbf{R}$  and  $t \in H$ . If  $b \in E$ , then  $b = \beta a + h$  for some  $\beta \in \mathbf{R}$  and  $h \in H$ , so

$$\begin{aligned}
u(b) &= u(\beta a + h) \\
&= \beta u(a) + u(h) \\
&= \beta(\gamma a + t) + h \\
&= \gamma(\beta a + h) + (1 - \gamma)h + \beta t \\
&= \gamma b + (1 - \gamma)h + \beta t
\end{aligned} \tag{1}$$

Therefore  $u(b) \in \gamma b + H$ . If also  $u(b) \in \gamma' b + H$ , then  $(\gamma' - \gamma)b \in H$ , which implies  $\gamma' = \gamma$  if  $b \notin H$ . Therefore  $\gamma$  is unique for  $b \notin H$ .

- Let  $x = a - (1 - \gamma)^{-1}t$ . Then  $x \neq 0$  since  $a \notin H$  and

$$\begin{aligned}
u(x) &= u(a - (1 - \gamma)^{-1}t) \\
&= u(a) - (1 - \gamma)^{-1}u(t) \\
&= \gamma a + t - (1 - \gamma)^{-1}t \\
&= \gamma[a - (1 - \gamma)^{-1}t] \\
&= \gamma x
\end{aligned}$$

Therefore  $x$  is an eigenvector of  $u$  with eigenvalue  $\gamma$ . Let  $S$  be the vector line through  $x$ . Then  $S \subseteq E(\gamma; u)$ . Conversely if  $b \in E(\gamma; u)$ , then  $u(b) = \gamma b$ , which implies  $(1 - \gamma)h + \beta t = 0$  in (1), so  $h = -\beta(1 - \gamma)^{-1}t$  and

$$b = \beta a + h = \beta[a - (1 - \gamma)^{-1}t] = \beta x \in S$$

Therefore  $S = E(\gamma; u)$ . By hypothesis  $H \subseteq E(1; u)$ . Conversely if  $b \in E(1; u)$ , then  $b = u(b) \in \gamma b + H$ , so  $(1 - \gamma)b \in H$ , so  $b \in H$ . Therefore  $H = E(1; u)$ .  $S$  is supplementary to  $H$  since  $x \notin H$ .

By hypothesis  $u(V) \subseteq V$  for any subspace  $V \subseteq H$ . If  $S \subseteq V$ , then  $V = S + V \cap H$  (Exercise 3.1.4), so clearly  $u(V) \subseteq V$ . Conversely if  $u(V) \subseteq V$  and  $v \in V - H$ , then  $v = s + h$  for some  $s \in S$  with  $s \neq 0$  and  $h \in H$ , so  $u(v) = \gamma s + h$  and  $v - u(v) = (1 - \gamma)s \in V$ , which implies  $s \in V$  and  $S = \mathbf{R}s \subseteq V$ .

- Fix  $e \in E$  with  $g(e) = 1$  and let  $c = u(e) - e$ . Since  $u(e) \in e + H$ ,  $g(c) = 0$  and  $c \in H$ . Now  $u(x) = x + g(x)c$  holds for  $x = e$ , and for  $x \in H$ , so by linearity it holds for all  $x \in \mathbf{R}e + H = E$ . Note  $c$  is unique since if  $u(e) = e + g(e)c'$ , then  $c' = u(e) - e = c$ .

The map  $x \mapsto x - g(x)c$  is clearly the inverse of  $u$ , so  $u$  is bijective.

If  $h(x) = 0$  is another equation of  $H$ , then by the above there exists  $c' \in H$  such that  $u(x) = x + h(x)c'$  for all  $x \in E$ . But  $h = \lambda g$  for some  $\lambda \neq 0$  (3.3.6), so  $u(x) = x + g(x)(\lambda c')$  for all  $x \in E$  and  $c = \lambda c'$  by uniqueness of  $c$ . If  $u \neq 1$ , then  $T = D_{0c} = D_{0c'}$  is independent of  $g$ .

If  $u(x) = x + g(x)c = \lambda x$ , then  $(\lambda - 1)x \in H$ . If  $\lambda \neq 1$ , then  $x \in H$ , so actually  $(\lambda - 1)x = 0$  and  $x = 0$ . If  $\lambda = 1$ , then  $g(x)c = 0$ , so either  $g(x) = 0$  and  $x \in H$ , or  $c = 0$  and  $u = 1$ .

As above,  $u(V) \subseteq V$  if  $V \subseteq H$  or  $T \subseteq V$ . Conversely if  $u(V) \subseteq V$  and  $x \in V - H$ , then  $g(x) \neq 0$  and  $g(x)c = u(x) - x \in V$ , so  $c \in V$  and  $T = \mathbf{R}c \subseteq V$ .  $\square$

- $\Gamma(E, H)$  is obviously a subgroup of  $\mathbf{GL}(E)$ . Define  $\varphi : \Gamma(E, H) \rightarrow \mathbf{R}^*$  by  $\varphi(u) = \gamma$ . Then  $\varphi$  is a well-defined, surjective homomorphism and  $\ker \varphi = \Theta(E, H)$ . It follows that  $\Theta(E, H)$  is a normal subgroup of  $\Gamma(E, H)$  and that  $\Gamma(E, H)/\Theta(E, H) \cong \mathbf{R}^*$ . Finally, the mapping  $c \mapsto (x \mapsto x + g(x)c)$  is an isomorphism  $H \cong \Theta(E, H)$ , so in particular  $\Theta(E, H)$  is abelian.

## Chapter IV

### Section 1

**Exercise (1).** If  $t \neq 1$  is a transvection in  $E$ , a basis for  $E$  may be chosen relative to which

$$M(t) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Conversely, matrices of the form

$$B_{12}(\lambda) = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \quad B_{21}(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

are matrices of transvections relative to some basis  $\{a_1, a_2\}$ . These transvections have the lines  $D_{0a_1}$  and  $D_{0a_2}$  respectively.

*Proof.* Write  $t(x) = x + g(x)c$ , where  $g \in E^*$  with  $g \neq 0$ , and  $c \neq 0$  with  $g(c) = 0$  (Exercise 3.3.6). Let  $a_1 = c$  and choose  $a_2$  such that  $g(a_2) = 1$ . Then  $\{a_1, a_2\}$  is the desired basis for  $E$ .

Conversely, if  $t$  is the transformation of  $B_{12}(\lambda)$  relative to  $\{a_1, a_2\}$ , then

$$\begin{aligned} t(x) &= t(\xi_1 a_1 + \xi_2 a_2) \\ &= \xi_1 t(a_1) + \xi_2 t(a_2) \\ &= \xi_1 a_1 + \xi_2 (\lambda a_1 + a_2) \\ &= \xi_1 a_1 + \xi_2 a_2 + \lambda \xi_2 a_1 \\ &= x + g(x)a_1 \end{aligned}$$

where  $g = \lambda a_2^* \in E^*$ . Therefore  $t$  is a transvection in the line  $D_{0a_1}$ . A similar argument applies to  $B_{21}(\lambda)$ .  $\square$

**Exercise (3).** If  $u \in \text{End}(E)$  and  $\text{rank}(u) = 1$ , there is a basis of  $E$  relative to which

$$M(u) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix} \quad (\delta \neq 0) \quad \text{or} \quad M(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The second case occurs if and only if  $u$  is nilpotent, in which case  $u^2 = 0$ .

