# Exercises from *Mathematical Logic*

John Peloquin

## Chapter II

### Section 1

EXERCISE 1.3. **R** is uncountable.

*Proof.* Let $a, b \in \mathbf{R}$ with $a < b$, and $I = [a, b]$. We show that for any given $\alpha : \mathbf{N} \to \mathbf{R}$, there exists an $r \in I$ such that $r \notin \{\alpha(n) \mid n \in \mathbf{N}\}$. Thus there is no surjection from **N** onto $I$, and therefore no surjection from **N** onto **R**. It follows from (1.1) that **R** is uncountable, since **R** is clearly not finite.

Let $\alpha : \mathbf{N} \to \mathbf{R}$. We define a sequence of subsets $I = I_0 \supseteq I_1 \supseteq \cdots$ inductively as follows:

$$I_0 = I$$
$$I_{n+1} = I_n - \{\alpha(n)\}$$

It follows that $\alpha(n) \notin I_{n+1}$ for all $n \in \mathbf{N}$. Now by the completeness of **R**, we know that $J = \bigcap_{n \in \mathbf{N}} I_n \neq \emptyset$. Choose $r \in J$. If there exists an $n \in \mathbf{N}$ such that $\alpha(n) = r$, then $\alpha(n) \in J \subseteq I_{n+1}$. So $\alpha(n) \in I_{n+1}$—a contradiction. Thus $r \notin \{\alpha(n) \mid n \in \mathbf{N}\}$ as desired. □

EXERCISE 1.4.

(a) If the sets $M_0, M_1, \ldots$ are at most countable, then the union $U = \bigcup_{n \in \mathbf{N}} M_n$ is at most countable.

(b) If $\mathscr{A}$ is an at most countable alphabet, then $\mathscr{A}^*$ (the set of all strings over $A$) is at most countable.

*Proof.* (a) From (1.1) we know that for each $M_i$ there exists a surjection $\alpha_i : \mathbf{N} \to M_i$. We can therefore enumerate each $M_i$ as

$$M_i = \{\alpha_{i0}, \alpha_{i1}, \alpha_{i2}, \ldots\}$$

and construct the following table:

$$\begin{array}{c|ccccc}
M_0 & \alpha_{00} & \alpha_{01} & \alpha_{02} & \alpha_{03} & \cdots \\
M_1 & \alpha_{10} & \alpha_{11} & \alpha_{12} & \alpha_{13} & \cdots \\
M_2 & \alpha_{20} & \alpha_{21} & \alpha_{22} & \alpha_{23} & \cdots \\
M_3 & \alpha_{30} & \alpha_{31} & \alpha_{32} & \alpha_{33} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}$$

Now we can construct a surjection $\alpha : \mathbf{N} \to U$ by proceeding along the diagonals:

$$U = \{\alpha_{00}, \alpha_{10}, \alpha_{01}, \alpha_{20}, \alpha_{11}, \alpha_{02}, \ldots\}$$

By (1.1), it follows that $U$ is at most countable.

(b) To prove this, we first claim that there are at most countably many strings over $\mathscr{A}$ of length $n$, for all $n \in \mathbf{N}$. We proceed by induction. This is trivially true for $n = 0$ and $n = 1$ since $\mathscr{A}$ is at most countable. Suppose the claim is true for strings of length $n$. Let $S_\alpha^{n+1}$ be the set of all strings of length $n+1$ over $\mathscr{A}$ ending in $\alpha$. Then $S_\alpha^{n+1}$ is at most countable by the induction hypothesis. Now the set $S^{n+1}$ of all strings of length $n+1$ over $\mathscr{A}$ is given by

$$S^{n+1} = \bigcup_{\alpha \in \mathscr{A}} S_\alpha^{n+1}$$

Since $\mathscr{A}$ is at most countable, it follows from part (a) that $S^{n+1}$ is at most countable. Thus our claim is proved by induction.

Now since each string in $\mathscr{A}^*$ is finite by definition, we have

$$\mathscr{A}^* = \bigcup_{n \in \mathbf{N}} S^n$$

Again by part (a), it follows that $\mathscr{A}^*$ is at most countable as desired.

□

EXERCISE 1.5. There is no surjective map from a set $M$ onto its power set $\mathscr{P}(M)$.

*Proof.* We show that for any map $\alpha : M \to \mathscr{P}(M)$, there exists a set $S \in \mathscr{P}(M)$ such that $S$ is not in the range of $\alpha$. Let $\alpha : M \to \mathscr{P}(M)$. Choose $S = \{a \in M \mid a \notin \alpha(a)\}$. Suppose there exists an $s \in M$ such that $\alpha(s) = S$. If $s \in S$, then by definition of $S$, $s \notin \alpha(s) = S$—a contradiction. On the other hand, if $s \notin S$, then $s \notin \alpha(s)$, so by definition of $S$, $s \in S$—a contradiction. In either case, we reach a contradiction, so $S$ is not in the range of $\alpha$ and $\alpha$ is not surjective. □

## Section 4

EXERCISE 4.7. If we alter our formula calculus by omitting parentheses around conjunctions—that is, by writing $\varphi \wedge \psi$ instead of $(\varphi \wedge \psi)$—then formulas do not necessarily have unique decompositions. In particular, the $\{P, Q\}$-formula

$$\chi := \exists v_0 P v_0 \wedge Q v_1$$

does not have a unique decomposition and thus does not have a unique set of subformulas.

*Proof.* We can construct two distinct derivations of $\chi$. The first one is

| | | |
|---|---|---|
| 1. | $Pv_0$ | (F2) using $P, v_0$. |
| 2. | $Qv_1$ | (F2) using $Q, v_1$. |
| 3. | $Pv_0 \wedge Qv_1$ | (F4) modified, applied to (1),(2) using $\wedge$. |
| 4. | $\exists v_0 Pv_0 \wedge Qv_1$ | (F5) applied to (3) using $\exists, v_0$. |

A second derivation uses steps (1) and (2) above, but then uses

| | | |
|---|---|---|
| $3'$. | $\exists v_0 Pv_0$ | (F5) applied to (1) using $\exists, v_0$. |
| $4'$. | $\exists v_0 Pv_0 \wedge Qv_1$ | (F4) modified, applied to (2),($3'$) using $\wedge$. |

Under the first decomposition, $\mathrm{SF}(\chi) = \{\chi, Pv_0 \wedge Qv_1, Qv_1, Pv_0\}$, and under the second decomposition, $\mathrm{SF}(\chi) = \{\chi, \exists v_0 Pv_0, Qv_1, Pv_0\}$. □

# Chapter III

## Section 1

EXERCISE 1.5. Let $A$ be a nonempty finite set and $S$ a finite symbol set. Then there are only finitely many $S$-structures with domain $A$.

*Proof.* Each $S$-structure over $A$ will contain assignments for the relation and function symbols in $S$ as well as the constants in $S$. Now there are only finitely many constant assignments possible; if $m$ is the number of constants in $S$, and $|A|$ is the cardinality of $A$, then there are $|A|^m$ possible constant assignments. Furthermore, there are only finitely many $n$-ary relations on $A$; since any $n$-ary relation $R$ on $A$ is a subset of $A^n$, there are $|P(A^n)| = 2^{|A^n|} = 2^{|A|^n}$ possible relations. Similarly, there are only finitely many $n$-ary functions on $A$. Thus there are only finitely many ways to construct an $S$-structure over $A$. □

## Section 3

EXERCISE 3.4. Every positive $S$-formula has an $S$-interpretation satisfying it.

*Proof.* Let $\varphi$ be a positive $S$-formula. We choose an $S$-interpretation $\mathfrak{I}$ with a domain $A = \{a_0\}$ consisting of one element. Use the constant assignment $\beta(x) = a_0$ for variables in $A_S$; also assign $a_0$ to each constant symbol in $S$. Assign to each $n$-ary relation symbol in $S$ the equality relation, and to each $n$-ary function symbol $f$ the constant function with value $a_0$. It follows by induction that for every $S$-term $t$, $\mathfrak{I}(t) = a_0$.

Now for any terms $t_1, \ldots, t_n \in T^S$, we have $\mathfrak{I} \models t_1 \equiv t_2$, and for any $n$-ary relation symbol $R \in S$, $\mathfrak{I} \models Rt_1, \ldots, t_n$. Thus $\mathfrak{I}$ satisfies all atomic $S$-formulas. Furthermore, if $\mathfrak{I} \models \psi$ and $\mathfrak{I} \models \gamma$, then $\mathfrak{I} \models (\psi \wedge \gamma)$ and $\mathfrak{I} \models (\psi \vee \gamma)$ by definition. Since $\varphi$ is positive, it follows that $\mathfrak{I} \models \varphi$. □

## Section 4

EXERCISE 4.12. Let $\phi$ and $\psi$ be logically equivalent formulas. Then for each formula $\chi$, we define $\chi'$ to be the result of replacing all occurrences of $\phi$ in $\chi$ with $\psi$. This is well-defined, and for all $\chi$, $\chi$ is equivalent to $\chi'$.

*Proof.* We define $'$ by induction on formulas. We use the developments on pp. 35–6 in order to reduce the induction cases considered.

$$\begin{aligned}
\chi = t_1 \equiv t_2,\ \chi = Rt_1,\ldots,t_n: &\quad \chi' = \begin{cases} \psi & \text{if } \chi = \phi \\ \chi & \text{otherwise} \end{cases} \\
\chi = \neg\gamma: &\quad \chi' = \neg\gamma'. \\
\chi = (\gamma \vee \xi): &\quad \chi' = (\gamma' \vee \xi'). \\
\chi = \exists x\gamma: &\quad \chi' = \exists x\gamma'.
\end{aligned}$$

It is easy to see by induction that $\chi$ is equivalent to $\chi'$ for all $\chi$. □

## Section 5

EXERCISE 5.11.

(a) The negation of of a universal sentence is logically equivalent to an existential sentence, and the negation of an existential sentence is logically equivalent to a universal sentence.

(b) If $\mathfrak{A} \subset \mathfrak{B}$ and $\varphi$ is an existential sentence, then if $\mathfrak{A} \models \varphi$, $\mathfrak{B} \models \varphi$.

*Proof.* (a) We prove the first claim by induction on universal sentences. Let $\varphi$ be a quantifier-free (universal) sentence. Then $\neg\varphi$ is a quantifier-free existential sentence, and trivially $\neg\varphi \sim \neg\varphi$. Let $\varphi, \psi$ be universal sentences whose negations are logically equivalent to existential sentences. Consider $(\varphi \vee \psi)$. We know $\neg(\varphi \vee \psi) \sim (\neg\varphi \wedge \neg\psi)$, and furthermore, by (4.12), $(\neg\varphi \wedge \neg\psi)$ is logically equivalent to an existential sentence since $\neg\varphi$ and $\neg\psi$ are by the induction hypothesis. By transitivity of equivalence, the result holds for $(\varphi \vee \psi)$. Similarly for $(\varphi \wedge \psi)$. Now consider $\forall x\varphi$. We know that $\neg\forall x\varphi \sim \exists x\neg\varphi$, and $\exists x\neg\varphi$ is equivalent to an existential sentence since $\neg\varphi$ is by hypothesis. Again by transitivity, the result holds for $\forall x\varphi$.

