

Notes and exercises from *Set Theory*

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Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (1.2). There is no set X such that $P(X) \subset X$.

Proof. By the axiom of regularity (1.8), X is \in -minimal in $\{X\}$, so $X \notin X$ and hence $P(X) \not\subset X$. \square

Exercise (1.3). If X is inductive, then the set $\{x \in X \mid x \subset X\}$ is inductive. Hence \mathbf{N} is transitive and for each $n \in \mathbf{N}$, $n = \{m \in \mathbf{N} \mid m < n\}$.

Proof. Let $S = \{x \in X \mid x \subset X\}$. By inductivity of X , $\emptyset \in S$, and if $x \in S$, then $x \cup \{x\} \in S$, so S is inductive. Taking $X = \mathbf{N}$, it follows that $S = \mathbf{N}$ since \mathbf{N} is the smallest inductive set. Hence $n \in \mathbf{N}$ implies $n \subset \mathbf{N}$, so \mathbf{N} is transitive and $n = \{m \in \mathbf{N} \mid m < n\}$. \square

Remark. We proved transitivity of \mathbf{N} “by induction” on \mathbf{N} : $0 \subset \mathbf{N}$ and if $n \subset \mathbf{N}$ then $n + 1 \subset \mathbf{N}$, so $n \subset \mathbf{N}$ for all $n \in \mathbf{N}$. The following exercises are similar.

Exercise (1.4). If X is inductive, then the set $\{x \in X \mid x \text{ is transitive}\}$ is inductive. Hence every $n \in \mathbf{N}$ is transitive.

Proof. The class C of transitive sets is inductive. Indeed, \emptyset is transitive, and if x is transitive then $x \cup \{x\}$ is transitive since $y \in x \cup \{x\}$ implies $y \subset x \subset x \cup \{x\}$. It follows that $\{x \in X \mid x \text{ is transitive}\} = X \cap C$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \mathbf{N}$, it follows as above that every $n \in \mathbf{N}$ is transitive. \square

Exercise (1.5). If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and } x \not\in x\}$ is inductive. Hence $n \not\in n$ and $n \neq n + 1$ for all $n \in \mathbf{N}$.

Proof. The class $C = \{x \mid x \text{ is transitive and } x \not\in x\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive (by inductivity of the class of transitive sets). Also $x \cup \{x\} \not\in x$, lest $x \cup \{x\} \subset x$ by transitivity of x and hence $x \in x$ —contradicting $x \not\in x$. Similarly $x \cup \{x\} \neq x$. Therefore $x \cup \{x\} \not\in x \cup \{x\}$. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive, and taking $X = \mathbf{N}$ that $n \not\in n$ and hence $n \neq n + 1$ for all $n \in \mathbf{N}$. \square

Remark. In order to prove that $n \not\in n$ for all $n \in \mathbf{N}$ by induction on \mathbf{N} , we “loaded the induction hypothesis” with transitivity.

Exercise (1.6). If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and regular}\}$ is inductive, where a set x is called *regular* if every nonempty subset of x has an \in -minimal element.

Proof. The class $C = \{x \mid x \text{ is transitive and regular}\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive. If $y \subset x \cup \{x\}$ is nonempty, let $z = y - \{x\} \subset x$. If $z = \emptyset$, then $y = \{x\}$, and x is \in -minimal in y by regularity of x . If $z \neq \emptyset$ and t is \in -minimal in z , then $x \not\in t$ by transitivity and regularity of x , so t is \in -minimal in y . Therefore $x \cup \{x\}$ is regular. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive. \square

Remark. Taking $X = \mathbf{N}$ it follows as above that every $n \in \mathbf{N}$ is regular.

Exercise (1.7). Every nonempty $X \subset \mathbf{N}$ has an \in -minimal element.

Proof. Choose $n \in X$. If n is not \in -minimal in X , then $n \cap X$ is a nonempty subset of n , which has an \in -minimal element m (Exercise 1.6). By transitivity of n (Exercise 1.4), m is \in -minimal in X . \square

Exercise (1.8). If X is inductive, then the set $\{x \in X \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is inductive. Hence for all $n \in \mathbf{N}$, $n = 0$ or $n = m + 1$ for some $m \in \mathbf{N}$.