*Proof.* Let  $N = u^{-1}(0)$  and  $R = u(E)$ . Then  $N$  and  $R$  are vector lines in  $E$  (4.1.7). If  $N \cap R = \{0\}$ , then  $E = N + R$  is a direct sum. Choose  $a_1 \in R$  and  $a_2 \in N$  with

$a_1, a_2 \neq 0$ . Then  $u(a_1) = \delta a_1$  with  $\delta \neq 0$  since  $u(a_1) \in R$  and  $a_1 \notin N$ , and also  $u(a_2) = 0$ . It follows that  $\{a_1, a_2\}$  is a basis of  $E$  relative to which

$$M(u) = \begin{pmatrix} \delta & 0 \\ 0 & 0 \end{pmatrix}$$

If  $N \cap R \neq \{0\}$ , then  $N = R$  (3.3.1). Choose  $a_1 \in N$ ,  $a_1 \neq 0$  and  $a_2$  with  $u(a_2) = a_1$ . Then  $\{a_1, a_2\}$  is a basis of  $E$  (since  $a_2 \notin N$ ) relative to which

$$M(u) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

If  $M(u)$  has this form, then  $M(u^2) = M(u)^2 = 0$ , so  $u^2 = 0$  and  $u$  is nilpotent. Conversely if  $u$  is nilpotent, there is  $k > 1$  least such that  $u^k = 0$ . Then  $u^{k-1}(E) \neq 0$  but  $u^{k-1}(E) \subseteq N$ , so  $u^{k-1}(E) = N$ . If  $k > 2$ , then  $u^{k-1}(E)$  is a proper subspace of  $u^{k-2}(E)$ , lest  $u^{k-2}(E) = N$  and  $u^{k-1} = 0$ . But then  $u^{k-2}(E) = E$ , impossible since  $\text{rank}(u) = 1$ . It follows that  $u^2 = 0$ , so  $N = R$  and  $M(u)$  has this form.  $\square$

**Exercise (7).** Let  $u : E \rightarrow E$  be an injective function such that  $E$  is the smallest variety containing  $u(E)$ , and  $u(a), u(b), u(c)$  are collinear whenever  $a, b, c \in E$  are collinear.

- If  $a, b, c \in E$  are not collinear, then  $u(a), u(b), u(c)$  are not collinear.
- For every line  $D$ , there is a unique line  $D'$  such that  $u(D) \subseteq D'$ . If  $u$  is bijective, then  $u(D) = D'$ ; moreover, if  $D_1, D_2$  are parallel (resp. distinct, not parallel), then so are  $u(D_1), u(D_2)$ .
- If  $u$  is bijective, there is  $v \in \mathbf{GA}(E)$  such that  $u_1 = v \circ u$  fixes the origin and basis vectors  $a_1, a_2 \in E$ .  $u_1$  maps lines to lines and preserves parallelism (resp. distinctness, non-parallelism); in particular,  $u_1$  preserves the lines  $D_{0a_1}, D_{0a_2}, D_{a_1a_2}$  and hence the directions of any lines parallel to these.
- Given the points  $\xi a_1, \eta a_1$ , it is possible to construct  $(\xi + \eta)a_1$  and  $\xi\eta a_1$  by intersecting lines with direction vectors derived from  $a_1, a_2$ .
- If  $\varphi$  is defined by  $u_1(\xi a_1) = \varphi(\xi)a_1$ , then  $\varphi$  is a field automorphism of  $\mathbf{R}$ , so  $\varphi = 1_{\mathbf{R}}$ .<sup>1</sup> It follows that  $u_1 = 1_E$ , so  $u = v^{-1}$  is an affine map.

*Proof.*

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<sup>1</sup>For this problem, we assume there are no nontrivial field automorphisms of  $\mathbf{R}$ .



- Suppose towards a contradiction that  $u(a), u(b), u(c)$  are on the line  $\Delta$ . Let  $x \in E$ . If  $x \in D_{ab} \cup D_{ac} \cup D_{bc}$ , then  $u(x) \in \Delta$ . Otherwise if, say,  $x$  and  $c$  are on opposite sides of  $D_{ab}$ , then  $D_{xc}$  and  $D_{ab}$  intersect at a unique point  $y$  by (3.3.9) and (4.1.6),  $u(y) \in \Delta$  since  $y \in D_{ab}$ , and  $u(x) \in \Delta$  since  $x \in D_{yc}$ . Similarly  $u(x) \in \Delta$  if  $x$  and  $b$  are on opposite sides of  $D_{ac}$ , or  $x$  and  $a$  are on opposite sides of  $D_{bc}$ . Finally, if none of these cases hold, then  $D_{xc}$  cannot be parallel to  $D_{ab}$ , because if  $x$  is in the direction of  $a - b$  from  $c$  (that is, if  $x = c + \xi(a - b)$  for  $\xi > 0$ ) then  $x$  and  $b$  are on opposite sides of  $D_{ac}$ , and if  $x$  is in the direction of  $b - a$  from  $c$  then  $x$  and  $a$  are on opposite sides of  $D_{bc}$ . Therefore  $D_{xc}$  and  $D_{ab}$  intersect at a unique point  $y$  by (4.1.6) and  $u(x) \in \Delta$  as above. Since  $x$  was arbitrary, this means  $u(E) \subseteq \Delta$ , contradicting the hypothesis about  $u(E)$ .
- By hypothesis,  $u(D_{ab}) \subseteq D_{u(a)u(b)}$ . If  $u$  is bijective and  $x \in D_{u(a)u(b)}$ , then  $x = u(c)$  for some  $c \in D_{ab}$  by the previous item, so  $D_{u(a)u(b)} \subseteq u(D_{ab})$ . If  $D_1, D_2$  are distinct and parallel, then  $D_1 \cap D_2 = \emptyset$ , so  $u(D_1) \cap u(D_2) = \emptyset$  by injectivity of  $u$ , so  $u(D_1), u(D_2)$  are distinct and parallel. If  $D_1, D_2$  are not parallel, they intersect at a unique point  $a$ . If  $b \in D_1$  and  $c \in D_2$  with  $b, c \neq a$ , then  $a, b, c$  are not collinear, so  $u(a), u(b), u(c)$  are not collinear by the previous item and  $u(D_1), u(D_2)$  are not parallel.
- Let  $a_1, a_2$  be basis vectors of  $E$ . Then  $0, a_1, a_2$  are not collinear (4.1.1), so  $u(0), u(a_1), u(a_2)$  are not collinear by a previous item, so  $a'_1 = u(a_1) - u(0)$  and  $a'_2 = u(a_2) - u(0)$  are basis vectors of  $E$ . Let  $w \in \mathbf{GL}(E)$  map  $a'_1 \mapsto a_1$  and  $a'_2 \mapsto a_2$  (4.1.10) and let  $v = w \circ t_{-u(0)} \in \mathbf{GA}(E)$  (3.2.19). Then  $u_1 = v \circ u$  fixes  $0, a_1, a_2$ .  $u_1$  operates as claimed on lines by (3.2.17) and the previous item, and the observation that, for example,

