Selected Exercises from *Finite Model Theory*

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Abstract

This paper contains selected exercises from the text *Finite Model Theory* by Ebbinghaus and Flum (see [1]). Exercises are organized by chapter and section, and are numbered as in the text.

Chapter 2

Section 2.3

EXERCISE 2.3.2. Let $(I_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$ and set

$$\overline{I}_j = \{\, q \in \mathrm{Part}(\mathcal{A},\mathcal{B}) \mid q \subseteq p \text{ for some } p \in I_j \,\}$$

Then $(\overline{I}_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$. Furthermore, $\overline{W}_j(\mathcal{A},\mathcal{B}) = W_j(\mathcal{A},\mathcal{B})$ (where $W_j(\mathcal{A},\mathcal{B})$ is defined as on p. 20).

Proof. Choose $p_j \in I_j$ for each $j \leq m$. Note that $\emptyset \mapsto \emptyset \in \mathrm{Part}(\mathcal{A},\mathcal{B})$, and $\emptyset \mapsto \emptyset \subseteq p_j$ for each j, so $\emptyset \mapsto \emptyset \in \overline{I}_j$ for all $j \leq m$. Now suppose $q \in \overline{I}_{j+1}$ for $j < m$ and $a \in A$. We have $q \subseteq p$ for some $p \in I_{j+1}$, and by the forth property of $(I_j)_{j\leq m}$, there exists a $p' \in I_j$ where $q \subseteq p \subseteq p'$ and $a \in \mathrm{dom}(p')$. Since $I_j \subseteq \overline{I}_j$, $p' \in \overline{I}_j$. Thus $(\overline{I}_j)_{j\leq m}$ satisfies the forth property. Similarly for the back property. Thus $(\overline{I}_j)_{j\leq m} : \mathcal{A} \cong_m \mathcal{B}$ as desired.

It is immediate that $W_j(\mathcal{A},\mathcal{B}) \subseteq \overline{W}_j(\mathcal{A},\mathcal{B})$. Suppose $\overline{a}' \mapsto \overline{b}' \in \overline{W}_j(\mathcal{A},\mathcal{B})$, so

$$\overline{a}' \mapsto \overline{b}' \subseteq \overline{a} \mapsto \overline{b}$$

for some $\overline{a} \mapsto \overline{b} \in W_j(\mathcal{A},\mathcal{B})$ where the duplicator wins $G_j(\mathcal{A},\overline{a},\mathcal{B},\overline{b})$. Then it is clear that the duplicator has a winning strategy for $G_j(\mathcal{A},\overline{a}',\mathcal{B},\overline{b}')$. Indeed, for any play, the duplicator can simply do what it would do in the corresponding play of $G_j(\mathcal{A},\overline{a},\mathcal{B},\overline{b})$. After j moves, the resulting map will be a subset of the partial isomorphism that would have resulted in the corresponding play of $G_j(\mathcal{A},\overline{a},\mathcal{B},\overline{b})$. Thus the resulting map will itself be a partial isomorphism, so the duplicator will win the play. $\square$

EXAMPLE 2.3.5. Let τ be an arbitrary vocabulary. We verify that $\mathrm{EVEN}[\tau]$ is not axiomatizable in $\mathrm{FO}[\tau]$.

Proof. For $m \geq 0$, let $\mathcal{A}$ be a τ-structure with $\|\mathcal{A}\| = m+1$ satisfying the following:

1. For each n-ary $R \in \tau$, $R^{\mathcal{A}} = \emptyset$, and
2. For some $a \in A$, $c^{\mathcal{A}} = a$ for all $c \in \tau$.

Denote by $\mathcal{B}$ the structure obtained from $\mathcal{A}$ by adjoining a new (unnamed) element u. Then $\|\mathcal{B}\| = m+2$, so $\mathcal{A} \in \text{EVEN}[\tau]$ iff $B \notin \text{EVEN}[\tau]$. We claim $\mathcal{A} \cong_m \mathcal{B}$. Define

$$I_j = \{\, p \in \text{Part}(\mathcal{A}, \mathcal{B}) \mid \|\text{dom}(p)\| \leq m - j + 1 \,\}$$

for $0 \leq j \leq m$. Note that $a \mapsto a \in I_j$ for all j. Furthermore, $(I_j)_{j \leq m}$ clearly satisfies the forth property, for any elements in A can simply be mapped to themselves. For the back property, suppose $p \in I_{j+1}$ for some $j+1 \leq m$ and $b \in B$, $b \notin \text{rng}(p)$. If $b \neq u$, simply choose $q = p \cup \{(b,b)\} \in I_j$. If $b = u$ and there are no constant symbols in τ, then choose any $a' \in A \setminus \text{dom}(p)$ (such an element must exist since $\|\text{dom}(p)\| \leq m$); it follows that $q = p \cup \{(a', b)\} \in I_j$. If there are constant symbols in τ, note that there are m unnamed elements in A, and the number of unnamed elements in $\text{dom}(p)$ is

$$\|\text{dom}(p)\| - 1 \leq m - j - 1 \leq m - 1$$

Thus we can choose an unnamed $a' \in A \setminus \text{dom}(p)$ and $q = p \cup \{(a', b)\} \in I_j$.

By Corollary 2.3.4, it follows that we can find, for arbitrary $m \geq 0$, two finite structures $\mathcal{A}$ and $\mathcal{B}$ with $\mathcal{A} \in \text{EVEN}[\tau]$, $\mathcal{B} \notin \text{EVEN}[\tau]$, and $\mathcal{A} \equiv_m \mathcal{B}$. By Theorem 2.2.12, it follows that $\text{EVEN}[\tau]$ is not axiomatizable in $\text{FO}[\tau]$ as claimed. ☐

EXAMPLE 2.3.7. Let $\tau = \{<, \min, \max\}$ and $\sigma = \tau \cup \{E\}$. For $n \geq 3$, let $\mathcal{A}_n$ be the ordered τ-structure with domain $A_n = \{0, \ldots, n\}$ and τ-symbols interpreted in the natural way. Define

$$E^{\mathcal{A}_n} = \{\, (i,j) \mid |i-j| = 2 \,\} \cup \{(0,n), (n,0), (1, n-1), (n-1, 1)\}$$

We verify that the graph $(\mathcal{A}_n, E^{\mathcal{A}_n})$ is connected iff n is odd.

Proof. Suppose n is odd and let $m_1, m_2 \in A_n$. If m_1 and m_2 are both even, then $|m_2 - m_1|$ is even, hence $m_1 = m_2$ or else there exists a sequence of edges connecting them; in either case, $m_1 \sim m_2$. Similarly if m_1 and m_2 are both odd. Suppose, say, m_1 is even and m_2 is odd. Since n is odd, $|n - m_2|$ is even, hence there exists a sequence of edges connecting m_2 to n, and then to 0 since $(n, 0) \in E^{\mathcal{A}_n}$; now $m_1 \sim 0$ by a previous case, so $m_1 \sim m_2$ as desired.

Conversely, suppose n is even. We claim that no even vertex is connected to an odd vertex. Indeed, looking at $E^{\mathcal{A}_n}$, we see that since n is even, all edges preserve the parity of the vertices they connect, so there are no paths between vertices of different parity. Since there exist both even and odd vertices in A_n (for example, 0 and 1), it follows that $(\mathcal{A}_n, E^{\mathcal{A}_n})$ is disconnected. ☐

Let $m \geq 2$ and $l, k \geq 2^m$. Let I_j be the set of partial isomorphisms from Example 2.3.6 from $\mathcal{A}_l|_\tau$ to $\mathcal{A}_k|_\tau$. We verify that for $j \geq 2$ and $p \in I_j$, p preserves E.