Proof. The class $C = \{x \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is obviously inductive, so $X \cap C$ is inductive. Taking $X = \mathbf{N}$, the rest follows, with $m \in \mathbf{N}$ by transitivity of \mathbf{N} (Exercise 1.3). \square

Exercise (1.9). Let $A \subset \mathbf{N}$ be such that $0 \in A$, and if $n \in A$ then $n + 1 \in A$. Then $A = \mathbf{N}$.

Proof. A is inductive, so $A = N$ since N is the smallest inductive set.

Alternately, if $A \neq N$, let m be \in -minimal in $N - A$ (Exercise 1.7). Then $m \neq 0$, and $m \neq n + 1$ for any $n \in N$, which is impossible (Exercise 1.8). \square

Exercise (1.10). Each $n \in N$ is T-finite.

Proof. By induction on N . First, $n = 0$ is T-finite since $P(\emptyset) = \{\emptyset\}$ and \emptyset is \subset -maximal in $\{\emptyset\}$. If n is T-finite, suppose $X \subset P(n \cup \{n\})$ is nonempty. Let

$$X' = \{u - \{n\} \mid u \in X\}$$

Then $X' \subset P(n)$ is nonempty and has a \subset -maximal element $u - \{n\}$ for some $u \in X$ by T-finiteness of n , where we may assume $n \in u$ if $u \cup \{n\} \in X$. We claim u is \subset -maximal in X . Indeed, if $v \in X$ and $u \subset v$, then $u - \{n\} \subset v - \{n\}$, so $u - \{n\} = v - \{n\}$ by \subset -maximality of $u - \{n\}$ in X' , so $u = v$ by the assumption about u . Therefore $n + 1$ is T-finite. \square

Exercise (1.11). N is T-infinite. In fact, $N \subset P(N)$ has no \subset -maximal element.

Proof. We know $N \subset P(N)$ by transitivity of N (Exercise 1.3), and for all $n \in N$ we have $n \subset n + 1 \in N$ but $n \neq n + 1$ (Exercise 1.5). \square

Remark. If A is T-finite and $\pi : A \rightarrow B$ is surjective (onto), then B is T-finite.

Proof. By pullback. If $X \subset P(B)$ is nonempty, let

$$X_{-1} = \pi_{-1}(X) = \{u_{-1} = \pi_{-1}(u) \mid u \in X\}$$

Then $X_{-1} \subset P(A)$ is nonempty and has a \subset -maximal element u_{-1} for some $u \in X$ by T-finiteness of A . We claim u is \subset -maximal in X . Indeed, if $v \in X$ and $u \subset v$, then $u_{-1} \subset v_{-1}$, so $u_{-1} = v_{-1}$ by \subset -maximality of u_{-1} in X_{-1} , so $u = v$ by surjectivity of π . Therefore B is T-finite. \square

Exercise (1.12). Every finite set is T-finite.

Proof. By definition, every finite set is the image of a T-finite natural number (Exercise 1.10), so is T-finite by the previous remark.

Alternately, if S is T-infinite, choose $X \subset P(S)$ nonempty with no \subset -maximal element. By induction on N , for each $n \in N$ there is a properly ascending chain $u_0 \subset \cdots \subset u_n \subset S$ of length $n + 1$ in X , and hence an injection $n \rightarrow S$ sending $m \in n$ into $u_{m+1} - u_m$. If S had k elements for some $k \in N$, there would then be an injection $k + 1 \rightarrow k$, which is impossible by an easy induction on k . Therefore S is infinite. \square

Exercise (1.13). Every infinite set is T-infinite.

Proof. If S is infinite, let X be the set of finite subsets of S . Clearly X is nonempty since $\emptyset \in X$. Also X has no \subset -maximal element. Indeed, if $u \in X$ then there is $s \in S - u$ since S is infinite, and $u \cup \{s\}$ is finite (if u has k elements, then $u \cup \{s\}$ has $k + 1$ elements), so $u \subset u \cup \{s\} \in X$ where the inclusion is proper. Therefore S is T-infinite. \square

Remark. The previous two exercises show that (Cantor) finiteness is equivalent to Tarski finiteness in ZF.

Chapter 2

Remark. In (2.1), $<$ is actually a *well-ordering* on Ord . Indeed, if $C \subset Ord$ is a nonempty class of ordinals and $\alpha \in C$ is not \in -minimal in C , then $\alpha \cap C$ is a nonempty subset of α which has an \in -minimal element β . By transitivity of α , β is \in -minimal in C .