$$u_1(D_{a_1 a_2}) = D_{u_1(a_1)u_1(a_2)} = D_{a_1 a_2}$$

- If  $L_1$  is the line through  $\eta a_1$  with direction vector  $a_2 - a_1$ , then  $L_1 \cap D_{0a_2} = \{\eta a_2\}$ , so  $\eta a_2$  is constructible. If  $L_2$  is the line through  $\xi a_1$  with direction vector  $a_2$ , and  $L_3$  is the line through  $\eta a_2$  with direction vector  $a_1$ , then  $L_2 \cap L_3 = \{\xi a_1 + \eta a_2\}$ , so  $\xi a_1 + \eta a_2$  is constructible. If  $L_4$  is the line through  $\xi a_1 + \eta a_2$  with direction vector  $a_2 - a_1$ , then  $L_4 \cap D_{0a_1} = \{(\xi + \eta)a_1\}$ , so  $(\xi + \eta)a_1$  is constructible. If  $L_5$  is the line through  $\eta a_2$  with direction vector  $a_2 - \xi a_1$ , then  $L_5 \cap D_{0a_1} = \{\xi \eta a_1\}$ , so  $\xi \eta a_1$  is constructible.
- By previous items,  $u_1(L_2)$  is the line passing through  $u_1(\xi a_1)$  parallel to  $L_2$  (and to  $D_{0a_2}$ ), and  $u_1(L_3)$  is the line passing through  $u_1(\eta a_2)$  parallel to  $L_3$

(and to  $D_{0a_1}$ ). Since  $u_1(\eta a_2) \in D_{0a_2}$  and  $u_1(\xi a_1) \in D_{0a_1}$ , it follows that  $u_1(\xi a_1) + u_1(\eta a_2)$  is the intersection of  $u_1(L_2)$  and  $u_1(L_3)$ . But we also have  $u_1(\xi a_1 + \eta a_2) \in u_1(L_2 \cap L_3) = u_1(L_2) \cap u_1(L_3)$ , so

$$u_1(\xi a_1 + \eta a_2) = u_1(\xi a_1) + u_1(\eta a_2)$$

Now  $u_1(L_4)$  is the line passing through this point and  $u_1((\xi + \eta)a_1)$  parallel to the line  $u_1(L_1)$ , which in turn passes through  $u_1(\eta a_1)$  and  $u_1(\eta a_2)$  parallel to  $L_1$  (and to  $D_{a_1 a_2}$ ). Therefore

$$u_1((\xi + \eta)a_1) = u_1(\xi a_1) + u_1(\eta a_1)$$

is the intersection of  $u_1(L_4)$  and  $D_{0a_1}$ . It follows that  $\varphi(\xi + \eta) = \varphi(\xi) + \varphi(\eta)$ .

We claim  $u_1(\eta a_2) = \varphi(\eta)a_2$ . Indeed,  $u_1(\eta a_2) = \lambda a_2$  for some  $\lambda$ , and also  $u_1(\eta a_2) = u_1(\eta a_1) + \mu(a_2 - a_1) = \varphi(\eta)a_1 + \mu(a_2 - a_1)$  for some  $\mu$  by facts about  $u_1(L_1)$  above. It follows that  $\lambda = \mu = \varphi(\eta)$  by linear independence of  $a_1, a_2$ , establishing the claim.

Now  $u_1(L_5)$  is the line passing through  $u_1(\xi \eta a_1)$  and  $u_1(\eta a_2)$ , and with the direction vector  $u_1(\xi a_1) - a_2$ . Therefore

$$u_1(\xi \eta a_1) = u_1(\eta a_2) + \varphi(\eta)[u_1(\xi a_1) - a_2] = \varphi(\eta)u_1(\xi a_1)$$

is the intersection of  $u_1(L_5)$  and  $D_{0a_1}$ . It follows that  $\varphi(\xi \eta) = \varphi(\xi)\varphi(\eta)$ . Since  $\varphi(1) = 1$ ,  $\varphi$  is a field automorphism of  $\mathbf{R}$  and hence  $\varphi = 1_{\mathbf{R}}$ .

By the above,  $u_1$  fixes  $D_{0a_1}$  and  $D_{0a_2}$  pointwise. If  $x = \xi_1 a_1 + \xi_2 a_2$ , then  $x$  is the intersection of the line through  $\xi_1 a_1$  parallel to  $D_{0a_2}$  and the line through  $\xi_2 a_2$  parallel to  $D_{0a_1}$ . It follows that  $u_1(x)$  is the intersection of the same lines, so  $u_1(x) = x$ . Therefore  $u_1 = 1_E$ , and  $u = v^{-1}$  is affine.  $\square$

**Exercise (8).** We have

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_1 & \beta_2 \end{pmatrix} = \begin{pmatrix} \alpha_1 \beta_1 & \alpha_1 \beta_2 \\ \alpha_2 \beta_1 & \alpha_2 \beta_2 \end{pmatrix}$$

This can be interpreted as the composite of a linear form  $E \rightarrow \mathbf{R}$  and a linear map  $\mathbf{R} \rightarrow E$ ; it maps the plane onto the origin or onto a vector line and hence is of rank 0 or 1.

**Exercise (9).** If  $f : E \rightarrow E$  is a function which commutes with all automorphisms in  $\mathbf{GL}(E)$ , then  $f = h_\lambda$  for some  $\lambda \in \mathbf{R}$ .

*Proof.* First, since  $f$  commutes with  $h_2 \in \mathbf{GL}(E)$ ,

$$f(0) = f(2 \cdot 0) = 2 \cdot f(0) = f(0) + f(0)$$

and it follows that  $f(0) = 0$ . Now  $f(\alpha x) = \alpha f(x)$  for all  $x \in E$  and  $\alpha \in \mathbf{R}$ , using the previous result for  $\alpha = 0$  and commutativity of  $f$  with  $h_\alpha \in \mathbf{GL}(E)$  for  $\alpha \neq 0$ .

Fix  $x \neq 0$  and let  $u \neq 1$  be a transvection in the line  $D = D_{0x}$ . Then  $u(f(x)) = f(u(x)) = f(x)$ , so  $f(x) \in D$  and hence  $f(x) = \lambda x$  for some  $\lambda \in \mathbf{R}$ . It follows that  $f(y) = \lambda y$  for all  $y \in D$ . Fix  $y \notin D$ . We may assume  $u(y) = x + y$ . By reasoning as above,  $f(y) = \lambda_y y$  and  $f(x + y) = \lambda_{x+y}(x + y)$  for some  $\lambda_y, \lambda_{x+y} \in \mathbf{R}$ . But also

$$f(x + y) = f(u(y)) = u(f(y)) = u(\lambda_y y) = \lambda_y u(y) = \lambda_y(x + y)$$

so  $\lambda_y = \lambda_{x+y}$  (since  $x + y \neq 0$ ). Now considering the transvection  $u'$  in the line  $D' = D_{0y}$  with  $u'(x) = x + y$ , it follows that  $\lambda = \lambda_{x+y} = \lambda_y$ . Therefore  $f(y) = \lambda y$  for all  $y \in E$ , so  $f = h_\lambda$ .  $\square$

## Section 2

*Remark.* In (4.2.5), if  $a \in E$ ,  $a \neq 0$  and  $f(x) = 0$  is an equation of  $D_{0a}$  with  $f \in E^*$ ,  $f \neq 0$  (3.3.6), choose  $b \in E$  with  $f(b) = 1$ . Then  $\{a, b\}$  is a basis of  $E$ . Let  $\Psi$  be the alternating bilinear form on  $E$  with  $\Psi(a, b) = 1$ . Then for all  $x = \xi a + \eta b$ ,

$$\Psi(a, x) = \eta = f(x)$$

Since  $\Psi$  is essentially the determinant (4.2.9) and the determinant measures (oriented) area, this just says that  $x$  is on the vector line determined by  $a$  if and only if the parallelogram determined by  $x$  and  $a$  has zero area.