The second claim is proved similarly.

(b) We know from (a) that $\neg\varphi$ is logically equivalent to a universal sentence. Now if $\mathfrak{A} \models \varphi$, then it is not the case that $\mathfrak{A} \models \neg\varphi$, but then it is not the case that $\mathfrak{B} \models \neg\varphi$ by (5.8), so $\mathfrak{B} \models \varphi$.

□

## Section 7

EXERCISE 7.5. Let $\Pi$ be the following set of second order $S_{\mathrm{ar}}$-sentences:

$$\forall x \neg x + 1 \equiv x$$
$$\forall x \forall y (x + 1 \equiv y + 1 \to x \equiv y)$$
$$\forall X((X0 \wedge \forall x(Xx \to Xx+1)) \to \forall y Xy)$$
$$\forall x\, x + 0 \equiv x$$
$$\forall x \forall y\, x + (y+1) \equiv (x+y) + 1$$
$$\forall x\, x \cdot 0 \equiv 0$$
$$\forall x \forall y\, x \cdot (y+1) \equiv (x \cdot y) + x$$

(a) If $\mathfrak{A} = (A, +^A, \cdot^A, 0^A, 1^A)$ satisfies $\Pi$, and if $\sigma^A : A \to A$ is defined by $\sigma^A(x) = x +^A 1^A$, then the structure $\mathfrak{A}' = (A, \sigma^A, 0^A)$ satisfies (P1)–(P3).

(b) $\mathfrak{N} = (\mathbf{N}, +, \cdot, 0, 1)$ is characterized by $\Pi$ up to isomorphism—that is, $\mathfrak{N}$ satisfies $\Pi$, and any structure $\mathfrak{A} = (A, +^A, \cdot^A, 0^A, 1^A)$ satisfying $\Pi$ is isomorphic to $\mathfrak{N}$.

*Proof.* (a) Suppose $\mathfrak{A}$ satisfies $\Pi$. Then we know that $\forall x \neg x +^A 1^A \equiv 0^A$, so $\forall x \neg \sigma^A(x) \equiv 0^A$; thus $\mathfrak{A}'$ satisfies (P1). We also know $\forall x \forall y (x +^A 1^A \equiv y +^A 1^A \to x \equiv y)$, so $\forall x \forall y (\sigma^A(x) \equiv \sigma^A(y) \to x \equiv y)$; thus $\sigma^A$ is injective and $\mathfrak{A}'$ satisfies (P2). Finally, we have

$$\forall X((X0^A \wedge (Xx \to Xx +^A 1^A)) \to \forall y Xy)$$

thus

$$\forall X((X0^A \wedge (Xx \to X\sigma^A(x))) \to \forall y Xy)$$

and so $\mathfrak{A}'$ satisfies (P3) as desired.

(b) First, $\mathfrak{N}$ satisfies $\Pi$. Now suppose $\mathfrak{A} = (A, +^A, \cdot^A, 0^A, 1^A)$ satisfies $\Pi$. We prove $\mathfrak{N} \cong \mathfrak{A}$. First, define $\pi : \mathbf{N} \to A$ inductively by

$$\pi(0) = 0^A$$
$$\pi(n+1) = \pi(n) +^A 1^A$$

We claim that $\pi$ is a bijection. First let $R$ be the range of $\pi$. Then $0^A \in R$ since $\pi(0) = 0^A$. If $a \in R$, then $\pi(n) = a$ for some $n \in \mathbf{N}$; but then $\pi(n+1) = \pi(n) +^A 1^A = a +^A 1^A$, so $a +^A 1^A \in R$. Since $\mathfrak{A}$ satisfies $\Pi$, it follows from the induction formula that $R = A$ and $\pi$ is surjective.

To verify injectivity, we must show that for all $m, n \in \mathbf{N}$, if $\pi(m) = \pi(n)$, then $m = n$. We proceed by induction on $n$. If $n = 0$, then $m = 0$; for if not, then $\pi(m) = \pi((m-1)+1) = \pi(m-1) +^A 1^A \neq 0^A = \pi(n)$ (by the first formula in $\Pi$)—a contradiction. So the claim is true in this case. Now suppose the claim holds for $n$; we prove it holds for $n+1$. If $\pi(m) = \pi(n+1)$, then $m \neq 0$ (by reasoning similar to that just given). Thus we have

$$\pi(m) = \pi((m-1)+1) = \pi(m-1) +^A 1^A = \pi(n) +^A 1^A = \pi(n+1)$$

which by the second formula in $\Pi$ yields $\pi(m-1) = \pi(n)$. By the induction hypothesis, $m-1 = n$, so $m = n+1$ and the claim holds for $n+1$ as desired.

Now that we know $\pi : \mathbf{N} \cong A$, we must show that for all $m, n \in \mathbf{N}$, $\pi(m+n) = \pi(m) +^A \pi(n)$; we proceed by induction on $n$. For $n = 0$, we have

$$\pi(m+0) = \pi(m) = \pi(m) +^A 0^A = \pi(m) +^A \pi(0)$$

the second equality holding by the fourth formula in $\Pi$. If the claim holds for $n$, then we have

$$\begin{aligned}
\pi(m+(n+1)) &= \pi((m+n)+1) \\
&= \pi(m+n) +^A 1^A \\
&= [\pi(m) +^A \pi(n)] +^A 1^A && \text{induction hypothesis} \\
&= \pi(m) +^A [\pi(n) +^A 1^A] && \text{fifth formula in } \Pi \\
&= \pi(m) +^A \pi(n+1)
\end{aligned}$$

so the claim holds for $n+1$ as desired.

Similarly, we verify that for all $m, n \in \mathbf{N}$, $\pi(m \cdot n) = \pi(m) \cdot^A \pi(n)$. For $n = 0$,

$$\pi(m \cdot 0) = \pi(0) = 0^A = \pi(m) \cdot^A 0^A = \pi(m) \cdot^A \pi(n)$$

the third equality holding by the sixth formula in $\Pi$. If the claim holds for $n$, then

$$\begin{aligned}
\pi(m \cdot (n+1)) &= \pi(m \cdot n + m) \\
&= \pi(m \cdot n) +^A \pi(m) && \text{from above result} \\
&= [\pi(m) \cdot^A \pi(n)] +^A \pi(m) && \text{induction hypothesis} \\
&= \pi(m) \cdot^A [\pi(n) +^A 1^A] && \text{seventh formula in } \Pi \\
&= \pi(m) \cdot \pi(n+1)
\end{aligned}$$

so the result holds for $n+1$.

Finally, we note that $\pi(0) = 0^A$ by definition, and

$$\pi(1) = \pi(0+1) = \pi(0) +^A 1^A = 0^A +^A 1^A = 1^A +^A 0^A = 1^A$$

Here we omit the proof of commutativity of $+^A$. We then have that $\mathfrak{N} \cong \mathfrak{A}$ as desired.

□

## Section 8

EXERCISE 8.10. Suppose $x_0, \dots, x_r$ are pairwise distinct and $x_0, \dots, x_r \not\in \mathrm{var}(t_0) \cup \dots \cup \mathrm{var}(t_r)$. Then

$$\varphi \frac{t_0, \dots, t_r}{x_0, \dots, x_r} \sim \forall x_0, \dots, \forall x_r (x_0 \equiv t_0 \land \dots \land x_r \equiv t_r \to \varphi)$$

*Proof.* The proof requires the following lemma: for any interpretation $\mathfrak{I} = ((A, \mathfrak{a}), \beta)$, $a \in A$, variable $x$, and term $t$, $\mathfrak{I}\frac{a}{x}(t) = \mathfrak{I}(t)$ if $x \notin \mathrm{var}(t)$. The proof is an easy induction on terms.

We only sketch the remainder of the proof. The idea is to proceed by induction on $r$ and then, in each part of this process, by induction on formulas.

Case $r = 0$: Suppose $\varphi = t_1' \equiv t_2'$. Then for any interpretation $\mathfrak{I}$ we have

$$\begin{array}{lll}
\mathfrak{I} \models \varphi\frac{t_0}{x_0} & \text{iff} & \mathfrak{I} \models [t_1' \equiv t_2']\frac{t_0}{x_0} \\
& \text{iff} & \mathfrak{I}\frac{\mathfrak{I}(t_0)}{x_0} \models t_1' \equiv t_2' \\
& & \text{(by Substitution Lemma)} \\
& \text{iff} & \mathfrak{I}\frac{\mathfrak{I}(t_0)}{x_0}(t_1') = \mathfrak{I}\frac{\mathfrak{I}(t_0)}{x_0}(t_2') \\
& \text{iff} & \text{For all } a_0 \in A, \text{ if } a_0 = \mathfrak{I}(t_0), \text{ then } \mathfrak{I}\frac{a_0}{x_0}(t_1') = \mathfrak{I}\frac{a_0}{x_0}(t_2') \\
& \text{iff} & \text{For all } a_0 \in A, \text{ if } \mathfrak{I}\frac{a_0}{x_0}(x_0) = \mathfrak{I}\frac{a_0}{x_0}(t_0), \text{ then } \mathfrak{I}\frac{a_0}{x_0}(t_1') = \\
& & \mathfrak{I}\frac{a_0}{x_0}(t_2') \\
& & \text{(by our lemma since } x_0 \notin \mathrm{var}(t_0)) \\
& \text{iff} & \text{For all } a_0 \in A, \text{ if } \mathfrak{I}\frac{a_0}{x_0} \models x_0 \equiv t_0, \text{ then } \mathfrak{I}\frac{a_0}{x_0} \models t_1' \equiv t_2' \\
& \text{iff} & \text{For all } a_0 \in A, \mathfrak{I}\frac{a_0}{x_0} \models x_0 \equiv t_0 \to t_1' \equiv t_2' \\
& \text{iff} & \mathfrak{I} \models \forall x_0(x_0 \equiv t_0 \to t_1' \equiv t_2') \\
& \text{iff} & \mathfrak{I} \models \forall x_0(x_0 \equiv t_0 \to \varphi)
\end{array}$$

So the result holds in this case. Similarly for the other formula cases. By induction on formulas, the result holds for all formulas when $r = 0$.

Now suppose the result holds for $r$ (for all formulas); we prove it holds for $r + 1$. Again, consider the case when $\varphi = t_1' \equiv t_2'$. Then we have

$$\begin{array}{lll}
\mathfrak{I} \models \varphi\frac{t_0,\dots,t_{r+1}}{x_0,\dots,x_{r+1}} & \text{iff} & \mathfrak{I} \models \left[\varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r}\right]\frac{t_{r+1}}{x_{r+1}} \\
& & \text{(since } x_{r+1} \neq t_i) \\
& \text{iff} & \mathfrak{I}\frac{\mathfrak{I}(t_{r+1})}{x_{r+1}} \models \varphi\frac{t_0,\dots,t_r}{x_0,\dots,x_r} \\
& & \text{(by Substitution Lemma)} \\
& \text{iff} & \mathfrak{I}\frac{\mathfrak{I}(t_{r+1})}{x_{r+1}} \models \forall x_0,\dots,\forall x_r(x_0 \equiv t_0 \wedge \cdots \wedge x_r \equiv t_r \to \varphi) \\
& & \text{(by induction hypothesis)} \\
& \text{iff} & \mathfrak{I} \models \forall x_0,\dots,\forall x_{r+1}(x_0 \equiv t_0 \wedge \cdots \wedge x_r \equiv t_r \wedge x_{r+1} \equiv \\
& & t_{r+1} \to \varphi) \\
& & \text{(as in case } r = 0)
\end{array}$$

So the result holds in this case. Similarly for other formula cases. By induction on formulas, the result holds for all formulas for $r + 1$.