This observation provides an alternative proof that Ord is a proper class: if Ord were a set, then because it is transitive and strictly well-ordered by \in , it would be an ordinal, and hence $Ord \in Ord$ —contradicting strictness.

Remark. In (2.13), an ordinal is “finite” if and only if it is a “finite ordinal”. In fact, if α is not a “finite ordinal” then $\omega \subset \alpha$, and it follows by induction on n that there is no surjection $n \rightarrow \omega$ (every function $n \rightarrow \omega$ attains a maximum), hence there is no surjection $n \rightarrow \alpha$, so α is not “finite”. The converse is trivial.

Exercise (2.2). α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$ for all β .

Proof. If α is a limit ordinal and $\beta < \alpha$, then $\beta + 1 \leq \alpha$ (2.5) and $\beta + 1 \neq \alpha$, so $\beta + 1 < \alpha$. If α is not a limit ordinal, then $\beta + 1 = \alpha$ for some $\beta < \alpha$. \square

Remark. It follows that α is a nonzero limit ordinal if and only if it is inductive.

Exercise (2.3). If X is inductive, then $X \cap Ord$ is inductive. N is the least nonzero limit ordinal, where $N = \bigcap \{X \mid X \text{ inductive}\}$.

Proof. Clearly Ord is inductive, so $X \cap Ord$ is inductive since the intersection of two inductive classes is inductive. Taking $X = N$, it follows that $N \subset Ord$, and since N is also transitive, N is an ordinal. By the previous remark, N is the least nonzero limit ordinal. \square

Exercise (2.4). (Without the axiom of infinity.) Let ω be the least nonzero limit ordinal, if it exists, or *Ord* otherwise. The following are equivalent:

- (i) There exists an inductive set.
- (ii) There exists an infinite¹ set.
- (iii) ω is a set.

Proof. (i) \iff (iii): The smallest inductive set is the least nonzero limit ordinal (Exercise 2.3).

(iii) \implies (ii): ω is infinite (Exercises 1.11–2).

(ii) \implies (i): For any finite set A , let $|A|$ denote the least $n \in \omega$ in bijective correspondence with A (the “number of elements” in A). If X is infinite, let

$$S = \{|A| \mid A \subset X \text{ finite}\}$$

Note S is a set by replacement, and S is inductive. Indeed, $0 \in S$ since $\emptyset \subset X$ and $|\emptyset| = 0$. If $n \in S$ and $A \subset X$ with $|A| = n$, then there must exist $x \in X - A$ since X is infinite, and $|A \cup \{x\}| = n + 1 \in S$. \square

Remark. This exercise shows that (i)–(iii) are equivalent forms of the axiom of infinity in ZF.

Exercise (2.5). If W is a well-ordered set, then there is no sequence $\langle a_n \mid n \in \mathbf{N} \rangle$ in W such that $a_0 > a_1 > \dots$.

Proof. If there were such a sequence, then $\{a_n \mid n \in \mathbf{N}\}$ would be a nonempty subset of W with no least element, contradicting well-ordering. \square

Exercise (2.6). There are arbitrarily large limit ordinals.

Proof. Given α , let $\beta = \alpha + \omega$. Clearly $\beta > \alpha$. If $\gamma < \beta$, then either $\gamma < \alpha$, in which case $\gamma + 1 < \alpha + 1 < \beta$, or $\gamma \geq \alpha$, in which case $\gamma = \alpha + n$ for some $n \in \omega$ (Lemma 2.25), so $\gamma + 1 = \alpha + n + 1 < \beta$. Thus β is a limit ordinal (Exercise 2.2). \square

Remark (Chain rule). If $f, g : \text{Ord} \rightarrow \text{Ord}$ are nondecreasing and continuous, then so is $f \circ g$. If f and g are normal, then so is $f \circ g$.

¹In this exercise, a set is *finite* if it is in bijective correspondence with some $n \in \omega$ and *infinite* otherwise, even if $\omega = \text{Ord}$.

