Notes and exercises from
Elementary Probability Theory

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Introduction

This document contains notes and exercises from [1].

Chapter 2

Exercise (3)**.** Let Ω be a sample space and P a probability measure on Ω. Let $A_1 + \cdots + A_n = \Omega$ be a partition and $a_1, \ldots, a_n > 0$. Define

$$Q(S) = \frac{a_1 P(SA_1) + \cdots + a_n P(SA_n)}{a_1 P(A_1) + \cdots + a_n P(A_n)} \qquad (S \subseteq \Omega)$$

Then Q is a probability measure on Ω.

Proof. By unity and additivity of P,

$$1 = P(\Omega) = P(A_1 + \cdots + A_n) = P(A_1) + \cdots + P(A_n)$$

so there is at least one $P(A_k) \neq 0$ and hence $a_k P(A_k) \neq 0$. By nonnegativity of P, $a_i P(SA_i) \geq 0$ for all $1 \leq i \leq n$ and $S \subseteq \Omega$. It follows (taking $S = \Omega$) that $\alpha = a_1 P(A_1) + \cdots + a_n P(A_n) > 0$, and $Q(S) \geq 0$ for all $S \subseteq \Omega$, so Q is well-defined and nonnegative. Also $Q(\Omega) = \alpha / \alpha = 1$, so Q is unital.

Finally, let $\{S_k\}$ be a countable family of disjoint subsets of Ω and $S = \sum S_k$.

Then

$$\begin{aligned}
\alpha Q(S) &= a_1 P(SA_1) + \cdots + a_n P(SA_n) \\
&= a_1 P\big(\sum S_k A_1\big) + \cdots + a_n P\big(\sum S_k A_n\big) && \text{by distributivity} \\
&= a_1 \sum P(S_k A_1) + \cdots + a_n \sum P(S_k A_n) && \text{by additivity of } P \\
&= \sum \big[a_1 P(S_k A_1) + \cdots + a_n P(S_k A_n)\big] \\
&= \sum \alpha Q(S_k) \\
&= \alpha \sum Q(S_k)
\end{aligned}$$

so $Q(S) = \sum Q(S_k)$ and Q is additive. □

Exercise (6)**.** Let Ω be a sample space, $P_1, \ldots, P_n$ probability measures on Ω, and $a_1, \ldots, a_n \geq 0$ with $a_1 + \cdots + a_n = 1$. Then

$$Q = a_1 P_1 + \cdots + a_n P_n$$

is a probability measure on Ω.

Proof. By nonnegativity of each P_i, $a_i P_i(S) \geq 0$ and hence $Q(S) \geq 0$ for all $S \subseteq \Omega$, so Q is nonnegative. By unity of each P_i,

$$Q(\Omega) = a_1 P_1(\Omega) + \cdots + a_n P_n(\Omega) = a_1 \cdot 1 + \cdots + a_n \cdot 1 = a_1 + \cdots + a_n = 1$$

so Q is unital. If $\{S_k\}$ is a countable family of disjoint subsets of Ω, then by additivity of each P_i,

$$\begin{aligned}
Q\big(\sum S_k\big) &= a_1 P_1\big(\sum S_k\big) + \cdots + a_n P_n\big(\sum S_k\big) \\
&= a_1 \sum P_1(S_k) + \cdots + a_n \sum P_n(S_k) \\
&= \sum \big[a_1 P_1(S_k) + \cdots + a_n P_n(S_k)\big] \\
&= \sum Q(S_k)
\end{aligned}$$

so Q is additive. □

Exercise (21)**.** Let Ω be a sample space, P a probability measure on Ω, and $\{A_k\}$ a countable family of subsets of Ω.

(a) If $A_k \subseteq A_{k+1}$ for all $k \geq 1$ and $A = \bigcup A_k$, then $P(A) = \lim_{k\to\infty} P(A_k)$.

(b) If $A_k \supseteq A_{k+1}$ for all $k \geq 1$ and $A = \bigcap A_k$, then $P(A) = \lim_{k\to\infty} P(A_k)$.

Proof. (a) Set $A_0 = \emptyset$. Then $A = \sum(A_k - A_{k-1})$, so by additivity of P,

$$\begin{aligned}
P(A) &= \sum P(A_k - A_{k-1}) \\
&= \sum \big[P(A_k) - P(A_{k-1})\big] \\
&= \lim_{n\to\infty} \sum_{k=1}^{n} \big[P(A_k) - P(A_{k-1})\big] \\
&= \lim_{n\to\infty} P(A_n)
\end{aligned}$$

(b) Observe $A_k^{\mathrm{c}} \subseteq A_{k+1}^{\mathrm{c}}$ for all $k \geq 1$ and $A^{\mathrm{c}} = \bigcup A_k^{\mathrm{c}}$. By part (a),

$$1 - P(A) = P(A^{\mathrm{c}}) = \lim_{k\to\infty} P(A_k^{\mathrm{c}}) = \lim_{k\to\infty} \big[1 - P(A_k)\big] = 1 - \lim_{k\to\infty} P(A_k)$$

so $P(A) = \lim_{k\to\infty} P(A_k)$. □

References

[1] Chung, K. L. *Elementary Probability Theory with Stochastic Processes*, 3rd ed. Springer, 1979.