By induction on **N**, then, the result holds for all $r$. □

# Chapter IV

## Section 2

EXERCISE 2.7(A). The following rule is correct:

$$
\begin{array}{ccc}
\Gamma & \varphi_1 & \psi_1 \\
\Gamma & \varphi_2 & \psi_2 \\
\hline
\Gamma & (\varphi_1 \vee \varphi_2) & (\psi_1 \vee \psi_2)
\end{array}
$$

*Proof.* Suppose $\Gamma\varphi_1 \models \psi_1$ and $\Gamma\varphi_2 \models \psi_2$. Let $\mathfrak{I}$ be any interpretation satisfying $\Gamma(\varphi_1 \vee \varphi_2)$. Then, by definition, $\mathfrak{I} \models \varphi_1$ or $\mathfrak{I} \models \varphi_2$. In the first case, $\mathfrak{I} \models \psi_1$ since $\Gamma\varphi_1 \models \psi_1$; similarly, in the second case, $\mathfrak{I} \models \psi_2$; so $\mathfrak{I} \models (\psi_1 \vee \psi_2)$ by definition. Thus $\Gamma(\varphi_1 \vee \varphi_2) \models (\psi_1 \vee \psi_2)$ and the result follows. $\square$

## Section 3

EXERCISE 3.6.

(a) We can justify the (derived) sequent rule

$$
\begin{array}{cc}
\Gamma & \varphi \\
\hline
\Gamma & \neg\neg\varphi
\end{array}
$$

as follows:

$$
\begin{array}{lllll}
1. & \Gamma & & \varphi & \text{Premise} \\
2. & \Gamma & \neg\varphi & \varphi & \text{(Ant) applied to 1.} \\
3. & \Gamma & \neg\varphi & \neg\varphi & \text{(Assm)} \\
4. & \Gamma & \neg\varphi & \neg\neg\varphi & \text{(Ctr') applied to 2,3.} \\
5. & \Gamma & \neg\neg\varphi & \neg\neg\varphi & \text{(Assm)} \\
6. & \Gamma & & \neg\neg\varphi & \text{(PC) applied to 4,5.}
\end{array}
$$

We can justify

$$
\begin{array}{cc}
\Gamma & \neg\neg\varphi \\
\hline
\Gamma & \varphi
\end{array}
$$

as follows:

$$
\begin{array}{lllll}
1. & \Gamma & & \neg\neg\varphi & \text{Premise} \\
2. & \Gamma & \neg\varphi & \neg\neg\varphi & \text{(Ant) applied to 1.} \\
3. & \Gamma & \neg\varphi & \neg\varphi & \text{(Assm)} \\
4. & \Gamma & & \varphi & \text{(Ctr) applied to 2,3.}
\end{array}
$$

(d) We can justify

$$
\begin{array}{cc}
\Gamma & (\varphi \wedge \psi) \\
\hline
\Gamma & \varphi
\end{array}
$$

as follows, recalling that $(\varphi \wedge \psi) = \neg(\neg\varphi \vee \neg\psi)$ since we have formally dropped the $\wedge$ connective from our logical symbols:

$$
\begin{array}{lllll}
1. & \Gamma & & \neg(\neg\varphi \vee \neg\psi) & \text{Premise} \\
2. & \Gamma & \neg\varphi & \neg\varphi & \text{(Assm)} \\
3. & \Gamma & \neg\varphi & (\neg\varphi \vee \neg\psi) & (\vee\text{S) applied to 2.} \\
4. & \Gamma & \neg\varphi & \neg(\neg\varphi \vee \neg\psi) & \text{(Ant) applied to 1.} \\
5. & \Gamma & & \varphi & \text{(Ctr) applied to 3,4.}
\end{array}
$$

# Chapter V

## Section 1

EXERCISE 1.13. Let $S$ be an arbitrary symbol set and suppose $\Phi$ is an inconsistent set of formulas. By IV.7.2(b), it follows that for any terms $t_1, t_2$, $\Phi \vdash t_1 \equiv t_2$. Thus $T^\Phi$, the domain of $\mathfrak{I}^\Phi$, consists of only one equivalence class. It follows that the interpretations of variable, constant, and function symbols are determined. For any $n$-ary relation symbol $R \in S$ and terms $t_1, \ldots, t_n$, $\Phi \vdash Rt_1 \cdots t_n$; thus $R^{\mathfrak{I}^\Phi} = (T^\Phi)^n$. This determines $\mathfrak{I}^\Phi$.

We see that $\mathfrak{I}^\Phi$ does not depend on $\Phi$ (only its inconsistency).

# Chapter VI

## Section 1

EXERCISE 1.3. Let $\Phi$ be a consistent, at most countable set of formulas, and suppose that $\Phi$ is satisfied by an infinite model. Then $\Phi$ is satisfied by a countable model.

*Proof.* Immediate by the Löwenheim-Skolem-Tarski theorem. □

## Section 3

EXERCISE 3.7. Suppose $\mathfrak{K}$ is $\Delta$-elementary. Then the class $\mathfrak{K}^\infty$ of infinite structures in $\mathfrak{K}$ is also $\Delta$-elementary.

*Proof.* Choose $\Phi$ such that $\mathfrak{K} = \mathrm{Mod}^S\Phi$. Define

$$\varphi^{\geq n} = \exists x_1 \cdots \exists x_n (\neg(x_1 \equiv x_2) \wedge \cdots \wedge \neg(x_1 \equiv x_n) \wedge \cdots \wedge \neg(x_{n-1} \equiv x_n))$$

which states that there exist (at least) $n$ distinct elements (for $n \geq 2$), and set

$$\Phi' = \{\varphi^{\geq n} \mid n \geq 2\}$$

Now define $\Phi^\infty = \Phi \cup \Phi'$. We claim that $\mathfrak{K}^\infty = \mathrm{Mod}^S\Phi^\infty$. Indeed, $\mathfrak{A} \in \mathfrak{K}^\infty$ if and only if $\mathfrak{A} \in \mathfrak{K}$ and $\mathfrak{A}$ is infinite, which holds if and only if $\mathfrak{A} \models \Phi$ and $\mathfrak{A} \models \Phi'$, which holds if and only if $\mathfrak{A} \models \Phi^\infty$, which holds if and only if $\mathfrak{A} \in \mathrm{Mod}^S\Phi^\infty$. □

EXERCISE 3.8. Let $\mathfrak{K}$ be a class of $S$-structures and $\Phi \subseteq L_0^S$. We say that $\Phi$ is a *system of axioms* for $\mathfrak{K}$ if and only if $\mathfrak{K} = \mathrm{Mod}^S\Phi$.

(a) A class $\mathfrak{K}$ is elementary if and only if it has a finite system of axioms.

*Proof.* If $\mathfrak{K}$ is elementary, then $\mathfrak{K} = \mathrm{Mod}^S\varphi$ for some sentence $\varphi$; set $\Phi = \{\varphi\}$. Conversely, suppose $\mathfrak{K} = \mathrm{Mod}^S\Phi$, where $\Phi = \{\varphi_1, \ldots, \varphi_n\}$. Define

$$\varphi = \varphi_1 \wedge \cdots \wedge \varphi_n$$

Then $\mathfrak{K} = \mathrm{Mod}^s\varphi$, so $\mathfrak{K}$ is elementary as desired. □

(b) If $\mathfrak{K}$ is elementary and $\mathfrak{K} = \mathrm{Mod}^S\Phi$, then there exists a finite subset $\Phi_0$ of $\Phi$ such that $\mathfrak{K} = \mathrm{Mod}^S\Phi_0$.

*Proof.* Let $\varphi$ be a sentence such that $\mathfrak{K} = \mathrm{Mod}^S\varphi$. Then $\mathrm{Mod}^S\Phi = \mathrm{Mod}^S\varphi$, so $\Phi \models \varphi$, and by completeness $\Phi \vdash \varphi$. But since proofs are finite, there exists a finite $\Phi_0 \subseteq \Phi$ such that $\Phi_0 \vdash \varphi$, and (by soundness) $\Phi_0 \models \varphi$. Thus $\mathrm{Mod}^S\Phi_0 \subseteq \mathrm{Mod}^S\varphi$. Noting that $\varphi \models \Phi \models \Phi_0$, it follows that $\mathfrak{K} = \mathrm{Mod}^S\Phi_0$ as desired. □

## Section 4

EXERCISE 4.8. Let $S = \{0, 1, +, \times, <\}$ and $\varphi$ be an $S$-sentence. If $\varphi$ is valid in all non-archimedean ordered fields, then $\varphi$ is valid in all ordered fields.

*Proof.* Let $\mathfrak{F}$ be an arbitrary ordered field. We can construct a non-archimedean ordered field $\mathfrak{F}'$ elementarily equivalent to $\mathfrak{F}$ (see proof of Theorem 4.5). Then $\mathfrak{F}' \models \varphi$ by hypothesis, so $\mathfrak{F} \models \varphi$. The result follows. □

# Chapter VII

## Section 4

EXERCISE 4.4. There is no circularity inherent in the relationship between ZFC and first-order logic. While it is *possible* to encode the axioms of ZFC in first-order logic and consider models of ZFC for object set theory, it is not *necessary* to do so in order to use ZFC for background set theory—in particular for the set theory required to construct the objects of first-order logic. The fallacy in the line of reasoning leading us to the purported circle is the claim that the machinery of first-order logic is *required* to use ZFC for a background set theory.

# Chapter VIII

## Section 2

EXERCISE 2.8. (We provide an alternate proof of Theorem 1.3(a) using Theorem 2.2.) Let $S$ be a symbol set and $S^r$ be the induced relational symbol set for $S$. For every $\psi \in L^S$, there exists a $\psi^r \in L^{S^r}$ such that for all $S$-interpretations $\mathfrak{I} = (\mathfrak{A}, \beta)$,

$$(\mathfrak{A}, \beta) \models \psi \quad \text{iff} \quad (\mathfrak{A}^r, \beta) \models \psi^r$$

where $\mathfrak{A}^r$ is the induced $S^r$-structure associated with $\mathfrak{A}$.

*Proof.* We define a syntactic interpretation $I$ of $S$ in $S^r$; that is, we define a mapping $I : S \cup \{S\} \to L^{S^r}$ as follows:

$$\begin{array}{rcl} S & \mapsto & v_0 \equiv v_0 \\ n\text{-ary } R & \mapsto & R v_0 \cdots v_{n-1} \\ n\text{-ary } f & \mapsto & F v_0 \cdots v_{n-1} v_n \\ c & \mapsto & C v_0 \end{array}$$

where (each) $F \in S^r$ corresponds to $f \in S$, and similarly $C \in S^r$ to $c \in S$.