Proof. If $\alpha < \beta$, then $g(\alpha) \leq g(\beta)$, so $f(g(\alpha)) \leq f(g(\beta))$. Let α be a nonzero limit ordinal. Clearly $f(g(\alpha)) \geq \lim_{\xi \rightarrow \alpha} f(g(\xi))$. If $g(\alpha) = g(\xi')$ for some $\xi' < \alpha$, then $f(g(\alpha)) = f(g(\xi')) \leq \lim_{\xi \rightarrow \alpha} f(g(\xi))$. Otherwise, $g(\alpha)$ is a limit ordinal. Indeed, $g(\alpha) = \lim_{\xi \rightarrow \alpha} g(\xi)$ by continuity of g , so if $\beta < g(\alpha)$, then $\beta < g(\xi)$ for some $\xi < \alpha$, and hence $\beta + 1 \leq g(\xi) < g(\alpha)$. But then $f(g(\alpha)) = \lim_{\zeta \rightarrow g(\alpha)} f(\zeta)$ by continuity of f . If $\zeta < g(\alpha)$, then $\zeta < g(\xi)$ for some $\xi < \alpha$, so $f(\zeta) \leq f(g(\xi))$, and hence $\lim_{\zeta \rightarrow g(\alpha)} f(\zeta) \leq \lim_{\xi \rightarrow \alpha} f(g(\xi))$. Therefore again $f(g(\alpha)) = \lim_{\xi \rightarrow \alpha} f(g(\xi))$ and $f \circ g$ is continuous.

If f and g are also increasing, then $f \circ g$ is increasing and hence normal. \square

Exercise (2.7). Every normal sequence $\langle \gamma_\alpha \mid \alpha \in \text{Ord} \rangle$ has arbitrarily large fixed points (that is, β such that $\gamma_\beta = \beta$).

Proof. Given α , let $\beta_0 = \gamma_\alpha$ and $\beta_{n+1} = \gamma_{\beta_n}$ for all $n \in \omega$. Note $\beta_{n+1} \geq \beta_n \geq \alpha$ for all $n \in \omega$ since γ is increasing (Lemma 2.4). Let $\beta = \lim_{n \rightarrow \omega} \beta_n$. Then

$$\gamma_\beta = \lim_{n \rightarrow \omega} \gamma_{\beta_n} = \lim_{n \rightarrow \omega} \beta_{n+1} = \beta$$

by the chain rule above (taking $\beta_\alpha = \beta$ for all $\alpha \geq \omega$). \square

Exercise (2.8). For all α, β, γ :

- (i) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$
- (ii) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$
- (iii) $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$

Proof. By induction on γ .

- (i) If $\gamma = 0$,

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + 0 = \alpha \cdot \beta + \alpha \cdot 0 = \alpha \cdot \beta + \alpha \cdot \gamma$$

If the result holds for γ , then

| | |
|---|--------------------------|
| $\alpha \cdot (\beta + (\gamma + 1)) = \alpha \cdot ((\beta + \gamma) + 1)$ | by associativity of $+$ |
| $= \alpha \cdot (\beta + \gamma) + \alpha$ | by definition of \cdot |
| $= (\alpha \cdot \beta + \alpha \cdot \gamma) + \alpha$ | by hypothesis |
| $= \alpha \cdot \beta + (\alpha \cdot \gamma + \alpha)$ | by associativity of $+$ |
| $= \alpha \cdot \beta + \alpha \cdot (\gamma + 1)$ | by definition of \cdot |

so the result holds for $\gamma + 1$. If γ is a nonzero limit ordinal and the result holds for all $\xi < \gamma$, then

$$\begin{aligned}\alpha \cdot (\beta + \gamma) &= \lim_{\xi \rightarrow \gamma} \alpha \cdot (\beta + \xi) && \text{by continuity of } \xi \mapsto \alpha \cdot (\beta + \xi) \\ &= \lim_{\xi \rightarrow \gamma} (\alpha \cdot \beta + \alpha \cdot \xi) && \text{by hypothesis} \\ &= \alpha \cdot \beta + \alpha \cdot \gamma && \text{by continuity of } \xi \mapsto \alpha \cdot \beta + \alpha \cdot \xi\end{aligned}$$

so the result holds for γ . Note that continuity of the composite mappings involved follows from continuity of addition and multiplication and the chain rule above.

(ii) Similar.