*Remark.* An alternating bilinear form  $\Psi \neq 0$  on  $E$  captures linear independence in that  $\Psi(x, y) \neq 0$  if and only if  $\{x, y\}$  is independent (4.2.5). It follows that if  $u \in \text{End}(E)$ , then  $\det(u) \neq 0$  if and only if  $u$  preserves linear independence (4.2.6.1), which is true if and only if  $u$  is bijective (4.1.8). This is just (4.2.8).

**Exercise (3).** Relative to a fixed basis of  $E$ , let  $u$  and  $v$  be defined by

$$M(u) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad M(v) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Then  $uv = v$ , so  $\text{rank}(uv) = 1$ , and  $vu = 0$ , so  $\text{rank}(vu) = 0$ , but  $\lambda = 0$  is the only eigenvalue of  $uv$  and  $vu$ , so  $\lambda^2 = 0$  is the characteristic equation of  $uv$  and  $vu$ .

**Exercise (5).** If  $u \in \text{End}(E)$  has eigenvalues, there is a basis of  $E$  relative to which

$$M(u) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad \text{or} \quad M(u) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

If, relative to some basis of  $E$ ,

$$M(u) = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$$

with  $\alpha \neq 0$ , then

$$M(u) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

relative to some basis of  $E$ , but there is no basis of  $E$  relative to which

$$M(u) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

*Proof.* If  $u$  has two distinct eigenvalues  $\lambda, \mu$  with eigenvectors  $a, b$  respectively, then  $\{a, b\}$  is a basis of  $E$  and

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$$

Otherwise,  $u$  has just one eigenvalue  $\lambda$ . If there are two linearly independent eigenvectors  $a, b$ , then

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$$

Note  $M(u)$  has this form only if  $u = h_\lambda$ . If there are not two linearly independent eigenvectors, let  $a$  be an eigenvector and  $\{a, x\}$  a basis of  $E$ . Then  $u(x) = \alpha a + \lambda x$  with  $\alpha \neq 0$  by assumption and (4.2.14), so

$$M(u, \{a, x\}) = \begin{pmatrix} \lambda & \alpha \\ 0 & \lambda \end{pmatrix}$$

If  $b = \alpha^{-1}x$ , then  $\{a, b\}$  is a basis of  $E$  and

$$M(u, \{a, b\}) = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Note  $M(u)$  has this form only if  $u \neq h_\lambda$ . □

# Chapter V

## Section 1

*Remark.* For all  $x, y \in E$ ,

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$$

*Proof.* Bilinearity of the inner product. □

This is called the *parallelogram law* because it shows that the sum of the squares of the lengths of the diagonals of a parallelogram is equal to the sum of the squares of the lengths of the sides.

*Remark.* If  $a, b \in E$  with  $a \neq b$  and  $x = \alpha a + \beta b$  with  $\alpha, \beta \geq 0$  and  $\alpha + \beta = 1$  is a point on the segment  $ab$  (see remark from (3.3) above), then

$$\alpha = \frac{\|x - b\|}{\|a - b\|} \quad \beta = \frac{\|x - a\|}{\|a - b\|}$$

In other words,  $\alpha$  is the ratio of the lengths of the segments  $xb$  and  $ab$ , and  $\beta$  that of  $xa$  and  $ab$ .

*Proof.* For example,  $x - b = \alpha a + (\beta - 1)b = \alpha(a - b)$ , so  $\|x - b\| = \alpha\|a - b\|$ . □

*Remark.* In (5.1.8),  $H(x) = x + H_0$  and  $D(x) = x + D_0$ . To see that the unique point  $y \in D \cap H(x)$  (3.3.8) satisfies the properties for  $x$ , let  $c \in D \cap H$  and observe that  $y - c$  is the orthogonal projection of  $x - c$  on  $D_0$ . Indeed,  $y - c \in D_0$  since  $y, c \in D$  and  $(x - c) - (y - c) = x - y \in H_0$  since  $y \in H(x)$ . It follows (5.1.7) that  $x - y$  is orthogonal to  $y' - y \in D_0$  for all  $y' \in D$  and

$$d(x, y) = d(x - c, y - c) \leq d(x - c, d) = d(x, d + c)$$

for all  $d \in D_0$ , so  $d(x, y) \leq d(x, y')$  for all  $y' \in D$  (since  $c + D_0 = D$ ). Moreover,  $y$  is unique in satisfying these properties for  $x$  since  $y - c$  is unique in satisfying them for  $x - c$  (5.1.7). A similar argument applies for the point  $z \in H \cap D(x)$ .

*Remark.* In (5.1.15), we initially suspect that  $u$  is a translation after a symmetry (5.1.13), and hence a symmetry itself. Since  $u(u(0)) = u^2(0) = 0$ , we suspect that  $u$  is a symmetry about a variety through  $a = \frac{1}{2}u(0)$  perpendicular to  $a$ , which turns out to be the case.

By (5.1.13),

$$\frac{1}{2}(x + u(x)) = a + \frac{1}{2}(x + v(x)) \in a + V$$

and

$$\frac{1}{2}(x + u(x)) - x = a - \frac{1}{2}(x - v(x)) \in a + W = W$$

since  $-v$  is the symmetry about  $W$  and  $a \in W$ . Therefore  $\frac{1}{2}(x + u(x))$  is the unique point in the intersection of  $a + V$  and  $x + W$  (3.3.8).

*Remark.* In (5.1.16),  $t_{2b} = (t_b u_1)(u_1 t_b)$  is the composite of two symmetries about varieties parallel to  $V$  differing by  $b$ . Therefore *translations are just composites of symmetries*.

**Exercise (1).** If  $D$  is a vector line in  $E$  and  $H$  is the orthogonal hyperplane, then  $D$  is the intersection of the vector hyperplanes orthogonal to vector lines in  $H$  (that is, the vector hyperplanes *perpendicular* to  $H$ ). Dually,  $H$  is the union of the vector lines orthogonal to  $D$ .