Now it is immediate from definitions that for an arbitrary $S$-interpretation $\mathfrak{I} = (\mathfrak{A}, \beta)$, we have $\mathfrak{A}^r \models \Phi_I$. We further claim that $(\mathfrak{A}^r)^{-I} = \mathfrak{A}$. Indeed, since $\mathfrak{A}^r \models (v_0 \equiv v_0)[a]$ for all $a \in A$, we have $A^{-I} = A$. For $n$-ary $f \in S$ and $a_0, \dots, a_n \in A$,

$$\begin{aligned} f^{(\mathfrak{A}^r)^{-I}}(a_0, \dots, a_{n-1}) = a_n \qquad & \text{iff} \qquad \mathfrak{A}^r \models F v_0 \cdots v_{n-1} v_n [a_0, \dots, a_{n-1}, a_n] \\ & \text{iff} \qquad F^{\mathfrak{A}^r}(a_0, \dots, a_{n-1}, a_n) \\ & \text{iff} \qquad f^{\mathfrak{A}}(a_0, \dots, a_{n-1}) = a_n \end{aligned}$$

So $f^{(\mathfrak{A}^r)^{-I}} = f^{\mathfrak{A}}$. Similarly for relation and constant symbols. Thus the claim holds.

Given that the claim holds, the desired result follows immediately from Theorem 2.2 with each $\psi^r = \psi^I$. □

## Section 3

EXERCISE 3.4. (An extension by definitions of an extension by definitions of a set $\Phi$ of $S$-sentences is an extension by definitions of $\Phi$.)

Let $S$ be a symbol set and $\Phi$ be a set of $S$-sentences. Suppose $s \notin S$ and $\delta_s$ is an $S$-definition of $s$ in $\Phi$. Define $S' = S \cup \{s\}$ and $\Phi' = \Phi \cup \{\delta_s\}$. Now suppose $s' \notin S'$ and $\delta_{s'}$ is an $S'$-definition of $s'$ in $\Phi'$, with associated formula $\varphi_{s'}(v_0, \dots, v_n) \in L^{S'}$. Assume $\varphi_{s'}$ is term-reduced (Theorem 1.2), and let $\varphi^I_{s'} \in L^S$ be the canonical interpretation of $\varphi_{s'}$ under the canonical syntactic interpretation $I$ of $S'$ in $S$—that is, using $\varphi_s$—with $\text{free}(\varphi^I_{s'}) = \text{free}(\varphi_{s'})$. Set

$$\delta^I_{s'} = \forall v_0 \cdots \forall v_n (\varphi_{\text{def}} \leftrightarrow \varphi^I_{s'}(v_0, \dots, v_n))$$

where

$$\varphi_{\text{def}} = \begin{cases} s' v_0 \cdots v_n & \text{if } s' \text{ is an } (n+1)\text{-ary relation symbol} \\ s' v_0 \cdots v_{n-1} \equiv v_n & \text{if } s' \text{ is an } n\text{-ary function symbol} \\ s' \equiv v_0 & \text{if } s' \text{ is a constant symbol } (n = 0) \end{cases}$$

Then $\Phi'' = \Phi \cup \{\delta_s, \delta^I_{s'}\}$ is an extension by definitions of $\Phi$.

*Proof.* We must prove that $\delta^I_{s'}$ is an $S$-definition of $s'$ in $\Phi$. If $s'$ is a relation symbol, it is sufficient to note that $\varphi^I_{s'} \in L^S$ and has the same free variables as $\varphi_{s'}$. Suppose $s'$ is an $n$-ary function symbol. Then we know, by definition,

$$\Phi' \models \forall v_0 \cdots \forall v_{n-1} \exists^{=1} v_n \varphi_{s'}$$

But then, by Theorem 3.2, and since $\varphi_{s'}$ is term-reduced,

$$\Phi \models [\forall v_0 \cdots \forall v_{n-1} \exists^{=1} v_n \varphi_{s'}]^I = \forall v_0 \cdots \forall v_{n-1} \exists^{=1} v_n \varphi^I_{s'}$$

Thus $\delta^I_{s'}$ is an $S$-definition of $s$ in $\Phi$ as desired. Similarly for the case when $s$ is a constant symbol. So the desired result holds. □

*Remark.* We cannot in general claim that $\Phi \cup \{\delta_s, \delta_{s'}\}$ is an extension by definitions of $\Phi$. This would require that $\delta_{s'}$ be an $S$-definition of $s'$ in $\Phi$, but there are many examples where this is not the case. Consider, for example, $S = \{\in\}$ and $\Phi = \mathrm{ZFC}$. Define symbol '$\subseteq$' using

$$\delta_{\subseteq} = \forall x \forall y (x \subseteq y \leftrightarrow \forall z (z \in x \rightarrow z \in y))$$

Now define '$\subset$' using

$$\delta_{\subset} = \forall x \forall y (x \subset y \leftrightarrow (x \subseteq y \wedge \neg(x \equiv y)))$$

It is impossible for $\delta_{\subset}$ to be an $S$-definition since it syntactically contains the symbol '$\subseteq$' in its associated formula.

## Section 4

EXERCISE 4.6. For the purposes of this exercise, let $\mathrm{SNF}(\varphi)$ denote the Skolem normal form of $\varphi$ as constructed in the proof of Theorem 4.5.

Let $\varphi \in L_0^S$ and $\psi = \mathrm{SNF}(\varphi)$. Choose $S' \supseteq S$ such that $\psi \in L_0^{S'}$. Then for every $S$-structure $\mathfrak{A}$, $\mathfrak{A} \models \varphi$ if and only if there exists an $S'$-extension $\mathfrak{A}'$ of $\mathfrak{A}$ such that $\mathfrak{A}' \models \psi$.

*Proof.* We may assume that $\varphi$ is in prenex normal form, for if $\mathrm{PNF}(\varphi)$ denotes the prenex normal form of $\varphi$ as in Theorem 4.4, we see from the proof of Theorem 4.5 that

$$\mathrm{SNF}(\varphi) = \mathrm{SNF}(\mathrm{PNF}(\varphi))$$

We proceed by induction on the number $n$ of existential quantifiers in $\varphi$. If $n = 0$, the result holds trivially, so suppose $n > 0$ and the result holds for $n-1$. Write

$$\varphi = \forall x_1 \cdots \forall x_k \exists x_{k+1} Q_{k+2} x_{k+2} \cdots Q_m x_m \varphi_0$$

where $x_1, \ldots, x_m$ are pairwise distinct, $Q_i \in \{\forall, \exists\}$ for $k+2 \leq i \leq m$, and $\varphi_0$ is quantifier-free. We execute one step in the procedure used to construct $\psi$. Set

$$\varphi_1 = Q_{k+2} x_{k+2} \cdots Q_m x_m \varphi_0$$

and choose $f_{k+1} \in S' - S$ appropriately as a $k$-ary function symbol if $k \neq 0$ or as a constant symbol if $k = 0$. Define

$$\psi' = [\forall x_1 \cdots \forall x_k \varphi_1] \frac{f_{k+1} x_1 \cdots x_k}{x_{k+1}}$$

Now let $\mathfrak{A}$ be an $S$-structure and suppose $\mathfrak{A} \models \varphi$. It is clear that we can extend $\mathfrak{A}$ to an $S \cup \{f_{k+1}\}$-structure $\mathfrak{A}'$ such that $\mathfrak{A}' \models \psi'$ by defining $f_{k+1}^{\mathfrak{A}'}$ appropriately. Note that $\psi'$ is a sentence whose number of existential quantifiers is $n-1$. By the induction hypothesis then, and the fact that $\psi = \mathrm{SNF}(\psi')$, we can extend $\mathfrak{A}'$ to an $S'$-structure $\mathfrak{A}''$ such that $\mathfrak{A}'' \models \psi$. Now $\mathfrak{A}''$ is the desired $S'$-extension of $\mathfrak{A}$.

Conversely, suppose $\mathfrak{A}''$ is an $S'$-extension of $\mathfrak{A}$ such that $\mathfrak{A}'' \models \psi$. Set

$$\mathfrak{A}' = \mathfrak{A}''|_{S\cup\{f_{k+1}\}} = (\mathfrak{A}, f_{k+1}^{\mathfrak{A}''})$$

Again by the induction hypothesis and the fact that $\psi = \mathrm{SNF}(\psi')$, $\mathfrak{A}' \models \psi'$. It is then immediate that $\mathfrak{A} \models \varphi$. Thus the desired result holds for $\varphi$. □

EXERCISE 4.8. Suppose $S$ is a relational symbol set and $\varphi \in L_0^S$ with

$$\varphi = \exists x_0 \cdots \exists x_n \forall y_0 \cdots \forall y_m \psi$$

where $\psi$ is quantifier-free. Then every model of $\varphi$ has a substructure containing at most $n+1$ elements which is also a model of $\varphi$.

*Proof.* Suppose $\mathfrak{A} \models \varphi$. Choose $a_0, \ldots, a_n \in A$ such that

$$\mathfrak{A} \models \forall y_0 \cdots \forall y_m \psi[a_0, \ldots, a_n]$$

Let $\mathfrak{B}$ be the induced substructure of $\mathfrak{A}$ on $B = \{a_0, \ldots, a_n\}$ (note that $\mathfrak{B}$ is well-defined since $S$ is relational). If $b_0, \cdots, b_m \in B$, we have

$$\mathfrak{A} \models \psi[a_0, \ldots, a_n, b_0, \ldots, b_m]$$

We prove by induction on quantifier-free $\psi$ that this holds if and only if

$$\mathfrak{B} \models \psi[a_0, \ldots, a_n, b_0, \ldots, b_m]$$

The atomic cases are immediate since $S$ is relational and thus all terms are variable symbols. The negation and disjunction steps are also immediate from the induction hypothesis. Since $\psi$ is quantifier-free, those are all the cases, and the claim holds.

Since $b_0, \ldots, b_n$ were arbitrary in $B$, it follows that

$$\mathfrak{B} \models \forall y_0 \cdots \forall y_m \psi[a_0, \ldots, a_n]$$

so, since $a_0, \ldots, a_n \in B$, $\mathfrak{B} \models \varphi$ as desired. □

*Remark.* We see that the sentence

$$\varphi' = \forall x \exists y Rxy$$

cannot be logically equivalent to a formula in the form of $\varphi$ above. For $\varphi'$ is satisfied in $\mathfrak{A} = (\mathbf{N}, <)$ with $R^{\mathfrak{A}} = <$, but there is no finite substructure of $\mathfrak{A}$ satisfying $\varphi'$.

# Chapter IX

## Section 1

EXERCISE 1.7. In the system $\mathscr{L}_{\mathrm{II}}^{w}$ of weak second-order logic, the following holds:

(a) There exists a sentence $\varphi$ and a structure $\mathfrak{A}$ such that $\mathfrak{A} \models_w \varphi$ but $\mathfrak{A} \not\models \varphi$.