(iii) Similar, using (i) and (ii) in the successor step. \square

Exercise (2.9).

$$(i) \quad (\omega + 1) \cdot 2 = \omega + 1 + \omega + 1 = \omega + \omega + 1 = \omega \cdot 2 + 1 < \omega \cdot 2 + 2 = \omega \cdot 2 + 1 \cdot 2$$

$$(ii) \quad (\omega \cdot 2)^2 = \omega \cdot 2 \cdot \omega \cdot 2 = \omega \cdot \omega \cdot 2 = \omega^2 \cdot 2 < \omega^2 \cdot 4 = \omega^2 \cdot 2^2$$

Remark. This result shows that $(\alpha + \beta) \cdot \gamma$ does not in general equal $\alpha \cdot \gamma + \beta \cdot \gamma$, and $(\alpha \cdot \beta)^\gamma$ does not in general equal $\alpha^\gamma \cdot \beta^\gamma$.

Exercise (2.10). If $\alpha < \beta$, then $\alpha + \gamma \leq \beta + \gamma$, $\alpha \cdot \gamma \leq \beta \cdot \gamma$, and $\alpha^\gamma \leq \beta^\gamma$ for all γ .

Proof. By induction on γ . \square

Exercise (2.11). $2 < 3$ but

$$(i) \quad 2 + \omega = \omega = 3 + \omega$$

$$(ii) \quad 2 \cdot \omega = \omega = 3 \cdot \omega$$

$$(iii) \quad 2^\omega = \omega = 3^\omega$$

Exercise (2.12). Let $\epsilon_0 = \lim_{n \rightarrow \omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$. Then ϵ_0 is the least ordinal ϵ such that $\omega^\epsilon = \epsilon$.

Proof. By continuity of exponentiation and the chain rule above (taking $\alpha_\beta = \epsilon_0$ for all $\beta \geq \omega$),

$$\omega^{\epsilon_0} = \lim_{n \rightarrow \omega} \omega^{\alpha_n} = \lim_{n \rightarrow \omega} \alpha_{n+1} = \epsilon_0$$

If $\omega^\epsilon = \epsilon$, we prove by induction that $\alpha_n \leq \epsilon$ for all $n \in \omega$, from which it follows that $\epsilon_0 \leq \epsilon$. Indeed, since $\omega^0 = 1 \neq 0$, we have $\epsilon \neq 0$ and hence $\alpha_0 = \omega \leq \omega^\epsilon = \epsilon$. If $\alpha_n \leq \epsilon$, then $\alpha_{n+1} = \omega^{\alpha_n} \leq \omega^\epsilon = \epsilon$. \square

Exercise (2.13). A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some $\alpha > 0$.

Proof. If $\gamma = \alpha + \beta$ with $\alpha, \beta < \gamma$, then $\alpha + \gamma > \alpha + \beta = \gamma$. Conversely, if $\alpha < \gamma$ and $\alpha + \gamma > \gamma$, fix β such that $\alpha + \beta = \gamma$ (Lemma 2.25). If $\beta \geq \gamma$, then $\gamma = \alpha + \beta \geq \alpha + \gamma > \gamma$, which is impossible, so $\beta < \gamma$.

The forward direction of the second equivalence follows from the Cantor normal form (Theorem 2.26). For the reverse direction, we prove by induction on $\alpha > 0$ that ω^α is indecomposable. The result holds for $\alpha = 1$ since $n + \omega = \omega$ for all $n \in \omega$. If ω^α is indecomposable and $\beta < \omega^{\alpha+1} = \omega^\alpha \cdot \omega$, then $\beta < \omega^\alpha \cdot n$ for some $n \in \omega$, so (Exercises 2.10 and 2.8)

$$\beta + \omega^{\alpha+1} \leq \omega^\alpha \cdot n + \omega^\alpha \cdot \omega = \omega^\alpha \cdot (n + \omega) = \omega^\alpha \cdot \omega = \omega^{\alpha+1}$$

and hence $\omega^{\alpha+1}$ is indecomposable. Finally if $\alpha > 0$ is a limit ordinal, ω^ξ is indecomposable for all $\xi < \alpha$, and $\beta < \omega^\alpha$, then by continuity

$$\beta + \omega^\alpha = \lim_{\xi \rightarrow \alpha} (\beta + \omega^\xi) = \lim_{\xi \rightarrow \alpha} \omega^\xi = \omega^\alpha$$

and hence ω^α is indecomposable. □

References

- [1] Jech, Thomas. *Set Theory*, 3rd ed. Springer, 2002.