Proof. Let $j \geq 2$, $p \in I_j$ and $m_1, m_2 \in \mathrm{dom}(p)$. Suppose that $(m_1, m_2) \in E^{\mathcal{A}_n}$. If $|m_1 - m_2| = 2 < 2^j$, then $|p(m_1) - p(m_2)| = 2$, so $(p(m_1), p(m_2)) \in E^{\mathcal{A}_n}$. If m_1 and m_2 are (respectively, in either order) the maximum and minimum elements of A_l, then $p(a_1)$ and $p(a_2)$ are the corresponding elements of A_k, so $(p(m_1), p(m_2)) \in E^{\mathcal{A}_n}$. Similarly if m_1 and m_2 are the immediate neighbors of the maximum and minimum elements, since their proximity ensures that $(p(m_1), p(m_2)) \in E^{\mathcal{A}_n}$. Note that these cases are exhaustive. The converse cases also clearly hold for all $(p(m_1), p(m_2))$. Thus p preserves E as desired. $\square$

EXERCISE 2.3.14. Let $\overline{a} \mapsto \overline{b} \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ and $m \geq 0$. Then

$$\mathcal{B} \models \varphi^m_{\mathcal{A},\overline{a}}[\overline{b}] \quad \text{iff} \quad (\mathcal{A}, \overline{a}) \cong_m (\mathcal{B}, \overline{b})$$[1]

Proof. By Theorem 2.3.3, $\mathcal{B} \models \varphi^m_{\mathcal{A},\overline{a}}[\overline{b}]$ iff there exists a sequence $(I_j)_{j \leq m}$ with $\overline{a} \mapsto \overline{b} \in I_m$ such that $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$. Clearly if this holds, then $\overline{a} \mapsto \overline{b}$ admits of extension using the back and forth properties m times (formally this is proved by induction on m).

Conversely, if $\overline{a} \mapsto \overline{b}$ admits of extension m times using the back and forth properties, we can construct such a sequence $(I_j)_{j \leq m}$. For $p \in \mathrm{Part}(\mathcal{A}, \mathcal{B})$ and $a \in A$, denote by $F(p, a)$ a partial isomorphism extending p to include a in its domain, when one exists; similarly define $B(p, b)$ for $b \in B$. Now set $I_m = \{\overline{a} \mapsto \overline{b}\}$. For $j < m$, set

$$I_j = \{\, F(p, a) \mid p \in I_{j+1}, a \in A \,\} \cup \{\, B(p, b) \mid p \in I_{j+1}, b \in B \,\}$$

It is immediate by induction on $i \leq m$ that each I_{m-i} is a nonempty set of partial isomorphisms from $\mathcal{A}$ to $\mathcal{B}$, and that every $p \in I_{m-i}$ can be extended (at least) $m - i$ times using the back and forth properties. From this it follows that $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$ as desired. $\square$

EXERCISE 2.3.15. Suppose $(I_j)_{j \leq m} : \mathcal{A} \cong_m \mathcal{B}$. Then $I_j \subseteq W_j(\mathcal{A}, \mathcal{B})$ (where the set $W_j(\mathcal{A}, \mathcal{B})$ is defined as on p. 20).

Proof. Let $p \in I_j$. Write $p = \overline{a} \mapsto \overline{b}$. As noted in a previous exercise, it is immediate by induction on $j \leq m$ that p can be extended (at least) j times in the hierarchy (I_j) using the back and forth properties. This provides a winning strategy for the duplicator in $G_j(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$. Thus p is a winning position for the duplicator in the j-game, that is, $p \in G_j(\mathcal{A}, \mathcal{B})$. Since p was arbitrary, $I_j \subseteq G_j(\mathcal{A}, \mathcal{B})$. $\square$

NOTE. Let $\mathcal{A}$ be a τ-structure and $\overline{a} = (a_1, \ldots, a_s) \in A^s$. Define recursively the m-isomorphism types of $\overline{a}$ in $\mathcal{A}$ in the following way:

$$\mathrm{IT}^0(\mathcal{A}, \overline{a}) = \{\, \varphi \mid \mathcal{A} \models \varphi[\overline{a}], \varphi(v_1, \ldots, v_s) \text{ atomic} \,\}$$

and

$$\mathrm{IT}^{m+1}(\mathcal{A}, \overline{a}) = \{\, \mathrm{IT}^m(\mathcal{A}, \overline{a}a) \mid a \in A \,\}$$

[1]We assume this notation means that $\overline{a} \mapsto \overline{b}$ admits of extension m times using the back and forth properties. This notation is not defined in the text.

We verify that for all τ-structures $\mathcal{B}$ and $\overline{b} \in B^s$,

$$\text{(i) } \mathrm{IT}^m(\mathcal{A}, \overline{a}) = \mathrm{IT}^m(\mathcal{B}, \overline{b}) \quad \text{iff} \quad \text{(ii) } \varphi^m_{\mathcal{A},\overline{a}} = \varphi^m_{\mathcal{B},\overline{b}}$$

Proof. We proceed by induction on m. For $m = 0$, note that (i) holds iff $(\mathcal{A}, \overline{a})$ and $(\mathcal{B}, \overline{b})$ agree on atomic τ-formulas (and hence also negated atomic τ-formulas) $\varphi(v_1, \ldots, v_s)$, which holds iff (ii) holds.

Now suppose $m > 0$ and the result holds for $m - 1$. Then (i) holds iff

$$\{ \mathrm{IT}^{m-1}(\mathcal{A}, \overline{a}a) \mid a \in A \} = \{ \mathrm{IT}^{m-1}(\mathcal{B}, \overline{b}b) \mid b \in B \}$$

which, by the induction hypothesis, holds iff

$$\{ \varphi^{m-1}_{\mathcal{A},\overline{a}a} \mid a \in A \} = \{ \varphi^{m-1}_{\mathcal{B},\overline{b}b} \mid b \in B \}$$

which holds iff, for some syntactic ordering,

$$\bigwedge_{a \in A} \exists v_{s+1} \varphi^{m-1}_{\mathcal{A},\overline{a}a}(\overline{v}, v_{s+1}) = \bigwedge_{b \in B} \exists v_{s+1} \varphi^{m-1}_{\mathcal{B},\overline{b}b}(\overline{v}, v_{s+1})$$
$$\bigvee_{a \in A} \varphi^{m-1}_{\mathcal{A},\overline{a}a}(\overline{v}, v_{s+1}) = \bigvee_{b \in B} \varphi^{m-1}_{\mathcal{B},\overline{b}b}(\overline{v}, v_{s+1})$$

which holds iff (ii) holds. $\square$

Section 2.4

NOTE. Let τ be relational, $\mathcal{A}$ and $\mathcal{B}$ be τ-structures, and $r \geq 0$. Suppose $a \in A$ and $b \in B$ have the same r-ball type, that is

$$\pi : (\mathcal{S}^{\mathcal{A}}(r, a), a) \cong (\mathcal{S}^{\mathcal{B}}(r, b), b)$$

We verify that π preserves sub-balls. More specifically, suppose $r' \leq r$ and $a' \in A$ with $\mathcal{S}^{\mathcal{A}}(r', a') \subseteq \mathcal{S}^{\mathcal{A}}(r, a)$. Then

$$\pi[\mathcal{S}^{\mathcal{A}}(r', a')] = S^{\mathcal{B}}(r', \pi(a'))$$

In particular, $\pi|_{\mathcal{S}^{\mathcal{A}}(r',a')} : \mathcal{S}^{\mathcal{A}}(r', a') \cong S^{\mathcal{B}}(r', \pi(a'))$.

Proof. Suppose $x \in \mathcal{S}^{\mathcal{A}}(r', a')$. If $x = a'$, the result holds trivially, so suppose $x \neq a'$. This means that for some $1 \leq k \leq r'$, there exist relation symbols $R_1, \ldots, R_k \in \tau$ and tuples $\overline{c}_1, \ldots, \overline{c}_k \in A$ satisfying the following properties:

1. $a' \in \overline{c}_1$, $x \in \overline{c}_k$
2. For all $1 \leq i \leq k$, $R_i^{\mathcal{A}} \overline{c}_i$
3. For all $1 < i \leq k$, there exists an element $c \in \overline{c}_{i-1} \cap \overline{c}_i$

It is immediate by induction on i that $\overline{c}_i \in \mathcal{S}^{\mathcal{A}}(r', a')$ for all $1 \leq i \leq k$. Thus, since π is an isomorphism, we have:

1. $\pi(a') \in \pi(\bar{c}_1)$, $\pi(x) \in \pi(\bar{c}_k)$
2. For all $1 \leq i \leq k$, $R_i^{\mathcal{B}} \pi(\bar{c}_i)$
3. For all $1 < i \leq k$, there exists an element $\pi(c) \in \pi(\bar{c}_{i-1}) \cap \pi(\bar{c}_i)$

Thus $\pi(x) \in S^{\mathcal{B}}(r', \pi(a'))$. Since x was arbitrary,

$$\pi[\mathcal{S}^{\mathcal{A}}(r', a')] \subseteq S^{\mathcal{B}}(r', \pi(a'))$$

The reverse inclusion is proved analogously. □

EXERCISE 2.4.7. The class of finite acyclic digraphs is not second-order axiomatizable by a sentence of the form

$$\varphi = \exists P_1 \cdots \exists P_r \psi$$

where $P_1, \ldots, P_r$ are unary relation variables and ψ is a first-order sentence over the vocabulary $\tau = \{E, P_1, \ldots, P_r\}$.