*Proof.* Let $\mathfrak{A}$ be any infinite structure. Consider the sentence

$$\varphi = \forall X((\forall x \exists^{=1} y Xxy \wedge \forall x \forall y \forall z((Xxz \wedge Xyz) \to x \equiv y)) \to \forall y \exists x Xxy)$$

(see p. 140). Now $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{A}$ is finite; thus $\mathfrak{A} \not\models \varphi$. But if $\forall X$ quantifies over only finite subsets of $A^2$, then we claim $\varphi$ is trivially satisfied in $\mathfrak{A}$; indeed, since $A$ is infinite, there is no finite $C \subseteq A^2$ such that

$$\mathfrak{A} \models \forall x \exists^{=1} y Xxy[C]$$

So the implication in $\varphi$ holds vacuously for all such assignments. $\square$

(b) For each sentence $\varphi \in L_{\mathrm{II}}^{w,S}$, there is a sentence $\psi \in L_{\mathrm{II}}^{S}$ such that for all $S$-structures $\mathfrak{A}$,

$$\mathfrak{A} \models_w \varphi \quad \text{iff} \quad \mathfrak{A} \models \psi$$

*Remark.* Note that the definition of $\models_w$ on p. 142 is ambiguous; the definition should really state

$$\mathfrak{I} \models_w \exists X^n \varphi \quad \text{:iff} \quad \text{there exists a finite } C \subseteq A^n \text{ such that } \mathfrak{I}\frac{C}{X^n} \models_w \varphi$$

where $\models_w$ is also used on the right-hand side.

*Proof.* We prove the stronger claim that for each formula $\varphi \in L_{\mathrm{II}}^{w,S}$, there exists a formula $\psi \in L_{\mathrm{II}}^{S}$ with $\mathrm{free}(\varphi) = \mathrm{free}(\psi)$ such that for all $S$-interpretations $\mathfrak{I} = (\mathfrak{A}, \gamma)$,

$$\mathfrak{I} \models_w \varphi \quad \text{iff} \quad \mathfrak{I} \models \psi$$

We proceed by induction on $\varphi$. The atomic cases are immediate since the relations $\models_w$ and $\models$ are defined in the same way there, so we can simply choose $\psi = \varphi$. The boolean cases are also immediate using the induction hypothesis and the definitions of weak and strong satisfaction. Suppose $\varphi = \exists x \varphi'$ and $\psi'$ corresponds to $\varphi'$ by the induction hypothesis. We have, for any $\mathfrak{I}$,

$$\begin{aligned}
\mathfrak{I} \models_w \exists x \varphi' \quad & \text{iff} \quad \text{there exists } a \in A \text{ such that } \mathfrak{I}\tfrac{a}{x} \models_w \varphi' \\
& \text{iff} \quad \text{there exists } a \in A \text{ such that } \mathfrak{I}\tfrac{a}{x} \models \psi' \\
& \text{iff} \quad \mathfrak{I} \models \exists x \psi'
\end{aligned}$$

Since $\mathrm{free}(\exists x \varphi') = \mathrm{free}(\exists x \psi')$, we set $\psi = \exists x \psi'$.

Finally, suppose $\varphi = \exists X^n \varphi'$, and let $\psi'$ correspond to $\varphi'$ by the induction hypothesis. We make use of a formula $\gamma(X^n)$ which states that $X^n$ is finite ($\gamma$ states that all injective functions on $X^n$ are surjective; see p. 140). We set $\psi = \exists X^n(\gamma(X^n) \wedge \psi')$. Then $\mathrm{free}(\varphi) = \mathrm{free}(\psi)$ and, for all $\mathfrak{I}$,

$$\begin{aligned}
\mathfrak{I} \models_w \exists X^n \varphi' \quad & \text{iff} \quad \text{there exists a finite } C \subseteq A^n \text{ such that } \mathfrak{I}\tfrac{C}{X^n} \models_w \varphi' \\
& \text{iff} \quad \text{there exists a } C \subseteq A^n \text{ such that } \mathfrak{I}\tfrac{C}{X^n} \models \gamma(X^n) \text{ and } \mathfrak{I}\tfrac{C}{X^n} \models \psi' \\
& \text{iff} \quad \text{there exists a } C \subseteq A^n \text{ such that } \mathfrak{I}\tfrac{C}{X^n} \models (\gamma(X^n) \wedge \psi') \\
& \text{iff} \quad \mathfrak{I} \models \psi
\end{aligned}$$

This completes the proof of our stronger claim. The original exercise is an immediate corollary. □

(c) The Compactness Theorem does not hold for $\mathscr{L}_{\mathrm{II}}^{w}$.

*Proof.* We construct a finitely weakly satisfiable set of sentences that is not weakly satisfiable. First define

$$\varphi = \exists X \forall x X x$$

Note that for an arbitrary $S$-structure $\mathfrak{A}$, $\mathfrak{A} \models_w \varphi$ if and only if $\mathfrak{A}$ is finite ($\varphi$ can be considered an $\mathscr{L}_{\mathrm{II}}^{w}$-correlate of $\varphi_{\mathrm{fin}}$; see p. 140). Now set

$$\Phi = \{\varphi\} \cup \{\varphi^{\geq n} \mid n \geq 2\}$$

It is clear that $\Phi$ is finitely weakly satisfiable, but not weakly satisfiable. □

## Section 2

EXERCISE 2.8(B). The isomorphism class of $(\mathbf{Z}, <)$ is axiomatizable by an $\mathscr{L}_{\omega_1\omega}$-sentence.

*Proof.* We construct an $\mathscr{L}_{\omega_1\omega}$-sentence $\varphi$ which states:

(∗) The relation < defines on the domain a linear ordering without endpoints such that between any two distinct elements there exist only finitely many elements.

Formally, set

$$\begin{aligned}\varphi_{\mathrm{lin}} = \forall x \neg(x < x) \wedge \forall x \forall y \forall z((x < y \wedge y < z) \to x < z) \\ \wedge \forall x \forall y(x < y \vee x \equiv y \vee y < x) \wedge \forall x \exists y(x < y) \wedge \forall x \exists y(y < x)\end{aligned}$$

Let

$$\varphi_{\mathrm{fin}} = \forall x \forall y(x < y \to \bigvee \Phi)$$

where

$$\Phi = \{\exists x_1 \cdots \exists x_n \forall z((x \leq z \wedge z \leq y) \to (z \equiv x_1 \vee \cdots \vee z \equiv x_n)) \mid n \geq 2\}$$

Finally, define $\varphi = \varphi_{\mathrm{lin}} \wedge \varphi_{\mathrm{fin}}$.

It is clear that if $\mathfrak{A} \cong (\mathbf{Z}, <)$, then $\mathfrak{A} \models \varphi$. Now let $\mathfrak{A} = (A, <_A)$ be a $\{<\}$-structure and suppose $\mathfrak{A} \models \varphi$. By (∗), we know that $<_A$ is a linear ordering without endpoints such that between any two distinct elements there are only finitely many elements. In particular, $A$ is countably infinite. Indeed, $A$ is infinite since there is no greatest (or least) element on the linear $<_A$, and $A$ is countable since it can be written as the union of all finite intervals centered about some point in $A$—a countable union of finite sets.

We can now easily define an isomorphism $\pi$ from $\mathfrak{A}$ to $(\mathbf{Z}, <)$. Let $a_0, a_1, a_2, \ldots$ be an enumeration of $A$. Define

$$\pi(a_i) = \begin{cases} 0 & \text{if } a_0 = a_i \\ n & \text{if } a_0 <_A a_i \text{ and } a_0 \text{ and } a_i \text{ are separated by } n-1 \text{ elements} \\ -n & \text{if } a_i <_A a_0 \text{ and } a_0 \text{ and } a_i \text{ are separated by } n-1 \text{ elements} \end{cases}$$

By the properties of $<_A$, it is immediate that $\pi$ is a well-defined bijection. It is easy to verify that $\pi$ preserves order. Thus $\pi$ is an isomorphism and the proof is complete. □

EXERCISE 2.9.

(a) For arbitrary $S$, $L^S_{\omega_1\omega}$ is uncountable.

*Proof.* We use a diagonal argument. Suppose $L^S_{\omega_1\omega}$ is at most countable for some $S$. Then the subset $L \subseteq L^S_{\omega_1\omega}$ consisting of all countable disjunctions is at most countable. In fact, $L$ is countable since

$$\{\bigvee \Phi_i \mid i \in \mathbf{N}\} \subseteq L$$

is infinite, where

$$\Phi_i = \{v_i \equiv v_j \mid j \in \mathbf{N}\}$$

Let $\varphi_1, \varphi_2, \ldots$ be an enumeration of $L$. We can write

$$\varphi_i = \psi_{i,1} \vee \psi_{i,2} \vee \cdots$$

Now define an infinite disjunction $\varphi = \psi_1 \vee \psi_2 \vee \cdots$ where $\psi_i = \neg\psi_{i,i}$. By hypothesis, we must have $\varphi = \varphi_j$ for some $j$. But $\varphi$ and $\varphi_j$ disagree (syntactically) at the $j$-th disjunct by construction—a contradiction. Thus our original assumption that $L^S_{\omega_1\omega}$ is at most countable is incorrect, as desired. □

# Chapter X

## Section 1

**Note:** In the following exercises, it is assumed that all alphabets are is finite.

EXERCISE 1.2. Let $\mathscr{A}$ be an alphabet and $W, W'$ be decidable subsets of $\mathscr{A}^*$. Then $W \cup W'$, $W \cap W'$, and $\mathscr{A}^* \setminus W$ ($\mathscr{A}^* \setminus W'$) are also decidable.

*Proof.* To decide $W \cup W'$: given $\zeta \in \mathscr{A}^*$, determine whether $\zeta \in W$. If so, halt; if not, determine whether $\zeta \in W'$. If so, halt; if not, print a nonempty string and halt.

To decide $W \cap W'$: given $\zeta \in \mathscr{A}^*$, determine whether $\zeta \in W$. If not, print a nonempty string and halt; if so, determine whether $\zeta \in W'$. If not, print a nonempty string and halt; if so, halt.

To decide $\mathscr{A}^* \setminus W$: given $\zeta \in \mathscr{A}^*$, simply decide whether $\zeta \in W$ and do the opposite action (that is, halt if $\zeta \notin W$, and print a nonempty string and halt if $\zeta \in W$). □

EXERCISE 1.3. For this exercise it is sufficient to note that we can construct decision procedures for variable symbols and formulas over $\mathscr{A}_0$, as well as for the set of free variables in a formula. The latter decision procedure relies on the recursive definition of free($\varphi$) for a formula $\varphi$.

EXERCISE 1.9. Let $W \subseteq U \subseteq \mathscr{A}^*$ and suppose $U$ is decidable. Then if $W$ and $U \backslash W$ are both enumerable, $W$ is decidable.