*Proof.* If  $H'$  is a vector hyperplane orthogonal to a vector line  $D' \subseteq H$ , then  $D$  is orthogonal to  $D'$  (since  $D$  is orthogonal to  $H$ ), so  $D \subseteq H'$  (since  $H'$  is the orthogonal subspace of  $D'$ ). Conversely, if  $x$  is in the intersection and  $h \in H$ ,  $h \neq 0$ , then  $x$  is orthogonal to  $h$  since  $x$  is in the vector hyperplane orthogonal to the vector line  $D_{0h}$ . The dual follows by taking orthogonal subspaces.  $\square$

**Exercise (2).** If  $a$  and  $a'$  are diametrically opposed points on a sphere  $S$ , then a necessary and sufficient condition for a point  $x$  to be on  $S$  is that  $x - a$  and  $x - a'$  be orthogonal.

*Proof.* Without loss of generality, we may assume that  $S$  is centered at the origin with radius  $\rho$ . Then  $a' = -a$ , so

$$(x - a \mid x - a') = \|x\|^2 - \rho^2 = 0 \iff \|x\| = \rho \iff x \in S \quad \square$$

**Exercise (4).** *Powers and coorthogonal spheres:*

- Let  $S$  be a sphere centered at  $c$  with radius  $\rho$ . If  $a \in E$  with  $\delta = d(a, c)$  and  $D$  is a line through  $a$  meeting  $S$  at distinct points  $x_1, x_2$ , then

$$(x_1 - a \mid x_2 - a) = \delta^2 - \rho^2$$

If instead  $D$  meets  $S$  at the single point  $x$ , then

$$d(a, x)^2 = \delta^2 - \rho^2$$

- If  $S_1, S_2$  are non-concentric spheres with respective centers  $c_1, c_2$  and radii  $\rho_1, \rho_2$ , then the set  $H$  of points whose powers with respect to  $S_1$  and  $S_2$  are equal is a hyperplane perpendicular to  $D_{c_1 c_2}$  and containing  $S_1 \cap S_2$ .

- Let  $\pi_k(x) = d(x, c_k)^2 - \rho_k^2$  denote the power of  $x$  with respect to  $S_k$  (for  $k = 1, 2$ ). Then the following are equivalent:

- (a)  $\pi_2(c_1) = \rho_1^2$
- (b)  $\pi_1(c_2) = \rho_2^2$
- (c)  $\emptyset \neq S_1 \cap S_2 \subseteq \{x \mid (x - c_1 \mid x - c_2) = 0\}$
- (d)  $\pi_1(x) + \pi_2(x) = 2(x - c_1 \mid x - c_2)$

*Proof.*

- Without loss of generality, we may assume that  $c = 0$ . In the first case, let  $b = \frac{1}{2}(x_1 + x_2)$  and  $y = \frac{1}{2}(x_1 - x_2)$ . Then

$$(b \mid y) = \frac{1}{4}(\|x_1\|^2 - \|x_2\|^2) = \frac{1}{4}(\rho^2 - \rho^2) = 0$$

Since  $y$  is a direction vector of  $D$ , it follows that  $b$  is perpendicular to  $D$ . Now  $x_1 = b + y$  and  $x_2 = b - y$ , so  $x_1 - a = (b - a) + y$  and  $x_2 - a = (b - a) - y$ , hence

$$\begin{aligned} (x_1 - a \mid x_2 - a) &= ((b - a) + y \mid (b - a) - y) \\ &= (b - a \mid b - a) - \|y\|^2 \\ &= \delta^2 - (\|b\|^2 + \|y\|^2) && \text{as } (b \mid b - a) = 0, (b \mid a) = \|b\|^2 \\ &= \delta^2 - \rho^2 && \text{by Pythagoras (5.1.5)} \end{aligned}$$

In the second case,  $D$  must be perpendicular to  $x$ . Indeed, if not, let  $d$  be a direction vector of  $D$  with  $(d \mid x) \neq 0$  and consider

$$z = x - \frac{2(d \mid x)}{\|d\|^2} d$$

Then  $z \in D$ ,  $z \neq x$ , and expansion of  $(z \mid z)$  shows that  $\|z\|^2 = \|x\|^2$ , so  $\|z\| = \rho$  and  $z \in S$ , contradicting that  $D$  is tangential to  $S$  at  $x$ . It follows that  $(x \mid a - x) = 0$ , and  $d(a, x)^2 = \delta^2 - \rho^2$  by Pythagoras (5.1.5).

- We first find  $c \in H \cap D_{c_1 c_2}$ . We must have  $c = \alpha c_1 + \beta c_2$  with  $\alpha, \beta \geq 0$  and  $\alpha + \beta = 1$ . In fact,  $\alpha = \|c - c_2\|/\delta$  and  $\beta = \|c - c_1\|/\delta$  where  $\delta = \|c_1 - c_2\| \neq 0$ ,

so that

$$\begin{aligned}
\alpha - \beta &= (\alpha - \beta)(\alpha + \beta) \\
&= \alpha^2 - \beta^2 \\
&= \frac{\|c - c_2\|^2 - \|c - c_1\|^2}{\delta^2} \\
&= \frac{(\|c - c_2\|^2 - \rho_2^2) + \rho_2^2 - (\|c - c_1\|^2 - \rho_1^2) - \rho_1^2}{\delta^2} \\
&= \frac{\rho_2^2 - \rho_1^2}{\delta^2}
\end{aligned}$$

since  $c \in H$ . It follows that

$$\alpha = \frac{1}{2} + \frac{\rho_2^2 - \rho_1^2}{2\delta^2} \quad \beta = \frac{1}{2} - \frac{\rho_2^2 - \rho_1^2}{2\delta^2} \quad c = \frac{1}{2}(c_1 + c_2) + \frac{\rho_2^2 - \rho_1^2}{2\delta^2}(c_1 - c_2)$$

Note  $c$  is the midpoint of  $c_1$  and  $c_2$  if and only if  $\rho_1 = \rho_2$ . Now

$$\begin{aligned}
x \in H &\iff \|x - c_1\|^2 - \rho_1^2 = \|x - c_2\|^2 - \rho_2^2 \\
&\iff (x - c_2 \mid x - c_2) - (x - c_1 \mid x - c_1) = \rho_2^2 - \rho_1^2 \\
&\iff (x \mid c_1 - c_2) = \frac{1}{2}(\|c_1\|^2 - \|c_2\|^2 + \rho_2^2 - \rho_1^2) = (c \mid c_1 - c_2)
\end{aligned}$$

Therefore  $H$  is the hyperplane through  $c$  perpendicular to  $D_{c_1 c_2}$ . Finally,  $S_1 \cap S_2 \subseteq H$  since  $S_1 \cap S_2$  consists of the points whose powers with respect to  $S_1$  and  $S_2$  are zero.