Proof. Let $\mathcal{H}_l = (H_l, E_l)$ be the finite acyclic digraph given by

$$H_l = \{0, \ldots, l\} \quad E_l = \{\, (i, i+1) \mid i < l \,\}$$

We first prove two lemmas:

Lemma. *Let $m \geq 0$. Then there exists an l_0 such that for all $l \geq l_0$ and all τ-structures $\mathfrak{H}_l = (\mathcal{H}_l, P_1, \ldots, P_r)$, there exist $a, b \in H_l$ with disjoint 3^m-balls of the same isomorphism type.*

Proof. Note that since $P_1, \ldots, P_r$ are unary, every 3^m-ball in a structure $\mathfrak{H}_l$ has cardinality at most $2 \cdot 3^m + 1$. As in the proof of Corollary 2.1.2, there are only finitely many pairwise nonisomorphic 3^m-balls (over τ-structures $\mathfrak{H}_l$), hence there are only finitely many 3^m-ball types. Let i be the number of 3^m-ball types. Then set

$$l_0 = (i+1)(2 \cdot 3^m + 1)$$

For any $l \geq l_0$, a structure $\mathfrak{H}_l$ must contain two disjoint 3^m-balls of the same isomorphism type. In fact, we see that such a structure must contain two such balls of cardinality $2 \cdot 3^m + 1$. □

Lemma. *Let $m \geq 0$ and suppose $\mathfrak{H}_l$ contains elements a, b with disjoint 3^m-balls of the same isomorphism type and of cardinality $2 \cdot 3^m + 1$. Let a_+ and b_+ be the successors of a and b, respectively—that is, the elements with $E_l a a_+$ and $E_l b b_+$.*

Construct $\mathfrak{H}'_l$ from $\mathfrak{H}_l$ by setting

$$E^{\mathfrak{H}'_l} = (E^{\mathfrak{H}_l} \setminus \{(a, a_+), (b, b_+)\}) \cup \{(a, b_+), (b, a_+)\}$$

Then $\mathfrak{H}'_l$ is cyclic and $\mathfrak{H}_l \equiv_m \mathfrak{H}'_l$.

Proof. To prove m-equivalence, we argue that for each 3^m-ball type Γ, $\mathfrak{H}_l$ and $\mathfrak{H}'_l$ both contain the same number of elements with 3^m-ball type Γ. The claim then follows from Hanf's Theorem (Theorem 2.4.1).

Indeed, each 3^m-ball in $\mathfrak{H}_l$ corresponds naturally, and injectively, to a 3^m-ball of the same isomorphism type in $\mathfrak{H}'_l$, and conversely. For example, given a 3^m-ball $S(3^m, a')$ of an element $a' \in H_l$, map any elements in $S(3^m, a')$ coinciding with $a_+, a_{++}, \ldots$ to $b_+, b_{++}, \ldots$, and conversely, and map according to the identity for the remaining elements. It follows from our assumptions that the map constructed is an isomorphism.

To see that $\mathfrak{H}'_l$ is cyclic, simply note that the endpoints from $\mathfrak{H}_l$ must both land together between a and b on one side of $\mathfrak{H}'_l$ or the other (where 'side' can be made precise in a natural way). Hence in the construction of $\mathfrak{H}'_l$ from $\mathfrak{H}_l$, a cycle was created on one side of a and b or the other. ☐

Now suppose that

$$\varphi = \exists P_1 \cdots \exists P_r \psi$$

axiomatizes the finite acyclic digraphs as above. Then for a finite digraph $\mathcal{D}$, $\mathcal{D}$ is acyclic iff $\mathcal{D} \models \varphi$, which holds iff there exist $P_1, \cdots, P_r \subseteq D$ such that

$$(\mathcal{D}, P_1, \ldots, P_r) \models \psi$$

Let m be the quantifier rank of ψ. Choose l_0 as in the first lemma and consider $\mathcal{H}_{l_0}$. Since $\mathcal{H}_{l_0}$ is acyclic, there exist $P_1, \ldots, P_r \subseteq H_l$ such that

$$(\mathcal{H}_{l_0}, P_1, \ldots, P_r) \models \psi$$

By the second lemma,

$$(\mathcal{H}'_{l_0}, P_1, \ldots, P_r) \models \psi$$

but H'_{l_0} is cyclic—a contradiction. ☐

We see, however, that the class of finite acyclic digraphs *can* be second-order axiomatized by a sentence of the form

$$\varphi = \forall P \psi$$

where P is unary and ψ is a first-order sentence over $\{E, P\}$. Indeed, this follows immediately from the following lemma:

Lemma. *The class of finite cyclic digraphs can be second-order axiomatized by a sentence of the form*

$$\varphi' = \exists P \psi'$$

where P is unary and ψ' is a first-order sentence over $\{E, P\}$.

Proof. Intuitively, ψ' says 'P is a cycle'. Formally, set

$$\begin{aligned}\psi' = \exists x P x \wedge \forall x (P x \rightarrow \\ &\exists y (P y \wedge E x y \wedge \forall z ((P z \wedge E x z) \rightarrow y = z)) \wedge \\ &\qquad \exists y (P y \wedge E y x \wedge \forall z ((P z \wedge E z x) \rightarrow y = z)))\end{aligned}$$

Let $\mathcal{D}$ be an arbitrary finite digraph. If $\mathcal{D} \models \varphi'$, then there exists some $P \subseteq D$ such that $(\mathcal{D}, P) \models \psi'$. Then P is nonempty, so choose $p \in P$. Since P is finite and ψ' holds for P, it is easy to verify that there exists a unique path through all other elements of P and returning to p. Thus P is a cycle, and $\mathcal{D}$ is cyclic as desired.

Conversely, if $\mathcal{D}$ is a finite cyclic digraph, let P be the elements in a cycle. $\square$

Note that a finite digraph $\mathcal{D}$ is acyclic iff $\mathcal{D} \not\models \varphi'$, which holds iff $\mathcal{D} \models \neg\varphi'$, which holds iff $\mathcal{D} \models \forall P \neg\psi'$.

EXERCISE 2.4.8. There exists a formula

$$\varphi(x, y) = \exists P \psi$$

where P is unary and ψ is first-order over $\{E, P\}$, such that for all finite graphs $\mathcal{G}$ and $a, b \in G$, $\mathcal{G} \models \varphi[a, b]$ iff $a \sim b$ in $\mathcal{G}$.

Proof. Intuitively, φ says 'x equals y or else there exists a path P from x to y'. Formally, set

$$\begin{aligned}\psi(x, y) = (x = y) \vee (Px \wedge Py \wedge \\ \exists w(Pw \wedge Exw \wedge \forall z(Pz \wedge Exz \to w = z)) \wedge \\ \exists w(Pw \wedge Eyw \wedge \forall z(Pz \wedge Eyz \to w = z)) \wedge \\ \forall z(Pz \wedge \neg(x = z) \wedge \neg(y = z) \to \\ \exists w_1 \exists w_2(\neg(w_1 = w_2) \wedge Pw_1 \wedge Pw_2 \wedge Ezw_1 \wedge Ezw_2 \wedge \\ \forall u(Pu \wedge Ezu \to (u = w_1 \vee u = w_2)))))\end{aligned}$$

$\square$

Note that the second-order sentence $\forall x \forall y \varphi(x, y)$ characterizes the finite connected graphs. Thus it cannot be logically equivalent to a sentence of the form

$$\exists P_1 \cdots \exists P_r \chi$$

with unary $P_1, \ldots, P_r$ and first-order χ over $\{E, P_1, \ldots, P_r\}$, for this would contradict Proposition 2.4.5.

Section 2.5

EXERCISE 2.5.3. Let τ be relational and let $\Phi \subseteq \mathrm{FO}[\tau]$ be the smallest set containing the atomic formulas and closed under conjunction, disjunction, and existential quantification. Now let EP be the set of sentences in Φ. We call EP the set of *existential positive sentences*.