*Proof.* We can construct a decision procedure for $W$ as follows: given $\zeta \in \mathscr{A}^*$, first determine whether $\zeta \in U$. If not, $\zeta \notin W$, so print a nonempty string and halt; if so, run the enumeration procedures for $W$ and $U \backslash W$ simultaneously until $\zeta$ appears in the output of one of them ($\zeta$ is guaranteed to appear in the output of one of the procedures, since $\zeta \in U$). If $\zeta \in W$, halt; if $\zeta \in U \backslash W$, print a nonempty string and halt. □

EXERCISE 1.10. Let $\mathscr{A}_1 \subseteq \mathscr{A}_2$ be alphabets and $W \subseteq \mathscr{A}_1^*$. Then $W$ is decidable in $\mathscr{A}_1^*$ if and only if $W$ is decidable in $\mathscr{A}_2^*$.

*Proof.* If $W$ is decidable in $\mathscr{A}_2^*$, it is immediate that $W$ is decidable in $\mathscr{A}_1^*$ since we have $\mathscr{A}_1^* \subseteq \mathscr{A}_2^*$. Now suppose $W$ is decidable in $\mathscr{A}_1^*$. By Theorem 1.8, it follows that $W$ and $\mathscr{A}_1^* \backslash W$ are both enumerable (note that this relies on the fact that $\mathscr{A}_1$ is finite; see note above). Now (again since $\mathscr{A}_1$ is finite), it is clear that $\mathscr{A}_1^*$ is decidable in $\mathscr{A}_2^*$. Thus it follows from EXERCISE 1.9 that $W$ is decidable in $\mathscr{A}_2^*$ as desired. □

## Section 2

EXERCISE 2.11. Suppose $W \subseteq \mathscr{A}^*$. Then $W$ is R-enumerable if and only if there exists a program $P$ such that $P : \zeta \mapsto \square$ if $\zeta \in W$ and $P : \zeta \mapsto \infty$ if $\zeta \notin W$.

*Proof.* Suppose $W$ is R-enumerable. Then there exists a program $P$ that, started with the empty input, eventually prints exactly the elements of $W$ (in other words, for any string $\zeta \in W$, $W$ prints $\zeta$ in finitely many steps, and $W$ prints only strings in $W$). We can hack $P$ to create the desired program $P'$. First, add code before the code of $P$ that copies $\zeta$ from $R_0$ into a register $R_i$ unused by $P$ (note that all labels for $P$ instructions, including those referenced in IF instructions, must be modified to preserve functionality). Now replace in $P$ all instructions of the form

$L$ PRINT

with code that compares $R_0$ and $R_i$ and does the following: if there is a match, it adds a character to another register $R_j$ unused by $P$ and jumps to the halt instruction of $P$, and if there is not a match, it simply proceeds to the next instruction (again all labels for $P$ instructions must be modified). Finally, replace the halt instruction of $P$ with code that checks whether $R_j$ is nonempty, halting if so and entering an infinite loop if not. We see that $P'$ is the desired program.[1]

Now suppose conversely that there exists such a program $P$. We sketch a 'multithreaded' enumeration procedure for $W$. First hack $P$ to a program $P'$ such that

[1] Note that the use of regiser $R_j$ was necessary to ensure that $P' : \zeta \mapsto \infty$ in the case that $W$ is finite.

$P' : \zeta \mapsto \zeta$ if $\zeta \in W$ and $P' : \zeta \mapsto \infty$, printing nothing, if $\zeta \notin W$ (this can be done by modifying the PRINT instructions of $P$). Let $Q$ be an enumeration procedure for $\mathscr{A}^*$. Hack $Q$ by replacing all PRINT instructions with code that starts a new instance of $P'$ with input from $R_0$ and continues running (in other words, $P'$ is run in a new thread each time). It is clear that this program is an enumeration procedure for $W$. $\square$

## Section 3

EXERCISE 3.5. (An abstract diagonal argument.)

(a) Let $M$ be a nonempty set and $R \subseteq M^2$. For $a \in M$, let

$$M_a = \{b \in M \mid Rab\}$$

Let $D = \{b \in M \mid \text{not } Rbb\}$. Then $D$ is distinct from each $M_a$.

*Proof.* Suppose $D = M_a$ for some $a \in M$. Then $Raa$ iff $a \in M_a$ by definition of $M_a$, which is true iff $a \in D$ by hypothesis, which is true iff not $Raa$ by definition of $D$. Thus

$$Raa \quad \text{iff} \quad \text{not } Raa$$

—a contradiction. $\square$

(b) Let $M = \mathscr{A}^*$ for a finite alphabet $\mathscr{A}$. Define $R \subseteq M^2$ by

$$R\xi\eta \quad \text{iff} \quad \xi \text{ Gödel-numbers a program enumerating a set containing } \eta$$

Then $D = \{\eta \mid \text{not } R\eta\eta\}$ is not R-enumerable.

*Proof.* Suppose $D$ is enumerated by a program $P$ and let $\xi_P$ be the Gödel number of $P$. Then we have $D = M_{\xi_P}$—contradicting (a). $\square$

(c) Again, let $M = \mathscr{A}^*$ for a finite alphabet $\mathscr{A}$ and $R \subseteq M^2$ be defined by

$$R\xi\eta \quad \text{iff} \quad \xi \text{ does not Gödel-number a program } P \text{ with } P : \eta \mapsto \text{halt}$$

Then all R-decidable subsets of $M$ occur among the $M_\xi$, and $D = \Pi'_{\text{halt}}$ (where $\Pi'_{\text{halt}}$ is as in Theorem 3.2).

*Proof.* For $\eta \in M$, not $R\eta\eta$ iff $\eta$ Gödel-numbers a program that halts on $\eta$. Thus $D = \Pi'_{\text{halt}}$. $\square$

## Section 4

EXERCISE 4.3. The set $\Phi$ of satisfiable $S_\infty$-sentences is not R-enumerable.

*Proof.* Suppose $\Phi$ is enumerated by a program $P_1$. Let $P_2$ be a program enumerating the $S_\infty$-validities (Theorem 2.8). Then we can construct the following decision procedure for the set of $S_\infty$-validities: given $\varphi \in L_0^{S_\infty}$, run $P_1$ and $P_2$ simultaneously until either $\varphi$ is printed by $P_2$ or $\neg\varphi$ is printed by $P_1$. One or the other must be printed in finitely many steps, for if $\varphi$ is not a validity, it is not satisfied by some $S_\infty$-structure, hence its negation is satisfiable. If $\varphi$ is printed by $P_2$, it is a validity; if it is printed by $P_1$, it is not a validity. Thus we have a decision procedure for the $S_\infty$-validities. But this contradicts Theorem 4.1, so our supposition is false. □

## Section 6.

EXERCISE 6.6. Let $T = \Phi^{\models}$ be a theory and suppose that $\Phi$ is R-enumerable. Then $T$ is R-axiomatizable.

*Proof.* We must construct an R-decidable set $\Phi'$ such that $T = \Phi'^{\models}$. Let $\varphi_0, \varphi_1, \ldots$ be an enumeration of $\Phi$ and set

$$\Phi' = \{\varphi_0 \wedge \cdots \wedge \varphi_n \mid n \geq 0\}$$

Note that $\Phi'$ is logically equivalent to $\Phi$, hence $T = \Phi'^{\models}$. But $\Phi'$ can be enumerated naturally in the order $\varphi_0, \varphi_0 \wedge \varphi_1, \ldots$ where the lengths of the successively enumerated sentences are strictly increasing. This means we can construct a decision procedure for $\Phi'$: given a sentence $\varphi$, calculate its length $l$. Now enumerate, in the natural order, the finitely many sentences of $\Phi'$ with length at most $l$, comparing each enumerated sentence with $\varphi$. If a match is found, $\varphi \in \Phi'$; if a match is not found, $\varphi \notin \Phi'$. Thus $\Phi'$ is R-decidable and $T$ is R-axiomatizable. □

EXERCISE 6.13. Let $\mathfrak{Z} = (\mathbf{Z}, +, \cdot, 0, 1)$ be the ring of integers (considered as an $S_{\mathrm{ar}}$-structure). Then $\mathrm{Th}(\mathfrak{Z})$ is not R-decidable.

*Proof.* We define a computable function $\pi$ on $L^{S_{\mathrm{ar}}}$ which maps a formula $\varphi$ to a formula $\pi(\varphi)$ with $\mathrm{free}(\varphi) = \mathrm{free}(\pi(\varphi))$ and such that, if $\mathrm{free}(\varphi) \subseteq \{x_0, \ldots, x_{n-1}\}$, then for all $m_0, \ldots, m_{n-1} \in \mathbf{N}$,

$$\mathfrak{N} \models \varphi[m_0, \ldots, m_{n-1}] \quad \text{iff} \quad \mathfrak{Z} \models \pi(\varphi)[m_0, \ldots, m_{n-1}]$$

In particular, for all $\varphi \in L_0^{S_{\mathrm{ar}}}$, $\mathfrak{N} \models \varphi$ iff $\mathfrak{Z} \models \pi(\varphi)$. Thus, if $\mathrm{Th}(\mathfrak{Z})$ is R-decidable, a program can be constructed that uses $\pi$ to decide $\mathrm{Th}(\mathfrak{N})$, contradicting Theorem 6.9.

We make use of the fact that an integer is a natural number iff it is the sum of four squares of integers. Define

$$\varphi_{\mathbf{N}}(x) = \exists x_1 \exists x_2 \exists x_3 \exists x_4 (x \equiv x_1 \cdot x_1 + x_2 \cdot x_2 + x_3 \cdot x_3 + x_4 \cdot x_4)$$

Then for all $z \in \mathbf{Z}$, $z \in \mathbf{N}$ iff $\mathfrak{Z} \models \varphi_{\mathbf{N}}[z]$.

We now define $\pi$ by induction on formulas. On the atomic formulas, $\pi$ is the identity. For the non-atomic formulas, we set

$$\begin{array}{rcl} \neg\psi & \mapsto & \neg\pi(\psi) \\ (\psi_1 \to \psi_2) & \mapsto & (\pi(\psi_1) \to \pi(\psi_2)) \\ \exists x\psi(x) & \mapsto & \exists x(\varphi_{\mathbf{N}}(x) \wedge \pi(\psi(x))) \end{array}$$

It is clear that $\pi$ is computable. (Note that the recursion used in the computation of $\pi$ is guaranteed to complete since $\pi$ is applied to shorter formulas each time.)