- Let  $\delta = \|c_1 - c_2\|$ . We have

$$\pi_2(c_1) = \delta^2 - \rho_2^2 = \rho_1^2 \iff \rho_2^2 = \delta^2 - \rho_1^2 = \pi_1(c_2)$$

so (a)  $\iff$  (b), and these are equivalent to  $\delta^2 = \rho_1^2 + \rho_2^2$ . If this condition holds, then  $\rho_1 - \delta \leq 0$ , so  $2\rho_1(\rho_1 - \delta) \leq 0$  and

$$\begin{aligned}
2\rho_1^2 - 2\rho_1\delta + \rho_2^2 &\leq \rho_2^2 \leq 2\rho_1^2 + 2\rho_1\delta + \rho_2^2 \\
\rho_1^2 - 2\rho_1\delta + \delta^2 &\leq \rho_2^2 \leq \rho_1^2 + 2\rho_1\delta + \delta^2 \\
(\rho_1 - \delta)^2 &\leq \rho_2^2 \leq (\rho_1 + \delta)^2 \\
|\rho_1 - \delta| &\leq \rho_2 \leq \rho_1 + \delta
\end{aligned}$$

so  $S_1 \cap S_2 \neq \emptyset$  (5.1.11.4). If  $x \in S_1 \cap S_2$ , then

$$\|x - c_1\|^2 + \|x - c_2\|^2 = \rho_1^2 + \rho_2^2 = \delta^2 = \|c_1 - c_2\|^2$$



so  $(x - c_1 \mid x - c_2) = 0$  (5.1.1.4). Therefore (a),(b)  $\implies$  (c). Conversely, if  $x \in S_1 \cap S_2$  and  $(x - c_1 \mid x - c_2) = 0$ , then by Pythagoras (5.1.5.1),

$$\delta^2 = \|c_1 - c_2\|^2 = \|x - c_1\|^2 + \|x - c_2\|^2 = \rho_1^2 + \rho_2^2$$

so (c)  $\implies$  (a),(b). Finally, by (5.1.1.4),

$$\begin{aligned} \pi_1(x) + \pi_2(x) &= \|x - c_1\|^2 + \|x - c_2\|^2 - (\rho_1^2 + \rho_2^2) \\ &= 2(x - c_1 \mid x - c_2) + \delta^2 - (\rho_1^2 + \rho_2^2) \end{aligned}$$

so clearly (a),(b)  $\iff$  (d).  $\square$

**Exercise (6).** *Characterization of similitudes:*

- If  $u : E \rightarrow E$  is a bijective function such that  $(u(x) \mid u(y)) = \alpha(x \mid y)$  for all  $x, y \in E$  ( $\alpha > 0$ ), then  $u$  is linear and consequently  $u \in \mathbf{GO}(E)$ .
- If  $u : E \rightarrow E$  is a bijective function such that  $d(u(x), u(y)) = \alpha d(x, y)$  for all  $x, y \in E$  ( $\alpha > 0$ ), then  $u$  is affine and consequently  $u \in \mathbf{Sm}(E)$ .
- If  $u \in \mathbf{GL}(E)$  is such that  $(x \mid y) = 0$  implies  $(u(x) \mid u(y)) = 0$  (in other words,  $u$  preserves orthogonality), then  $u \in \mathbf{GO}(E)$ .

*Proof.*

- By direct computation,

$$\begin{aligned} (u(x + y) - u(x) - u(y) \mid u(x + y) - u(x) - u(y)) &= \\ \alpha(x + y - x - y \mid x + y - x - y) &= 0 \end{aligned}$$

so  $u(x + y) - u(x) - u(y) = 0$  by positive definiteness of the inner product. Similarly  $u(\xi x) - \xi u(x) = 0$ . Therefore  $u$  is linear.

- Let  $v = t_{-u(0)} u$ . Then  $v : E \rightarrow E$  is a bijective function and (5.1.1.5)

$$\begin{aligned} 2(v(x) \mid v(y)) &= \|v(x)\|^2 + \|v(y)\|^2 - \|v(x) - v(y)\|^2 \\ &= \|u(x) - u(0)\|^2 + \|u(y) - u(0)\|^2 - \|u(x) - u(y)\|^2 \\ &= \alpha^2(\|x\|^2 + \|y\|^2 - \|x - y\|^2) \\ &= 2\alpha^2(x \mid y) \end{aligned}$$

Therefore  $v \in \mathbf{GO}(E)$  by the previous item and  $u \in \mathbf{Sm}(E)$  (5.1.14).

- First observe that  $(u(x) | u(y)) = 0$  implies  $(x | y) = 0$ . Indeed, if  $y = 0$  this is trivial. If  $y \neq 0$ , then  $x = \xi y + z$  with  $(z | y) = 0$  (5.1.7) and

$$\begin{aligned}
0 &= (u(x) | u(y)) \\
&= (u(\xi y + z) | u(y)) \\
&= (\xi u(y) + u(z) | u(y)) \\
&= \xi \|u(y)\|^2 && \text{since } (u(z) | u(y)) = 0
\end{aligned}$$

Since  $u(y) \neq 0$ , this implies  $\xi = 0$ , so  $x = z$  and  $(x | y) = 0$ .

Now if  $y \neq 0$ , it follows that  $(u(x) | u(y)) = 0$  is another equation of the hyperplane  $(x | y) = 0$  orthogonal to  $y$ , so (3.3.6) there is  $\mu_y > 0$  with

$$(u(x) | u(y)) = \mu_y (x | y)$$

for all  $x \in E$ . We claim that  $\mu_y$  is independent of  $y$ . Indeed, if  $y' = 0$  then trivially  $(u(x) | u(y')) = \mu_y (x | y')$  for all  $x \in E$ . If  $(y | y') \neq 0$ , then

$$\mu_y = \frac{(u(y) | u(y'))}{(y | y')} = \mu_{y'}$$

Finally, if  $y' \neq 0$  and  $(y | y') = 0$ , then  $(y | y + y') \neq 0$  and  $(y' | y + y') \neq 0$ , so

$$\mu_y = \mu_{y+y'} = \mu_{y'}$$

by the previous case. □

**Exercise (7).** In  $\mathbf{GL}(E)$ , the normalizer of  $\mathbf{O}(E)$  is  $\mathbf{GO}(E)$ .

*Proof.* The normalizer contains  $\mathbf{GO}(E)$  since if  $v \in \mathbf{GO}(E)$  and  $u \in \mathbf{O}(E)$ , then  $\mu(vuv^{-1}) = 1$  (5.1.12.3), so  $vuv^{-1} \in \mathbf{O}(E)$ .

Conversely, suppose  $v \in \mathbf{GL}(E)$  normalizes  $\mathbf{O}(E)$ . If  $(x | y) = 0$  and  $x = 0$ , then trivially  $(v(x) | v(y)) = 0$ . If  $x \neq 0$ , let  $u \in \mathbf{O}(E)$  be the symmetry about the hyperplane orthogonal to  $x$ , so  $u(x) = -x$  and  $u(y) = y$ . Then  $vuv^{-1} \in \mathbf{O}(E)$  by hypothesis and  $vuv^{-1}v = vu$ , so

$$(v(x) | v(y)) = (v(u(x)) | v(u(y))) = -(v(x) | v(y))$$

and hence  $(v(x) | v(y)) = 0$ . Since  $x, y$  were arbitrary, it follows that  $v$  preserves orthogonality, and therefore  $v \in \mathbf{GO}(E)$  (Exercise 6). □

## Section 2

*Remark.* In the plane, two lines are parallel if and only if there is a line to which they are both perpendicular; in this case, they are perpendicular to the same lines.