Then EP is preserved under homomorphisms.

Proof. We prove the stronger claim that Φ is preserved under homomorphisms in the following precise sense: for all $\varphi \in \Phi$, and for all τ-structures $\mathcal{A}$ and $\mathcal{B}$ where $h : A \to B$ is a homomorphism, if $\overline{a} \in A$, then

$$\mathcal{A} \models \varphi[\overline{a}] \quad \text{implies} \quad \mathcal{B} \models \varphi[h(\overline{a})]$$

The desired result is an immediate corollary.

We proceed by induction on φ. If φ is atomic, then $\varphi = Rx_1 \cdots x_n$ for some $R \in \tau$ (recall τ is relational). Now $\mathcal{A} \models \varphi[\overline{a}]$ iff $\overline{a}' \in R^{\mathcal{A}}$ (where $\overline{a}'$ is determined from $\overline{a}$ by $x_1, \ldots, x_n$), which, by the homomorphism property, implies $h(\overline{a}') \in R^{\mathcal{B}}$, which holds iff $\mathcal{B} \models \varphi[h(\overline{a})]$. The conjunction and disjunction cases are immediate by induction. Suppose $\varphi = \exists x \psi$. Then $\mathcal{A} \models \varphi[\overline{a}]$ iff there exists $a \in A$ such that $\mathcal{A} \models \psi[\overline{a}, a]$. By the induction hypothesis then, $\mathcal{B} \models \psi[h(\overline{a}), h(a)]$, which implies $\mathcal{B} \models \varphi[h(\overline{a})]$. $\square$

EXERCISE 2.5.4. Let φ be a first-order sentence. Then every (finite) model of φ contains a minimal model of φ.

Proof. Suppose not, and let $\mathcal{A}$ be a finite model of φ containing no minimal model of φ. This means that every submodel of φ in $\mathcal{A}$ (including $\mathcal{A}$ itself) contains a proper submodel of φ. Thus we can construct a properly decreasing sequence

$$\mathcal{A} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots$$

of submodels of φ. Now it is immediate by induction on n that

$$\|A_n\| \leq \|A\| - n$$

But $\|A\| = n$ for some n, hence $\|A_n\| = 0$—contradicting the fact that $\mathcal{A}_n$ is a structure (which must have a nonempty universe). Thus our original supposition is false, and the desired result holds. $\square$

NOTE. Let $\Phi \subseteq \mathrm{FO}_0[\tau]$ (where $\mathrm{FO}_0[\tau]$ denotes the set of first-order τ-sentences). Let Φ^B be the smallest set containing Φ that is closed under the boolean operations ($\neg$, $\wedge$, $\vee$, $\to$, $\leftrightarrow$). We call Φ^B the *boolean closure* of Φ.

Suppose $\mathcal{A}$ and $\mathcal{B}$ agree on Φ—that is, for all $\varphi \in \Phi$,

$$\mathcal{A} \models \varphi \quad \text{iff} \quad \mathcal{B} \models \varphi$$

Then $\mathcal{A}$ and $\mathcal{B}$ agree on Φ^B.

Proof. Proceed by (closure) induction on $\varphi \in \Phi^B$. For $\varphi \in \Phi$, the result holds by assumption. If the result holds for $\varphi \in \Phi^B$, then

$$\begin{array}{llll} \mathcal{A} \models \neg\varphi & \text{iff} & \text{not } \mathcal{A} \models \varphi & \text{by definition} \\ & \text{iff} & \text{not } \mathcal{B} \models \varphi & \text{by induction} \\ & \text{iff} & \mathcal{B} \models \neg\varphi & \text{by definition} \end{array}$$

Thus the result holds for $\neg\varphi$. Similarly for the other cases.

Note that formally, what we have shown is that the subset Φ_0^B of Φ^B for which the result holds contains Φ, and is closed under the boolean operations. Since Φ^B is the smallest such set, $\Phi^B \subseteq \Phi_0^B$, hence $\Phi^B = \Phi_0^B$ as desired. $\square$

Chapter 3

Section 3.1

NOTE. Let τ be an arbitrary symbol set. We give an alternate $\mathrm{SO}[\tau]$-axiomatization of $\mathrm{EVEN}[\tau]$ (see p. 37). Note that the authors construct a sentence φ which states that there exists a binary equivalence relation all of whose equivalence classes contain exactly two elements. Thus a (finite) structure satisfies φ just in case it can be partitioned into n pairs of elements for some n, which holds just in case its universe has even cardinality (namely $2n$).

Another natural approach to this problem is to construct a sentence stating that the universe can be partitioned into two sets of the same cardinality. It is clear that a (finite) structure satisfies this sentence just in case it has even cardinality.

We define

$$\begin{aligned}\varphi = \exists X \exists Y \exists F(&\exists x Xx \wedge \exists y Yy \wedge \forall x(Xx \leftrightarrow \neg Yx)\wedge\\ &\forall x(\exists y Fxy \to Xx) \wedge \forall y(\exists x Fxy \to Yy)\wedge\\ &\forall x(Xx \to \exists y(Fxy \wedge \forall z(Fxz \to y = z)))\wedge\\ &\forall y(Yy \to \exists x(Fxy \wedge \forall z(Fzy \to x = z))))\end{aligned}$$

Note that this sentence and that used by the authors are both Σ^1_1.

PROPOSITION 3.1.3. Let τ be a finite vocabulary and $m \geq 0$. The relation $\equiv^{\mathrm{MSO}}_m$ is an equivalence relation with finitely many equivalence classes.

Proof. The fact that $\equiv^{\mathrm{MSO}}_m$ is an equivalence relation is immediate from the definition. To prove that there are only finitely many equivalence classes, we claim that for all $r, s, j \geq 0$,

$$\Psi_{r,s,j} = \{\, \psi^j_{\mathcal{A},\overline{a},\overline{P}} \mid \mathcal{A} \text{ a } \tau\text{-structure, } \overline{a} \in A^r,\, \overline{P} \in \mathcal{P}(A)^s \,\}$$

is finite. Indeed, this follows by induction on j. For $j = 0$, since τ is finite, the set

$$\Phi_{r,s} = \{\, \varphi(x_1, \ldots, x_r, X_1, \ldots, X_s) \in \mathrm{FO}[\tau] \mid \varphi \text{ atomic or negated atomic} \,\}$$

is finite for all r, s. Hence there are only finitely many conjunctions over $\Phi_{r,s}$, and thus $\Psi_{r,s,j}$ is finite for all r, s as desired.

Suppose the claim holds for j—that is, $\Psi_{r,s,j}$ is finite for all r, s. It is then easy to verify by the definition that for all $\mathcal{A}, \overline{a}, \overline{P}$, $\psi^{j+1}_{\mathcal{A},\overline{a},\overline{P}}$ is in fact a first-order τ-sentence, and there are only finitely many such sentences. Thus $\Psi_{r,s,j+1}$ is finite for all r, s. By induction, the claim holds for all j.

A corollary of this claim (set $r, s = 0$ and $j = m$) is that the set

$$\Psi_m = \{\, \psi^m_{\mathcal{A}} \mid \mathcal{A} \text{ a } \tau\text{-structure} \,\}$$

is finite. We claim that the finite set

$$P_m = \{\, \mathrm{Mod}(\psi^m_{\mathcal{A}}) \mid \psi^m_{\mathcal{A}} \in \Psi_m \,\}$$

is the set of equivalence classes for $\equiv_m^{\mathrm{MSO}}$. Indeed, this is now immediate from Exercise 3.1.2, since for all τ-structures $\mathcal{B}$,

$$\mathcal{B} \models \psi_{\mathcal{A}}^m \quad \text{iff} \quad \mathcal{A} \equiv_m^{\mathrm{MSO}} \mathcal{B}$$

Thus $\mathrm{Mod}(\psi_{\mathcal{A}}^m)$ is precisely the $\equiv_m^{\mathrm{MSO}}$-equivalence class of $\mathcal{A}$. $\square$

Section 3.2

NOTE. Every subformula of an $\mathrm{L}_{\infty\omega}$-sentence contains only finitely many free variables.