We now prove the claims made about $\pi$ above. It is immediate by induction on formulas that for all $\varphi \in L^{S_{\mathrm{ar}}}$, $\mathrm{free}(\varphi) = \mathrm{free}(\pi(\varphi))$. The satisfaction claim is also verified by induction on $\varphi$. The atomic case (there is only the equality case since $S_{\mathrm{ar}}$ contains no relation symbols) is immediate after verifying by induction on terms that for all $S_{\mathrm{ar}}$-terms $t$, if all the variable symbols in $t$ are assigned to elements of $\mathbf{N}$, then $\mathfrak{Z}(t) = \mathfrak{N}(t)$ (under that assignment). The boolean cases are also immediate. Finally, suppose the claim holds for $\psi(x, x_0, \ldots, x_{n-1})$ and consider $\varphi = \exists x\psi$. We have, for all $m_0, \ldots, m_{n-1} \in \mathbf{N}$,

$$\begin{aligned} &\mathfrak{N} \models \exists x\psi[m_0, \ldots, m_{n-1}] \\ &\quad \text{iff there is } m \in \mathbf{N} \text{ such that } \mathfrak{N} \models \psi[m, m_0, \ldots, m_{n-1}] \\ &\quad \text{iff there is } m \in \mathbf{Z} \text{ such that } \mathfrak{Z} \models \varphi_{\mathbf{N}}[m] \text{ and } \mathfrak{Z} \models \pi(\psi)[m, m_0, \ldots, m_{n-1}] \\ &\quad \text{iff there is } m \in \mathbf{Z} \text{ such that } \mathfrak{Z} \models (\varphi_{\mathbf{N}} \wedge \pi(\psi))[m, m_0, \ldots, m_{n-1}] \\ &\quad \text{iff } \mathfrak{Z} \models \exists x(\varphi_{\mathbf{N}} \wedge \pi(\psi))[m_0, \ldots, m_{n-1}] \\ &\quad \text{iff } \mathfrak{Z} \models \pi(\exists x\psi)[m_0, \ldots, m_{n-1}] \end{aligned}$$

Thus the desired claim holds. □

# Chapter XII

## Section 1.

EXERCISE 1.9. Let $S = \emptyset$. Then any two infinite $S$-structures are partially isomorphic.

*Proof.* Let $\mathfrak{A}$ and $\mathfrak{B}$ be two infinite $S$-structures. Set

$$I = \{\, p \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B}) \mid \mathrm{dom}(p) \text{ is finite} \,\}$$

We claim $I : \mathfrak{A} \cong_p \mathfrak{B}$. Indeed, $I$ is nonempty since $\emptyset \in I$. Suppose $p \in I$ and $a \in A$. If $a \notin \mathrm{dom}(p)$, note that since $\mathrm{dom}(p)$ is finite, $\mathrm{ran}(p)$ is also finite, and hence $B \setminus \mathrm{ran}(p)$ is nonempty. Choose $b \in B \setminus \mathrm{ran}(p)$ and set $q = p \cup \{(a, b)\}$. We see that $q$ is injective and thus, since $S$ is empty, $q$ is a partial isomorphism. Since $\mathrm{dom}(q)$ is finite, $q \in I$, and $q$ is the desired extension of $p$. The back-property is proved similarly. □

EXERCISE 1.10.

(a) Consider $\mathbf{N}$ and $\mathbf{R}$ as $\emptyset$-structures. By EXERCISE 1.9, $\mathbf{N} \cong_p \mathbf{R}$, but $\mathbf{N} \not\cong \mathbf{R}$ since there exists no bijection $\pi : \mathbf{N} \to \mathbf{R}$ (see EXERCISE I.1.3).

(b) Let $S = \{\sigma, 0\}$ and let $\Phi_\sigma$ consist of the successor axioms as in Example 1.8. Using compactness, construct a nonstandard model $\mathfrak{N}'$ of $\Phi_\sigma$ such that $\mathfrak{N}' \not\cong \mathfrak{N}_\sigma$, and assume $\mathfrak{N}'$ is countable by Lowenheim-Skolem. Since $\mathfrak{N}_\sigma$ and $\mathfrak{N}'$ are both at most countable and nonisomorphic, Lemma 1.5(d) gives $\mathfrak{N}_\sigma \not\cong_p \mathfrak{N}'$. But since both structures are models of $\Phi_\sigma$, it follows from Example 1.8 that $\mathfrak{N}_\sigma \cong_f \mathfrak{N}'$.

## Section 2

Exercise 2.5. Let $S = \emptyset$ and $T = \{\varphi^{\geq n} \mid n \geq 2\}^{\models}$ be the theory of infinite structures. Then $T$ is complete and R-decidable.

*Proof.* From Exercise 1.9, we know that any two infinite $S$-structures are partially isomorphic and hence finitely isomorphic (Lemma 1.5(b)). By Fraïssé's Theorem (Theorem 2.1) then, any two infinite $S$-structures are elementarily equivalent. Noting that an $S$-structure $\mathfrak{A}$ is infinite iff $\mathfrak{A} \models T$, it follows from Lemma 2.3 that $T$ is complete. Since $T$ is R-axiomatizable by construction, it follows from Theorem X.6.5(a) that $T$ is R-decidable as desired. □

Exercise 2.6. Let $S = \{P_n \mid n \in \mathbf{N}\}$ be a set of unary relation symbols. Define $S$-structures $\mathfrak{A}$ and $\mathfrak{B}$ where $A = \mathbf{N}$, $B = \mathbf{N} \cup \{\infty\}$, and

$$P_n^{\mathfrak{A}} = \{m \in \mathbf{N} \mid m \geq n\} \qquad P_n^{\mathfrak{B}} = \{m \in \mathbf{N} \mid m \geq n\} \cup \{\infty\}$$

Then $\mathfrak{A} \equiv \mathfrak{B}$ but not $\mathfrak{A} \cong_f \mathfrak{B}$.

*Proof.* We claim that for all $\varphi \in L_n^S$ and $a_0, \ldots, a_{n-1} \in A$,

$$\mathfrak{A} \models \varphi[a_0, \ldots, a_{n-1}] \quad \text{iff} \quad \mathfrak{B} \models \varphi[a_0, \ldots, a_{n-1}]$$

(In other words, $\mathfrak{A}$ is an *elementary substructure* of $\mathfrak{B}$.) In particular, we obtain $\mathfrak{A} \equiv \mathfrak{B}$. The atomic cases are immediate from the fact that $\mathfrak{A} \subseteq \mathfrak{B}$, and the boolean cases are immediate from the induction hypothesis. If the induction hypothesis holds for $\psi(x, x_0, \ldots, x_{n-1})$, then from

$$\mathfrak{A} \models \exists x \psi[a_0, \ldots, a_{n-1}]$$

we obtain

$$\mathfrak{B} \models \exists x \psi[a_0, \ldots, a_{n-1}]$$

trivially. Conversely, suppose $\mathfrak{B} \models \exists x \psi[a_0, \ldots, a_{n-1}]$, so there exists a $b \in B$ such that $\mathfrak{B} \models \psi[b, a_0, \ldots, a_{n-1}]$. If $b \in A$, we are done; if $b = \infty$, we claim there exists an $a \in A$ such that $\mathfrak{B} \models \psi[a, a_0, \ldots, a_{n-1}]$ (without proof at the moment).[2]

[2] Note that an analysis of the definable sets in $B$ using automorphisms is not helpful here since there are no nontrivial automorphisms of $\mathfrak{B}$; this follows from the fact that each natural $n$ is definable in $\mathfrak{B}$ by the formula $\varphi_n(x) = P_n x \wedge \neg P_{n+1} x$.

Suppose now towards a contradiction that $(I_n)_{n\in\mathbf{N}} : \mathfrak{A} \cong_f \mathfrak{B}$. Choose some $n$ and $p \in I_{n+1}$. By the back property of partially isomorphic structures, there exists a $q \in I_n$, $p \subseteq q$, such that $q(m) = \infty$ for some $m \in \mathrm{dom}(q)$. But note that

$$m \notin P^{\mathfrak{A}}_{m+1} \quad \text{and} \quad q(m) \in P^{\mathfrak{B}}_{m+1}$$

—a contradiction. Thus $\mathfrak{A} \not\cong_f \mathfrak{B}$. □

## Section 3

EXERCISE 3.12. Let $S$ be finite and relational and let $\mathfrak{B}$ be an $S$-structure whose domain $B$ contains exactly $n$ elements. Then for all $S$-structures $\mathfrak{A}$,

$$\mathfrak{A} \models \varphi^{n+1}_{\mathfrak{B}} \quad \text{iff} \quad \mathfrak{A} \cong \mathfrak{B}$$

(In other words, $\varphi^{n+1}_{\mathfrak{B}}$ characterizes $\mathfrak{B}$ up to isomorphism.)

*Proof.* Since $S$ is finite and relational and $n+1 \geq 1$, it follows from Theorem 3.10 that

$$\mathfrak{A} \models \varphi^{n+1}_{\mathfrak{B}} \quad \text{iff} \quad \mathfrak{A} \cong_{n+1} \mathfrak{B}$$

We claim that $\mathfrak{A} \cong_{n+1} \mathfrak{B}$ iff $\mathfrak{A} \cong \mathfrak{B}$. First, if $\mathfrak{A} \cong \mathfrak{B}$, then $\mathfrak{A} \cong_f \mathfrak{B}$ by Lemma 1.5 (a) and (b), and it is immediate that $\mathfrak{A} \cong_{n+1} \mathfrak{B}$. Conversely, suppose $(I_m)_{m\leq n+1} : \mathfrak{A} \cong_{n+1} \mathfrak{B}$ and choose $p \in I_{n+1}$. Extend $p$ using the back property $n$ times to a partial isomorphism $q \in I_1$ such that $\mathrm{ran}(q) = B$. We must have $\mathrm{dom}(q) = A$, for otherwise there exists a proper extension $q'$ of $q$ in $I_0$ which is injective, contradicting the fact that $\mathrm{ran}(q) = B$. Thus $q : \mathfrak{A} \cong \mathfrak{B}$ and $\mathfrak{A} \cong \mathfrak{B}$ as claimed. (Compare this proof with the proof of Lemma 1.5(c).) □

EXERCISE 3.17. Let $S = \{P_1, \ldots, P_r\}$ consist of unary relation symbols. Then for every $S$-structure $\mathfrak{A}$ and every $m \geq 1$, there exists an $S$-structure $\mathfrak{B}$ whose domain contains at most $m \cdot 2^r$ elements such that $\mathfrak{A} \cong_m \mathfrak{B}$.

*Proof.* Suppose $\mathfrak{A}$ is given. Let $\mathscr{C}^{\mathfrak{A}}$ denote the collection of all subsets of $A$ of the form

$$C^{\mathfrak{A}}_k = A_{k,1} \cap \cdots \cap A_{k,r} \qquad A_{k,i} = P^{\mathfrak{A}}_i \text{ or } A_{k,i} = \overline{P^{\mathfrak{A}}_i}$$

(For $X \subseteq A$, we denote $A \setminus X$ by $\overline{A}$.) Note that $|\mathscr{C}^{\mathfrak{A}}| \leq 2^r$. Also, $\mathscr{C}^{\mathfrak{A}}$ forms a 'quasi-partition' of $A$. We have

$$A = \bigcup_k C^{\mathfrak{A}}_k$$

and we claim that if $C^{\mathfrak{A}}_i \neq C^{\mathfrak{A}}_j$, then $C^{\mathfrak{A}}_i \cap C^{\mathfrak{A}}_j = \emptyset$. Indeed, if $C^{\mathfrak{A}}_i \neq C^{\mathfrak{A}}_j$, then we must have $A_{i,l} \neq A_{j,l}$ for some $1 \leq l \leq r$. But then $A_{i,l} = \overline{A_{j,l}}$, and since $A_{j,l} \cap \overline{A_{j,l}} = \emptyset$, it follows that $C^{\mathfrak{A}}_i \cap C^{\mathfrak{A}}_j = \emptyset$ as claimed. (Note that $\mathscr{C}^{\mathfrak{A}}$ may contain an empty set, so it is not in general a partition of $A$.)