*Proof.* Parallel lines have the same direction, hence are perpendicular to the same lines, in particular the vector line orthogonal to that direction. Conversely, if two lines are both perpendicular to a third, then their direction is the vector line orthogonal to the direction of the third, hence they are parallel (5.2.2).  $\square$

*Remark.* In the plane, a line is tangential to a circle at a point if and only if the line intersects the circle at the point and is perpendicular to the line passing through the point and the center of the circle.

*Proof.* The forward direction is proved in Exercise 5.1.4. The reverse direction follows from Pythagoras (5.1.5).  $\square$

*Remark.* In (5.2.3), the mapping  $f \mapsto a$  is actually a vector space *isomorphism* from the dual space  $E^*$  to  $E$ . In fact, if  $\{a_1, a_2\}$  is an orthonormal basis of  $E$ , then the isomorphism is just that determined by  $a_i^* \mapsto a_i$ , where  $\{a_1^*, a_2^*\}$  is the dual basis (4.1.15).

*Remark.* In (5.2.4), we have vector space isomorphisms

$$\mathcal{B}(E, E; \mathbf{R}) \cong \text{Hom}(E, E^*) \cong \text{End}(E)$$

*Remark.* In the complex plane  $\mathbf{C}$ , a number  $z \in \mathbf{C}$  induces an endomorphism through multiplication, and its conjugate  $\bar{z}$  induces the adjoint endomorphism. Note  $\bar{\bar{z}} = z$  as in (5.2.5.2),  $\overline{wz} = \bar{w} \bar{z}$  as in (5.2.5.3),  $\bar{z}z = |z|^2$  is the “multiplier” of  $z$  as in (5.2.5.5), and  $z$  is “orthogonal” if it is on the unit circle. If  $z \neq 0$ , then  $z = \rho u$  with  $\rho = |z| > 0$  and  $u = z/|z|$  on the unit circle uniquely determined, as in the polar decomposition (5.1.12).

Endomorphisms of a plane can therefore be viewed as generalized complex numbers, with adjoints as generalized complex conjugates. This idea is made precise in (5.5).

*Remark.* In (5.2.8),  $M(w) = (\alpha_{ij})$  with  $\alpha_{11} = \alpha_{22} = 0$  and  $\alpha_{12} = -\alpha_{21}$  relative to an orthonormal basis (5.2.6), so  $\det(w) = \alpha_{12}^2$  (4.2.9.2). Since  $w \neq 0$ ,  $\det(w) > 0$  and  $w$  is invertible (4.2.8).

**Exercise (1).** If  $u \in \mathbf{GL}(E)$  is an involution, then  $u \in \mathbf{O}(E)$  if and only if  $u$  is self-adjoint. If  $p \in \text{End}(E)$  is idempotent, then  $p$  is an orthogonal projection (5.1.6) if and only if  $p$  is self-adjoint.

*Proof.* By (5.2.5.6),  $u \in \mathbf{O}(E)$  if and only if  $u^* = u^{-1} = u$ . By Exercise 3.2.1,  $p$  is the projection onto  $p(E)$  in the direction of  $p^{-1}(0)$ . If  $p(E)$  is orthogonal to  $p^{-1}(0)$ , then

$$\begin{aligned} (p(x) | y) &= (p(x) | p(y) + (y - p(y))) \\ &= (p(x) | p(y)) && \text{since } (p(x) | y - p(y)) = 0 \\ &= (p(x) + (x - p(x)) | p(y)) && \text{since } (x - p(x) | p(y)) = 0 \\ &= (x | p(y)) \end{aligned}$$

for all  $x, y \in E$ . Therefore  $p^* = p$ . Conversely if  $p^* = p$ , then for all  $x \in E$  and  $y \in p^{-1}(0)$ ,

$$(p(x) | y) = (x | p(y)) = (x | 0) = 0$$

so  $p(E)$  is orthogonal to  $p^{-1}(0)$ . □

**Exercise (8).** Suppose  $u \in \text{End}(E)$  has eigenvalues  $\lambda_1 \neq \lambda_2$  with corresponding eigenvectors  $a_1, a_2$ . If  $b_1$  (resp.  $b_2$ ) is nonzero and orthogonal to  $a_1$  (resp.  $a_2$ ), then  $b_1$  (resp.  $b_2$ ) is an eigenvector of  $u^*$  corresponding to  $\lambda_2$  (resp.  $\lambda_1$ ).

*Proof.* Since  $\lambda_1 \neq \lambda_2$ ,  $\{a_1, a_2\}$  must be linearly independent and hence a basis. If  $(a_1 | x) = (a_1 | y)$  and  $(a_2 | x) = (a_2 | y)$ , then for any  $z = \alpha_1 a_1 + \alpha_2 a_2$ ,

$$(z | x) = \alpha_1 (a_1 | x) + \alpha_2 (a_2 | x) = \alpha_1 (a_1 | y) + \alpha_2 (a_2 | y) = (z | y)$$

and it follows that  $x = y$ .

Now observe that for  $z = a_1, a_2$ ,

$$(z | u^*(b_1)) = (u(z) | b_1) = \lambda_2 (z | b_1) = (z | \lambda_2 b_1)$$

so  $u^*(b_1) = \lambda_2 b_1$ . Similarly  $u^*(b_2) = \lambda_1 b_2$ . □

### Section 3

*Remark.* In (5.3.1), if we fix an *orientation* of the plane (4.3.1), then a similitude preserves orientation if it is direct and reverses orientation if it is inverse (4.3.2). In other words,  $\mathbf{GO}^+(E) = \mathbf{GO}(E) \cap \mathbf{GL}^+(E)$ . If  $u \in \mathbf{GO}(E)$ , then the following are equivalent:

- $u$  is a rotation
- $u$  preserves length and orientation
- $u$  preserves length and oriented area<sup>2</sup>
- $u$  preserves orientation and oriented area

*Remark.* By the polar decomposition (5.1.12) and (5.3.2), an affine similitude of the plane is the composite of a translation after a linear scaling after a linear rotation or reflection. If the scaling is trivial, then the similitude is an affine isometry; otherwise, it is a linear similitude about another origin (5.3.6).

*Remark.* We consider products of affine isometries  $u, v \in \mathbf{Is}(E)$  with  $u, v \neq 1$ :

- If  $u$  is a translation and  $v$  is a rotation, then  $uv$  is a rotation about another origin (5.3.7).
- If  $u$  is a translation and  $v$  is a reflection, then  $uv$  is a glide reflection (5.3.7).
- If  $u$  and  $v$  are rotations, then  $uv$  is a translation or a rotation (5.3.7).
- If  $u$  is a rotation and  $v$  is a reflection, then  $uv$  is a glide reflection (5.3.7).
- If  $u$  and  $v$  are reflections in parallel lines, then  $uv$  is a translation in a direction orthogonal to that of the lines (5.1.16).
- If  $u$  and  $v$  are reflections in non-parallel lines, then  $uv$  is a rotation about the point of intersection of the lines. This follows from (5.3.2) after taking the origin at the point of intersection.

**Exercise (2).** If  $u \in \text{End}(E)$  is normal, then  $vu v^{-1}$  is normal for all  $v \in \mathbf{GO}(E)$ . Also,  $u$  is self-adjoint or a direct similitude.