Proof. Let φ be an $\mathrm{L}_{\infty\omega}$-sentence and suppose towards a contradiction that ψ is a subformula of φ containing infinitely many free variables $x_1, x_2, \ldots$ We note that ψ must occur within an infinite nested quantification over $x_1, x_2, \ldots$ in φ. But it is immediate by induction on $\mathrm{L}_{\infty\omega}$-formulas that no formula contains an infinite nested quantification. Thus the claim holds. $\square$

NOTE. Let T be a theory such that all models of T are elementarily equivalent. Then for every sentence φ, $T \models \varphi$ or $T \models \neg\varphi$.

Proof. If T is not satisfiable, the result holds trivially, so suppose $\mathcal{A} \models T$. Note that every model of T satisfies precisely the same sentences as $\mathcal{A}$. Thus for any sentence φ, if $\mathcal{A} \models \varphi$, then every model of T satisfies φ, so $\Phi \models \varphi$. If not $\mathcal{A} \models \varphi$, then (by definition) $\mathcal{A} \models \neg\varphi$, so $T \models \neg\varphi$. $\square$

EXERCISE 3.2.14. Let $\mathcal{A}$ and $\mathcal{B}$ be τ-structures. Then

$$W_0(\mathcal{A}, \mathcal{B}) \supseteq \cdots \supseteq W_m(\mathcal{A}, \mathcal{B}) \supseteq \cdots \supseteq W_\infty(\mathcal{A}, \mathcal{B})$$

Proof. We claim first that for all $m > 0$, $W_{m-1} \supseteq W_m$. Suppose $\overline{a} \mapsto \overline{b} \in W_m$. Then by definition the duplicator wins $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$. By Lemma 2.2.4(c), the duplicator wins $G_{m-1}(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$. Thus $\overline{a} \mapsto \overline{b} \in W_{m-1}$ as desired.

Now we claim that for all $m \geq 0$, $W_m \supseteq W_\infty$. Let $\overline{a} \mapsto \overline{b} \in W_\infty$. To win any play of $G_m(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$, the duplicator simply moves as it would in $G_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$. Then (by the definition of winning in G_∞) the duplicator wins the play in G_m. Thus $\overline{a} \mapsto \overline{b} \in W_m$ as desired.

These two claims imply the desired result. $\square$

Suppose now that A or B is finite. Then there exists $m_0 \leq 1 + \min\{\|A\|, \|B\|\}$ such that

$$W_0(\mathcal{A}, \mathcal{B}) \supset \cdots \supset W_{m_0}(\mathcal{A}, \mathcal{B}) = W_\infty(\mathcal{A}, \mathcal{B})$$

Proof. Suppose (say) that A is finite and $\|A\| \leq \|B\|$. Let $m_0' = 1 + \|A\|$. We first show that $W_{m_0'} \subseteq W_\infty$. Let $\overline{a} \mapsto \overline{b} \in W_{m_0'}$. Thus the duplicator has a winning strategy for $G_{m_0'}(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$. In particular, the duplicator can win any play in which the spoiler chooses all of the elements in A within its first $m_0' - 1$ moves. Note that the resulting map $\pi \supseteq \overline{a} \mapsto \overline{b}$ after $m_0' - 1$ moves must be surjective onto B, for otherwise the spoiler could choose an element $b \in B \setminus \mathrm{rng}(\pi)$ on the m_0'-th move, to which the duplicator would have no winning response—a contradiction. Thus π is an isomorphism. This provides a winning strategy for the duplicator in $G_\infty(\mathcal{A}, \overline{a}, \mathcal{B}, \overline{b})$: the duplicator simply moves according to π.

By this and the result above, we have

$$W_0 \supseteq \cdots \supseteq W_{m_0'} = W_\infty$$

We claim (without proof at the moment) that there exists $0 \leq m_0 \leq m_0'$ giving the desired result. $\square$

Section 3.3

NOTE. Let $\tau = \{<\}$ and define $\mathrm{FO}^2[\tau]$-formula $\psi_n(x)$ inductively as follows:

$$\psi_0(x) = \forall y \neg\, y < x \quad \psi_{n+1} = \forall y (y < x \leftrightarrow \bigvee_{i \leq n} \exists x (x = y \wedge \psi_i(x)))$$

We verify that for all orderings $\mathcal{A}$ and $a \in A$, and all $n \geq 0$,

$$\mathcal{A} \models \psi_n[a] \quad \text{iff} \quad a \text{ is the } n\text{-th element of } <^{\mathcal{A}}$$

Proof. Let $\mathcal{A}$ be an ordering. We proceed by induction on n. Case $n = 0$ is trivial. Now suppose the claim holds for n. If a is the $(n+1)$-th element of $<^{\mathcal{A}}$, then the elements less than a on $<^{\mathcal{A}}$ are precisely the i-th elements, for $i \leq n$. Thus $\mathcal{A} \models \psi_{n+1}[a]$.

Conversely, suppose $\mathcal{A} \models \psi_{n+1}[a]$ and let j be the position of a in $<^{\mathcal{A}}$ (note that j is well-defined since $\mathcal{A}$ is an ordering). If $j < n+1$, then by the induction hypothesis $\mathcal{A} \models \psi_i[a]$ for some $i \leq n$. But then (by way of ψ_{n+1}) $a < a$—contradicting the fact that $\mathcal{A}$ is an ordering. Thus $j \geq n+1$. If $j > n+1$, then in particular there exists an $(n+1)$-th element of $<^{\mathcal{A}}$, say a', where $a' <^{\mathcal{A}} a$. Then (again by way of ψ_{n+1}) we must have $\mathcal{A} \models \psi_i[a']$ for some $i \leq n$—contradicting the induction hypothesis. Thus $j = n+1$ as desired. $\square$

Note that all properties of an ordering (irreflexivity, transitivity, and trichotomy) were used in the proof.

EXAMPLE 3.3.6. In the following, for a pebble α in a pebble game, let α' denote the object marked by α if α marks an object, or else let $\alpha' = *$.

(a) Let $\tau = \emptyset$ and $\mathcal{A}$ and $\mathcal{B}$ be τ-structures with $\|A\|, \|B\| \geq s$. Then the duplicator wins $G^s_\infty(\mathcal{A}, \mathcal{B})$.

Proof. We describe a winning strategy for the duplicator in $G^s_\infty(\mathcal{A}, \mathcal{B})$. Suppose that on its j-th move the spoiler places a pebble α_i on the board in $\mathcal{A}$. Then on its j-th move, the duplicator considers two cases:

(i) If $\alpha_i' = \alpha_k'$ for some $k \neq i$, then the duplicator chooses $\beta_i' = \beta_k'$.

(ii) If $\alpha'_i \neq \alpha'_k$ for all $k \neq i$, note that there are at most $s-1$ pebbles other than α_k on the board in $\mathcal{A}$ and thus (trivially by induction on j) at most $s-1$ pebbles β_k on the board in $\mathcal{B}$. Thus there are at most $s-1$ pebbled elements in B. Now $\|B\| \geq s$, so there exists an unpebbled $b \in B$. The duplicator chooses $\beta'_i = b$.

The duplicator uses an analogous strategy in case the spoiler places a pebble β_i on the board in $\mathcal{B}$ on its j-th move.

It is easily verified by induction on j that this is a winning strategy for the duplicator. Indeed, for $j=0$ this holds trivially since $\emptyset \mapsto \emptyset$ is a partial isomorphism. If $j>0$ and the pebble configuration after the $(j-1)$-th moves induces an s-partial isomorphism, then the above strategy of the duplicator preserves well-definedness and injectivity for the map induced after the j-th moves. Since $\tau = \emptyset$, this map is an s-partial isomorphism.

Thus the duplicator wins $G^s_\infty(\mathcal{A},\mathcal{B})$. □

(b) Let $l \geq 3$ and let $\mathcal{A} = \mathcal{G}_l$ and $\mathcal{B} = \mathcal{G}_l \uplus \mathcal{G}_l$ be graphs consisting of one and two cycles of length $l+1$, respectively. Then the duplicator wins $G^2_\infty(\mathcal{A},\mathcal{B})$.

Proof. Recall that in G^2_∞, we are only working with pebbles α_1, α_2 in $\mathcal{A}$ and β_1, β_2 in $\mathcal{B}$. We describe part of a winning strategy for the duplicator; the remaining parts are similar.