Now given $m \geq 1$, construct for each set $C^{\mathfrak{A}}_k \in \mathscr{C}^{\mathfrak{A}}$ a corresponding set $C^{\mathfrak{B}}_k$ as follows: if $|C^{\mathfrak{A}}_k| \leq m$, set $C^{\mathfrak{B}}_k = C^{\mathfrak{A}}_k$; otherwise, let $C^{\mathfrak{B}}_k$ be an arbitrary $m$-element subset of $C_k^{\mathfrak{A}}$. Define $B = \bigcup_k C_k^{\mathfrak{B}}$ and set $P_i^{\mathfrak{B}} = P_i^{\mathfrak{A}} \cap B$, forming an $S$-structure $\mathfrak{B}$. Note that $|B| \leq m \cdot 2^r$.

We claim that $\mathfrak{A} \cong_m \mathfrak{B}$. To prove this, we construct a sequence $I_0, \dots, I_m$, where for $0 \leq n \leq m$,

$$I_n = \{ p \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B}) \mid |\mathrm{dom}(p)| \leq m - n \}$$

Note that $I_n \subseteq \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$ and $\emptyset \in I_n$ for each $n$. We verify the forth property (the back property can be proved similarly). Suppose $n + 1 \leq m$, $p \in I_{n+1}$, and $a \in A$ where $a \notin \mathrm{dom}(p)$. We note that $a \in C_k^{\mathfrak{A}}$ for some $k$. Now since

$$|\mathrm{ran}(p)| = |\mathrm{dom}(p)| \leq m - (n+1) \leq m - 1$$

there exists an element $b \in C_k^{\mathfrak{B}}$ such that $q = p \cup \{(a, b)\}$ is an injection. Furthermore, it can be seen that $q \in \mathrm{Part}(\mathfrak{A}, \mathfrak{B})$, and since $|\mathrm{dom}(q)| = |\mathrm{dom}(p)| + 1 \leq m - n$, we have $q \in I_n$ as desired. Thus $(I_n)_{n \leq m} : \mathfrak{A} \cong_m \mathfrak{B}$. □

# Chapter XIII

## Section 1

EXERCISE 1.6. $\mathscr{L}_Q \leq \mathscr{L}_{\mathrm{II}}$, not $\mathscr{L}_{\mathrm{II}}^w \leq \mathscr{L}_Q$, and not $\mathscr{L}_Q \leq \mathscr{L}_{\mathrm{II}}^w$.

*Proof.* We omit the details of the verification but note that $\mathscr{L}_Q \leq \mathscr{L}_{\mathrm{II}}$ since uncountability can be expressed in $\mathscr{L}_{\mathrm{II}}$ (see p. 140). In $\mathscr{L}_{\mathrm{II}}^w$, we can characterize the finite structures with a sentence (see EXERCISE IX.1.7(C)), but since compactness holds for $\mathscr{L}_Q$ (Theorem IX.3.2) this is not possible in $\mathscr{L}_Q$. Conversely, we can characterize uncountable structures with a sentence in $\mathscr{L}_Q$, but since Löwenheim-Skolem holds in $\mathscr{L}_{\mathrm{II}}^w$ (see EXERCISE IX.2.7), this is not possible in $\mathscr{L}_{\mathrm{II}}^w$. □

## Section 3

EXERCISE 3.7. (Beth's Definability Theorem.) Let $S$ be finite and relational, and let $P$ be a $k$-ary relation symbol not in $S$. Let $\Phi \subseteq L_0^{S \cup \{P\}}$. Then $\Phi$ defines $P$ explicitly if and only if $\Phi$ defines $P$ implicitly.

In order to prove this result, we first prove the following lemma:
**Lemma.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be $S$-structures with $\pi : \mathfrak{A} \cong \mathfrak{B}$. Then for all $\varphi \in L_n^{S \cup \{P\}}$,

$$(\mathfrak{A}, P^{\mathfrak{A}}) \models \varphi[\overline{a}] \quad \text{iff} \quad (\mathfrak{B}, \pi(P^{\mathfrak{A}})) \models \varphi[\pi(\overline{a})]$$

*Proof.* Proceed by induction on $\varphi$. The atomic $S$-cases hold since $\pi$ preserves satisfiability between $\mathfrak{A}$ and $\mathfrak{B}$ for all $S$-formulas. If $\varphi = P x_1 \cdots x_n$, then

$$\begin{aligned} (\mathfrak{A}, P^{\mathfrak{A}}) \models \varphi[\overline{a}] \quad & \text{iff} \quad \overline{a} \in P^{\mathfrak{A}} \\ & \text{iff} \quad \pi(\overline{a}) \in \pi(P^{\mathfrak{A}}) \\ & \text{iff} \quad (\mathfrak{B}, \pi(P^{\mathfrak{A}})) \models \varphi[\pi(\overline{a})] \end{aligned}$$

The boolean and quantifier cases follow easily. □

An immediate corollary of the above lemma is that if $\pi : \mathfrak{A} \cong \mathfrak{B}$, then $(\mathfrak{A}, P^{\mathfrak{A}})$ and $(\mathfrak{B}, \pi(P^{\mathfrak{A}}))$ agree on $S \cup \{P\}$-sentences. In particular, if $(\mathfrak{A}, P^{\mathfrak{A}}) \models \Phi$, then $(\mathfrak{B}, \pi(P^{\mathfrak{A}})) \models \Phi$. Thus, in the case that $\Phi$ defines $P$ implicitly, if $(\mathfrak{A}, P^{\mathfrak{A}}) \models \Phi$ and $(\mathfrak{B}, P^{\mathfrak{B}}) \models \Phi$, then $P^{\mathfrak{B}} = \pi(P^{\mathfrak{A}})$.

Now we proceed with the original proof.

*Proof.* First suppose that there exists an explicit definition $\psi \in L_k^S$ of $P$ in $\Phi$, that is

$$\Phi \models \forall x_1 \cdots x_k (P x_1 \cdots x_k \leftrightarrow \psi)$$

If $\mathfrak{A}$ is an $S$-structure, $P_1, P_2 \subseteq A^k$, and

$$(\mathfrak{A}, P^1) \models \Phi \quad \text{and} \quad (\mathfrak{A}, P^2) \models \Phi$$

then for all $\overline{a} \in A^k$,

$$\begin{array}{lll} \overline{a} \in P^1 & \text{iff} & (\mathfrak{A}, P^1) \models P x_1 \cdots x_k [\overline{a}] \\ & \text{iff} & (\mathfrak{A}, P^1) \models \psi[\overline{a}] \\ & \text{iff} & (\mathfrak{A}, P^2) \models \psi[\overline{b}] \qquad\qquad \text{since } \psi \in L_k^S \text{ (coincidence)} \\ & \text{iff} & \overline{a} \in P^2 \end{array}$$

Thus $P^1 = P^2$ as desired.

Conversely, suppose $P$ is defined implicitly in $\Phi$. For $n \geq 0$ set

$$\chi^n = \bigvee \{\varphi^n_{\mathfrak{A}, \overline{a}} \mid \mathfrak{A} \text{ an } S\text{-structure, } (\mathfrak{A}, P^{\mathfrak{A}}) \models \Phi, \text{ and } P^{\mathfrak{A}} \overline{a}\}$$

We claim that there exists an $n$ such that

$$\Phi \models \forall x_1 \cdots x_k (P x_1 \cdots x_k \leftrightarrow \chi^n)$$

so that $P$ is explicitly defined in $\Phi$, as desired.

First note that, for all $n \geq 0$, we have

$$\Phi \models \forall x_1 \cdots x_k (P x_1 \cdots x_k \rightarrow \chi^n)$$

Indeed, if $n \geq 0$ and $(\mathfrak{A}, P^{\mathfrak{A}}) \models \Phi$, then for all $\overline{a} \in A^k$, if $(\mathfrak{A}, P^{\mathfrak{A}}) \models P x_1 \cdots x_k [\overline{a}]$, then $P^{\mathfrak{A}} \overline{a}$. Further, we know $\mathfrak{A} \models \varphi^n_{\mathfrak{A}, \overline{a}}[\overline{a}]$ (see XII.3.5(b)). Thus $\mathfrak{A} \models \chi^n[\overline{a}]$, so by coincidence since $\chi^n \in L_k^S$, $(\mathfrak{A}, P^{\mathfrak{A}}) \models \chi^n[\overline{a}]$.

Now suppose there does not exist an $n$ satisfying the claim. Then for each $n \geq 0$, there exist $S$-structures $\mathfrak{A}$ and $\mathfrak{B}$, $\overline{a} \in A^k$ and $\overline{b} \in B^k$, such that $(\mathfrak{A}, P^{\mathfrak{A}}) \models \Phi$ and $(\mathfrak{B}, P^{\mathfrak{B}}) \models \Phi$, $\mathfrak{B} \models \varphi^n_{\mathfrak{A}, \overline{a}}[\overline{b}]$, but

$$(\mathfrak{A}, P^{\mathfrak{A}}) \models P x_1 \cdots x_k [\overline{a}] \quad \text{and} \quad (\mathfrak{B}, P^{\mathfrak{B}}) \models \neg P x_1 \cdots x_k [\overline{b}]$$

In other words, for each $n \geq 0$, there exist $\mathfrak{A}, \mathfrak{B}$ and $\overline{a}, \overline{b}$ such that $(\mathfrak{A}, P^{\mathfrak{A}}) \models \Phi$, $(\mathfrak{B}, P^{\mathfrak{B}}) \models \Phi$, $(\mathfrak{A}, \overline{a}) \cong_n (\mathfrak{B}, \overline{b})$ but $P^{\mathfrak{A}} \overline{a}$ and not $P^{\mathfrak{B}} \overline{b}$.

Now the latter result can be formalized in terms of the satisfiability of a certain set of sentences, as in the proof of Lindström's Theorem (we omit the many details). We obtain from this development two $S$-structures $\mathfrak{A}$ and $\mathfrak{B}$ and tuples $\overline{a}, \overline{b}$ such that $(\mathfrak{A}, P^{\mathfrak{A}}) \models \Phi$ and $(\mathfrak{B}, P^{\mathfrak{B}}) \models \Phi$, $\pi : (\mathfrak{A}, \overline{a}) \cong (\mathfrak{B}, \overline{b})$ but $P^{\mathfrak{A}} \overline{a}$ and not $P^{\mathfrak{B}} \overline{b}$. This contradicts the results of our lemma above, since $P^{\mathfrak{B}} = \pi(P^{\mathfrak{A}})$ and $\overline{b} = \pi(\overline{a})$. Thus our supposition is false and there exists an $n$ satisfying the claim above. $\square$

## References

[1] Ebbinghaus, H.–D. and J. Flum and W. Thomas. *Mathematical Logic*, 2nd ed. New York: Springer, 1994.