*Proof.* By (5.2.5.5),  $v^* v = \alpha \cdot 1$  for some  $\alpha > 0$ , so  $v^* = \alpha v^{-1}$ ,  $(v^{-1})^* = \alpha^{-1} v$ , and

$$\begin{aligned}
 (vu v^{-1})^* vu v^{-1} &= (v^{-1})^* u^* v^* vu v^{-1} \\
 &= vu^* v^{-1} vu v^{-1} \\
 &= vu^* uv^{-1} \\
 &= vu u^* v^{-1} && \text{since } u \text{ is normal} \\
 &= vu v^{-1} vu^* v^{-1} \\
 &= vu v^{-1} (vu v^{-1})^*
 \end{aligned}$$

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<sup>2</sup>By definition,  $u$  preserves oriented area if  $u \in \mathbf{SL}(E)$ .

Therefore  $vu v^{-1}$  is normal.

If  $M(u) = (\alpha_{ij})$  relative to some fixed orthonormal basis, then  $M(u^*) = (\alpha_{ji})$  (5.2.6.3) and

$$M(u^*)M(u) = M(u^*u) = M(uu^*) = M(u)M(u^*)$$

implies  $\alpha_{12} = \pm\alpha_{21}$  and  $\alpha_{11}(\alpha_{12} - \alpha_{21}) = \alpha_{22}(\alpha_{12} - \alpha_{21})$ . If  $\alpha_{12} = \alpha_{21}$ , then  $M(u)$  is symmetric and  $u$  is self-adjoint. If  $\alpha_{12} \neq \alpha_{21}$ , then  $\alpha_{12} = -\alpha_{21}$  and  $\alpha_{11} = \alpha_{22}$ , so

$$M(u^*u) = (\alpha_{11}^2 + \alpha_{12}^2) \cdot 1 = \det(u) \cdot 1$$

Therefore  $u^*u = \det(u) \cdot 1$ , so  $u$  is a direct similitude.  $\square$

**Exercise (5).** Every affine isometry of the plane is the product of at most three reflections in lines. The product of two affine rotations is a translation if and only if the associated linear rotations are mutually inverse.

*Proof.* If  $u \in \mathbf{Is}(E)$ , then by cases (5.3.7):

- If  $u$  is the identity, then  $u$  is trivially the product of zero reflections.
- If  $u$  is a nonzero translation, then  $u$  is the product of two reflections about lines perpendicular to the direction of translation (5.1.16).
- If  $u$  is a rotation, then  $u$  is the product of two reflections about lines intersecting at the center of the rotation (5.3.2).
- If  $u$  is a glide reflection, then  $u$  is a product of one or three reflections depending as the translation is zero or nonzero.

If  $u_1, u_2 \in \mathbf{Is}(E)$  are affine rotations, then  $u_1 = t_1 v_1$  and  $u_2 = t_2 v_2$  where  $t_1, t_2$  are translations and  $v_1, v_2 \in \mathbf{O}^+(E)$ . By (3.2.19.1),  $u_1 u_2 = t v_1 v_2$ , where  $t$  is a translation, so  $u_1 u_2$  is a translation if and only if  $v_1 v_2$  is, which is true if and only if  $v_1 v_2 = 1$ .  $\square$

## Section 4

*Remark.* In (5.4.3), recall that  $\Psi(x, y)$  is the *oriented area* of the parallelogram determined by  $x$  and  $y$  relative to the fixed orientation of the plane (see the remark on (4.2.5) above). Therefore if  $u$  is a rotation and  $x$  is a unit vector, then  $\Psi(x, u(x))$  is the signed distance from  $u(x)$  to the line determined by  $x$ —that is, the sine of the angle  $\widehat{(x, u(x))}$  of rotation.

*Remark.* In (5.4.10), a line  $D'$  passing through  $-a_1$  is tangential to  $\mathbf{U}$  if and only if it is parallel to  $D$ , by remarks from (5.2) above. Therefore  $D'$  meets  $\mathbf{U}$  at a unique point  $x$  distinct from  $-a_1$  if and only if it meets  $D$  at a unique point  $y$ .

For  $x \in \mathbf{U}'$ ,  $(x \mid x + a_1) = (a_1 \mid x + a_1)$ , so it follows from (5.1.7) and (5.4.8) that the direction of  $D_{-a_1, x}$  bisects  $\Delta_{0a_1}$  and  $\Delta_{0x}$  and  $\widehat{(a_1, x)} = 2\widehat{(a_1, x + a_1)}$ . It is then immediate that  $x = \cos 2\theta \cdot a_1 + \sin 2\theta \cdot a_2$  and  $y = \tan \theta \cdot a_2$ , where  $\theta = \widehat{(a_1, x + a_1)}$ .

*Remark.* In (5.4.13), where  $\theta', \theta''$  correspond to  $\Delta', \Delta''$ , we have  $\theta' = \widehat{(-\Delta_0, \Delta')}$  and  $\theta'' = \widehat{(-\Delta_0, \Delta'')}$  (5.4.12), so

$$\theta'' - \theta' = -\theta' + \theta'' = \widehat{(\Delta', -\Delta_0)} + \widehat{(-\Delta_0, \Delta'')} = \widehat{(\Delta', \Delta'')}$$

Let  $x', x''$  be unit vectors in  $\Delta', \Delta''$ . Then by (4.3.5) and (5.4.9),  $S^\circ(\Delta', \Delta'')$  is minor if and only if

$$\sin(\theta'' - \theta') = \sin \widehat{(\Delta', \Delta'')} = \sin \widehat{(x', x'')} = \Psi(x', x'') > 0$$

which is equivalent to  $\theta'' - \theta' > 0$  (5.4.12). Similarly,  $S^\circ(\Delta', \Delta'')$  is major if and only if  $\theta'' - \theta' < 0$ .

**Exercise (7).** In the set of pairs of vector half-lines (resp. vector lines), the orbits under the action of  $\mathbf{O}^+(E)$  are the sets of pairs with the same angle; under  $\mathbf{O}(E)$ , they are the sets of pairs with the same or opposite angle.

*Proof.* By (5.4.7) and (5.3.2). □

## Section 5

**Exercise (3).** *Cayley's parametric representation:* If  $z \in \mathbf{C}$  with  $|z| = 1$  and  $z \neq -1$ , then there exists  $\xi \in \mathbf{R}$  unique such that

$$z = (1 + \xi i)(1 - \xi i)^{-1}$$

*Proof.* Let  $\theta = \arg z$ ,  $\xi = \tan(\theta/2)$ , and  $w = 1 + \xi i$ . By (5.4.5.3),

$$\begin{aligned} w\overline{w}^{-1} &= \frac{w^2}{|w|^2} \\ &= \frac{(1 + \xi i)^2}{1 + \xi^2} \\ &= \frac{1 - \xi^2}{1 + \xi^2} + \frac{2\xi}{1 + \xi^2}i \\ &= \cos \theta + i \sin \theta = z \end{aligned}$$

If  $z = (1 + \xi' i)(1 - \xi' i)^{-1}$ , direct computation shows  $\xi' = \xi$ . □

## References

- [1] Dieudonné, J. *Linear Algebra and Geometry*. Hermann, 1969.