(a) If the spoiler places β_1 on the board in $\mathcal{B}$, the duplicator moves as follows:
   i. If β_2 is off the board, the duplicator places α_1 anywhere in $\mathcal{A}$.
   ii. If β_2 is on the board and $\beta'_1 = \beta'_2$, the duplicator chooses $\alpha'_1 = \alpha'_2$.
   iii. If β_2 is on the board and $\beta'_1 \neq \beta'_2$, the duplicator moves as follows:
      A. If $E^{\mathcal{B}}\beta'_1\beta'_2$, the duplicator places α_1 such that $E^{\mathcal{A}}\alpha'_1\alpha'_2$.
      B. If not $E^{\mathcal{B}}\beta'_1\beta'_2$, then note that since $l \geq 3$, there exists $a \in A$ such that $a \neq \alpha'_2$ and not $E^{\mathcal{A}}a\alpha'_2$. The duplicator places α_1 on a.

As in the exercise above, it is verified by induction on the number of moves in a play that the duplicator wins $\mathcal{G}^2_\infty$ as desired. □

Note that the spoiler wins $\mathcal{G}^3_\infty(\mathcal{A},\mathcal{B})$. Indeed, to win, the spoiler first places β_1 and β_2 on different cycles in $\mathcal{B}$. The spoiler then chooses a direction in which to 'approach' α_1 with α_2 and α_3 in $\mathcal{A}$, as follows: the spoiler places α_3 on the vertex adjacent to α_2 in the chosen direction towards α_1. Note that the duplicator must place β_3 on same cycle as β_2 in $\mathcal{B}$. The spoiler then moves α_2 to the vertex adjacent to α_3 in the chosen direction. Again the duplicator must keep β_2 on the same cycle. The spoiler continues in this manner until α_2 (or α_3) is adjacent to α_1 and also to α_3 (respectively, α_2). The duplicator will have no winning response for β_2 (respectively, β_3) since only one edge connection can be preserved in $\mathcal{B}$ among $\beta_1, \beta_2, \beta_3$.

EXERCISE 3.3.7.

(a) Let $\tau = \{<, \ldots\}$ consist of relation symbols at most binary. Let $\mathcal{A}$ and $\mathcal{B}$ be finite ordered τ-structures. Then $\mathcal{A} \cong \mathcal{B}$ iff the duplicator wins $G^2_\infty(\mathcal{A}, \mathcal{B})$.

Proof. One direction is immediate: if $\pi : \mathcal{A} \cong \mathcal{B}$, then a winning strategy for the duplicator in $G^2_\infty(\mathcal{A}, \mathcal{B})$ is given by π.

For the converse, suppose the duplicator wins $G^2_\infty(\mathcal{A}, \mathcal{B})$. Write

$$\mathcal{A} = a_1 <^{\mathcal{A}} \cdots <^{\mathcal{A}} a_m \qquad \mathcal{B} = b_1 <^{\mathcal{B}} \cdots <^{\mathcal{B}} b_n$$

where $m = \|A\|$ and $n = \|B\|$. Assume without loss of generality that $m \leq n$.

We claim that in any play of G^2_∞ in which the duplicator wins, the following holds: for $k \leq m$, the spoiler places pebble α_i on a_k (or β_i on b_k) on his j-th move iff the duplicator places pebble β_i on b_k (respectively, α_i on a_k) on his j-th move. This is verified by induction on k. For $k = 1$, if the spoiler places (say) α_1 on a_1 but the duplicator fails to place β_1 on b_1, then in his next move the spoiler could place β_2 on b_1 such that $\beta'_2 <^{\mathcal{B}} \beta'_1$. The duplicator would be forced in his next move to place α_2 such that $\alpha'_1 <^{\mathcal{A}} \alpha'_2$—contradicting the assumption that the duplicator wins the play. Similarly for the converse, and for the other cases.

Now suppose $k > 1$ and the claim holds for values less than k. Suppose the spoiler places (say) α_1 on a_k but the duplicator places β_1 on b_l where $l \neq k$. By the induction hypothesis we must have $l > k$. Now in his next move the spoiler could place β_2 on b_k. In this case, by the induction hypothesis again and the fact that $\alpha'_1 = a_k$, the duplicator must place α_2 such that $\alpha'_1 <^{\mathcal{A}} \alpha'_2$. But $\beta'_2 <^{\mathcal{B}} \beta'_1$—a contradiction. Again a similar argument verifies the converse and other cases. Thus by induction the claim holds for all $k \leq m$.

Note that if $m < n$, then the spoiler could first place α_1 on a_m so that (by our claim) the duplicator must place β_1 on b_m; the spoiler could then place β_2 on b_{m+1}, leaving the duplicator with no winning response—contradicting our assumption. Thus $m = n$.

We claim that $\pi : a_i \mapsto b_i$ is an isomorphism. Clearly π is well-defined, bijective, and preserves order. By our claim and the assumption that the duplicator wins G^2_∞, it is immediate that π also preserves the other relations in τ. Thus π is an isomorphism. $\square$

(b) Let $m \geq s$. Then the duplicator wins $G^s_m(\mathcal{A}, \mathcal{B})$ iff the duplicator wins $G^s_m(\mathcal{A}, \mathcal{B})$ with the additional requirement that during the first s moves, distinct pebbles must be chosen. (Formally, for a given play of G^s_m, set

$$P_0 = \emptyset \qquad P_{j+1} = P_j \cup \{\alpha_i, \beta_i\} \quad (j < m)$$

where α_i and β_i are the pebbles chosen on the j-th move (disregarding which player choses which pebble). Thus P_j is the set of all pebbles chosen during the first j moves of the play. The additional requirement above states that for any play of G^s_m, for all $j < s$, $P_{j+1} \supset P_j$.)

Proof. One direction is trivial: if the duplicator wins G^s_m, then in particular the duplicator wins any play of G^s_m in which distinct pebbles are chosen during the first s moves.

Conversely, suppose the duplicator wins any play in the modified G^s_m. To win a play p of G^s_m, the duplicator constructs (and moves according to) a 'parallel' play in which distinct pebbles are chosen during the first s moves. For $n \le 2m$, denote by I_n the initial segment of pebble/element selection pairs in p up to n moves (where the moves of each player are counted separately). The duplicator initially sets $I'_0 = I_0$. For $1 \le j \le s$, assume that I'_{2j-2} is defined and is an initial segment of a play in the modified G^s_m in which the duplicator has been moving according to his winning strategy. Suppose that on his j-th move the spoiler places pebble γ_i somewhere. Then on his j-th move, the duplicator proceeds as follows:

(a) If γ_i does not appear in I'_{2j-2}, the duplicator defines

$$I'_{2j-1} = (I'_{2j-2}, (\gamma_i, \gamma'_i))$$

He then moves in p as he would in the modified G^s_m (according to his winning strategy) in a play with initial segment I'_{2j-1}. Suppose in doing so places pebble ρ_i somewhere. He then defines $I'_{2j} = (I'_{2j-1}, (\rho_i, \rho'_i))$.

(b) If γ_i *does* appear in I'_{2j-2}, the duplicator finds a new pebble γ_k to 'substitute' for γ_i in the parallel play. More specifically, the duplicator finds $\gamma_k \notin I'_{2j-2}$ (such a pebble exists since $j \le s$) and defines

$$I'_{2j-1} = (I'_{2j-2}, (\gamma_k, \gamma'_i))$$

Now I'_{2j-1} is an initial segment in the modified G^s_m, so the duplicator can respond in p as he would (according to his winning strategy) in the modified G^s_m with initial segment I'_{2j-1}. If in doing so he places ρ_i somewhere, he sets $I'_{2j} = (I'_{2j-1}, (\rho_k, \rho'_i))$.

It is now easy to verify by induction on j that this provides a winning strategy for the duplicator in G^s_m during the first s moves. (Formally, one proves that for $1 \le j \le s$, the map π induced by the pebble configuration in p after j moves is a subset of the map induced by the pebble configuration in the parallel play of the modified G^s_m after j moves (which the duplicator wins), and hence π is an s-partial isomorphism.)

Note that after s moves, all pebbles are on the board in the parallel play. If all pebbles are also on the board in p, then p corresponds to its own parallel play, and the duplicator can simply move according to its winning strategy for the modified G^s_m for the remainder of the moves in p.

If not all pebbles are on the board in p after s moves, then at least one pebble required a substitute in the parallel play. Associate with each pebble on the board in p its most recent substitute in the parallel play, if it required one, or itself if not; call these the *associate pebbles*. Note that the pebbles off the board in p can be put in bijective correspondence with the non-associate pebbles in the parallel play. Thus for the remainder of p, the duplicator can move pebbles on the board as he would their associates in the parallel play, and can handle new pebbles (in p) with the non-associates in the parallel play. $\square$

NOTE. It is immediate by induction on m that the free variables in ${}^s\psi^m_{\mathcal{A},\overline{a}}$ have indices in $\mathrm{supp}(\overline{a})$ for all structures $\mathcal{A}$ and $\overline{a} \in (A \cup \{*\})^s$. In particular, $\psi^m_{\mathcal{A}}$ is a sentence.

NOTE. For structures $\mathcal{A}$ and $\mathcal{B}$, if the duplicator wins $G^s_\infty(\mathcal{A}, \mathcal{B})$, then the duplicator wins $G^s_m(\mathcal{A}, \mathcal{B})$ for all $m \geq 0$. The converse holds if $\mathcal{A}$ and $\mathcal{B}$ are finite.

Proof. The first claim is immediate by definitions. If $\mathcal{A}$ and $\mathcal{B}$ are finite and the duplicator wins G^s_m for all $m \geq 0$, then by Corollary 3.3.10(a), $\mathcal{A} \equiv^s_m \mathcal{B}$ for all $m \geq 0$. Thus $\mathcal{A} \equiv^s \mathcal{B}$. By Corollary 3.3.3, $\mathcal{A} \equiv^{\mathrm{L}^s_{\infty\omega}} \mathcal{B}$, so by Corollary 3.3.10(b), the duplicator wins G^s_∞. $\square$

EXERCISE 3.3.14.

(a) Let K be a class of finite structures. Then (i) K is not axiomatizable in FO^s iff (ii) for every $m \geq 1$ there exist finite structures $\mathcal{A}$ and $\mathcal{B}$ such that

$$\mathcal{A} \cong^s_m \mathcal{B} \text{ but } \mathcal{A} \in K, \mathcal{B} \notin K$$

Proof. Suppose towards a contradiction that (ii) holds but (i) fails to hold. Thus there exists a $\varphi \in \mathrm{FO}^s_0$ such that $K = \mathrm{Mod}(\varphi)$. Set $m = \mathrm{qr}(\varphi)$. Choose $\mathcal{A}$ and $\mathcal{B}$ as in (ii). By Corollary 3.3.10(a), $\mathcal{A} \equiv^s_m \mathcal{B}$, hence $\mathcal{A} \models \varphi$ iff $\mathcal{B} \models \varphi$. Since $\mathcal{A} \in K$, $\mathcal{A} \models \varphi$, so $\mathcal{B} \models \varphi$. But $\mathcal{B} \notin K$ by hypothesis, so $\mathcal{B} \not\models \varphi$—a contradiction. Thus (ii) implies (i).

Conversely, suppose (ii) fails to hold, so there exists some $m \geq 1$ such that for all finite $\mathcal{A}$ and $\mathcal{B}$,

$$\mathcal{A} \in K \text{ and } \mathcal{A} \cong^s_m \mathcal{B} \text{ implies } \mathcal{B} \in K$$

Define $\varphi = \bigvee_{\mathcal{A} \in K} \psi^m_{\mathcal{A}}$. Note that $\varphi \in \mathrm{FO}^s_0$. We claim that $K = \mathrm{Mod}(\varphi)$. Indeed, for all $\mathcal{B}$, $\mathcal{B} \models \psi^m_{\mathcal{B}}$, thus $\mathcal{B} \models \varphi$ if $\mathcal{B} \in K$. Conversely, if $\mathcal{B} \models \varphi$, then $\mathcal{B} \models \psi^m_{\mathcal{A}}$ for some $\mathcal{A} \in K$. Again by Corollary 3.3.10(a), $\mathcal{A} \cong^s_m \mathcal{B}$, hence $\mathcal{B} \in K$ as desired. $\square$

(b) Let K be a class of finite structures and suppose Γ is a global n-ary relation on K. Then the following are equivalent for $s \geq n$:

(i) Γ is $\mathrm{L}^s_{\infty\omega}$-definable—that is, there exists $\varphi \in \mathrm{L}^s_{\infty\omega}$ such that for all $\mathcal{A} \in K$ and $\overline{a} \in A$,

$$\mathcal{A} \models \varphi[\overline{a}] \quad \text{iff} \quad \overline{a} \in \Gamma(\mathcal{A})$$

(ii) Γ is closed under G^s_∞—that is, for all $\mathcal{A}, \mathcal{B} \in K$, $\overline{a} \in \Gamma(\mathcal{A})$ and $\overline{b} \in B$, if the duplicator wins $G^s_\infty(\mathcal{A}, \overline{a} * \cdots *, \mathcal{B}, \overline{b} * \cdots *)$, then $\overline{b} \in \Gamma(\mathcal{B})$.

Proof. Suppose (i) holds. Let $\mathcal{A}, \mathcal{B} \in K$, $\bar{a} \in \Gamma(\mathcal{A})$, and $\bar{b} \in B$. By (i), $\mathcal{A} \models \varphi[\bar{a}]$. If the duplicator wins $G^s_\infty(\ldots)$, then by Theorem 3.3.9(b), $\bar{a}$ satisfies in $\mathcal{A}$ the same $\mathrm{L}^s_{\infty\omega}$-formulas as $\bar{b}$ does in $\mathcal{B}$. Thus $\mathcal{B} \models \varphi[\bar{b}]$, and by (i) again, $\bar{b} \in \Gamma(\mathcal{B})$. Thus (i) implies (ii).

Conversely, suppose (ii) holds. Define

$$\varphi = \bigvee \{ \bigwedge_{m \geq 0} \psi^m_{\mathcal{A},\bar{a}} \mid \mathcal{A} \in K, \bar{a} \in \Gamma(\mathcal{A}) \}$$

Note that $\varphi \in \mathrm{L}^s_{\infty\omega}$. We claim that for all $\mathcal{B} \in K$ and $\bar{b} \in B$, $\mathcal{B} \models \varphi[\bar{b}]$ iff $\bar{b} \in \Gamma(\mathcal{B})$. Indeed, if $\mathcal{B} \in K$ and $\bar{b} \in \Gamma(\mathcal{B})$, then (trivially) $\mathcal{B} \models \psi^m_{\mathcal{B},\bar{b}}[\bar{b}]$ for all $m \geq 0$. Hence $\mathcal{B} \models \varphi[\bar{b}]$. Conversely, if $\mathcal{B} \models \varphi[\bar{b}]$, then for some $\mathcal{A} \in K$ and $\bar{a} \in \Gamma(\mathcal{A})$, $\mathcal{B} \models \bigwedge_{m \geq 0} \psi^m_{\mathcal{A},\bar{a}}[\bar{b}]$. By Theorem 3.3.9(a), the duplicator wins $G^s_m(\ldots)$ for all $m \geq 0$. Thus (since $\mathcal{A}$ and $\mathcal{B}$ are finite!), the duplicator wins $G^s_\infty(\ldots)$. Hence by (ii), $\bar{b} \in \Gamma(\mathcal{B})$ as claimed. Thus (ii) implies (i). □

EXERCISE 3.3.26. $\mathrm{L}^\omega_{\infty\omega}$ has more expressive power than FO on both the class of finite orderings and the class of finite graphs.

Proof. We use formulas from Example 3.3.1. For the class ORD of finite orderings, set

$$\varphi_E = \bigvee_{n \geq 0} \chi_{2n}$$

Then for all $\mathcal{A} \in$ ORD, $\mathcal{A} \models \varphi_E$ iff $\mathcal{A}$ is even. Thus the even finite orderings are axiomatizable in $\mathrm{L}^\omega_{\infty\omega}$, which is not the case in FO by Example 2.3.6.

For the class GRAPH of finite graphs, set

$$\varphi_C = \forall x \forall y (x = y \bigvee_{n \geq 1} \varphi_n(x, y))$$

Then for all $\mathcal{G} \in$ GRAPH, $\mathcal{G} \models \varphi_C$ iff $\mathcal{G}$ is connected, so the class CONN of finite connected graphs is axiomatizable (relative to GRAPH) in $\mathrm{L}^\omega_{\infty\omega}$. This is not the case in FO by Example 2.3.8. Note also that by a simple compactness argument, CONN is not axiomatizable in FO relative to the class of all graphs (including infinite graphs); of course, φ_C still works in $\mathrm{L}^\omega_{\infty\omega}$ relative to the class of all graphs. □

Section 3.4

NOTE. For $l \geq 1$, $\models \exists^{\geq l} x \varphi(x) \leftrightarrow \neg \bigvee_{j<l} \exists^{=j} x \varphi(x)$.

Proof. This follows immediately from cardinality results. For $l \geq 1$, $\mathcal{A} \models \exists^{\geq l} x \varphi(x)$ iff there exist at least l elements in the set of solutions to $\varphi(x)$ in $\mathcal{A}$, which holds iff it is not the case for any $j < l$ that there exist exactly j elements in the set of solutions, which holds iff $\mathcal{A} \models \neg \bigvee_{j<l} \exists^{=j} x \varphi(x)$. □

PROPOSITION 3.4.4. Let $\mathcal{G} = (G, E^G, C^G_1, \ldots, C^G_r)$ be a stable colored graph and $a, b \in G$. Then a and b have the same color iff the duplicator wins C-$G^2_\infty(\mathcal{G}, a*, \mathcal{G}, b*)$.

Proof. First suppose the duplicator wins $C\text{-}G^2_\infty(\ldots)$. Then from Theorem 3.4.3 it follows that for all $\varphi(x) \in \mathrm{C}^s_{\infty\omega}$,

$$\mathcal{G} \models \varphi[a] \quad \text{iff} \quad \mathcal{G} \models \varphi[b]$$

In particular, for all $j = 1, \ldots, r$,

$$\mathcal{G} \models C_j x[a] \quad \text{iff} \quad \mathcal{G} \models C_j x[b]$$

Thus $a \in C_j^G$ iff $b \in C_j^G$, so a and b have the same color.

Conversely, suppose a and b have the same color. We describe a winning strategy for the duplicator in $C\text{-}G^2_\infty(\ldots)$. Suppose that on his j-th move the spoiler selects (say) α_1 and $X \subseteq G$. The duplicator responds on his j-th move as follows:

1. If α_2 is not on the board, the duplicator simply responds with $Y = X$. When the spoiler then places β_1 on an element in Y, the duplicator moves according to the identity and places α_1 on the same element.

2. If α_2 is on the board, we assume that α_2' and β_2' were previously chosen to have the same color, and thus the same color type since $\mathcal{G}$ is stable. The duplicator chooses Y to have the same number of elements of each color directly connected to β_2' as X has for α_2', as well as the same number of elements of each color not directly connected (note that this is possible by the assumption of color type). Then $\|X\| = \|Y\|$. When the spoiler places β_1 on an element in Y, the duplicator responds as follows:

   (a) If $\beta_1' = \beta_2'$, then the duplicator sets $\alpha_1' = \alpha_2'$.

   (b) If $\beta_1' \neq \beta_2'$ and $(\beta_1', \beta_2') \in E^G$, then the duplicator places α_1 on an element with the same color as β_1' and directly connected to α_2' (this is possible by construction of Y).

   (c) If $\beta_1' \neq \beta_2'$, and $(\beta_1', \beta_2') \notin E^G$, then the duplicator places α_1 on an element with the same color as β_1' but not directly connected to α_2' (again possible by construction of Y).

The duplicator responds similarly for other moves of the spoiler.

It is verified by induction on j (the number of moves) that this provides a winning strategy for the duplicator. Indeed, for $j = 0$ note that by assumption a and b have the same color and thus $a \mapsto b$ is a partial isomorphism. For $j > 0$, if after the completion of $j-1$ moves $\alpha_i' \mapsto \beta_i'$ is a 2-partial isomorphism, then the above strategy can be used by the duplicator and preserves the 2-partial isomorphism property. $\square$

Section 3.5

EXERCISE 3.5.3. In the finite, $\mathrm{L}_{\omega_1\omega}$ has the Beth property, the Craig interpolation property, and is closed under order-invariant sentences.

Proof. For the Beth property, let τ be a (finite) symbol set and $R \notin \tau$ an n-ary relation symbol. Suppose $\varphi \in \mathrm{L}_{\omega_1\omega}[\tau \cup \{R\}]$ defines R implicitly in the finite—that is, for all finite τ-structures $\mathcal{A}$ and $R_1^A, R_2^A \subseteq A^n$

$$(\mathcal{A}, R_1^A) \models \varphi \text{ and } (\mathcal{A}, R_2^A) \models \varphi \text{ implies } R_1^A = R_2^A$$

Then we claim there exists an explicit τ-definition of R relative to φ in the finite—that is, there exists a formula $\psi(\overline{x}) \in \mathrm{L}_{\omega_1\omega}[\tau]$ such that

$$\varphi \models_{\text{fin}} \forall \overline{x}(R\overline{x} \leftrightarrow \psi(\overline{x}))$$

To prove this, set

$$\psi(\overline{x}) = \bigvee \{\, \varphi_{\mathcal{A},\overline{a}}^{\|A\|+1} \mid \mathcal{A} \text{ a finite } \tau\text{-structure}, R^A \subseteq A^n, (\mathcal{A}, R^A) \models \varphi, \overline{a} \in R^A \,\}$$

Note that $\psi \in \mathrm{L}_{\omega_1\omega}[\tau]$. Now suppose that $\mathcal{B}$ is a finite τ-structure, $R^B \subseteq B^n$, and $(\mathcal{B}, R^B) \models \varphi$. If $\overline{b} \in R^B$, then since (trivially) $\mathcal{B} \models \varphi_{\mathcal{B},\overline{b}}^{\|B\|+1}[\overline{b}]$, $\mathcal{B} \models \psi[\overline{b}]$. Conversely, suppose $\mathcal{B} \models \psi[\overline{b}]$, so for some finite τ-structure $\mathcal{A}$, $R^A \subseteq A^n$, and $\overline{a} \in R^A$ with $(\mathcal{A}, R^A) \models \varphi$, $\mathcal{B} \models \varphi_{\mathcal{A},\overline{a}}^{\|A\|+1}[\overline{b}]$. Thus $(\mathcal{A}, \overline{a}) \cong_{\|A\|+1} (\mathcal{B}, \overline{b})$, so $(\mathcal{A}, \overline{a}) \cong (\mathcal{B}, \overline{b})$, say by way of π. It follows (by induction on φ) that $(\mathcal{B}, \pi(R^A)) \models \varphi$, so by the implicit definition of R, $\pi(R^A) = R^B$. Since $\pi(\overline{a}) = \overline{b}$, we have $\overline{b} \in R^B$ as desired.

For the interpolation property, suppose now that φ is an $\mathrm{L}_{\omega_1\omega}[\sigma]$-sentence, ψ is an $\mathrm{L}_{\omega_1\omega}[\tau]$-sentence, and $\varphi \models_{\text{fin}} \psi$. We claim there exists an $\mathrm{L}_{\omega_1\omega}[\sigma \cap \tau]$-sentence χ (the interpolant) satisfying

$$\varphi \models_{\text{fin}} \chi \quad \text{and} \quad \chi \models_{\text{fin}} \psi$$

By Example 3.2.1(b), any class K of finite structures is axiomatizable in $\mathrm{L}_{\omega_1\omega}$. Thus let χ axiomatize

$$K = \{\, \mathcal{A}|_{(\sigma\cap\tau)} \mid \mathcal{A} \text{ a finite } \sigma\text{-structure}, \mathcal{A} \models \varphi \,\}$$

We claim that χ is the desired interpolant. Indeed, if $\mathcal{A}$ is a finite σ-structure and $\mathcal{A} \models \varphi$, then $\mathcal{A}|_{(\sigma\cap\tau)} \in K$ and hence $\mathcal{A} \models \chi$. If $\mathcal{B}$ is a finite τ-structure and $\mathcal{B} \models \chi$, then $\mathcal{B}|_{(\sigma\cap\tau)} \in K$, which by definition of K implies that $\mathcal{B}$ can be extended to a $(\sigma \cup \tau)$-structure $\mathcal{B}'$ with $\mathcal{B}' \models \varphi$. Now since $\varphi \models_{\text{fin}} \psi$, $\mathcal{B}' \models \psi$ and hence $\mathcal{B} \models \psi$ as desired.

In light of the interpolation property, it follows from the remarks on p. 64 that $\mathrm{L}_{\omega_1\omega}$ is closed under order-invariant sentences. $\square$

References

[1] Ebbinghaus, H.–D. and J. Flum. *Finite Model Theory*, 2nd ed. New York: Springer, 1999.