

# Notes and exercises from *Set Theory*

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## Introduction

This document contains notes and exercises from [1].

## Chapter 1

**Exercise (2).** There is no set  $X$  such that  $P(X) \subset X$ .

*Proof.* By the axiom of regularity (1.8),  $X$  is  $\in$ -minimal in  $\{X\}$ , so  $X \notin X$  and hence  $P(X) \not\subset X$ .  $\square$

**Exercise (3).** If  $X$  is inductive, then the set  $\{x \in X \mid x \subset X\}$  is inductive. Hence  $N$  is transitive and for each  $n \in N$ ,  $n = \{m \in N \mid m < n\}$ .

*Proof.* Let  $S = \{x \in X \mid x \subset X\}$ . By inductivity of  $X$ ,  $\emptyset \in S$ , and if  $x \in S$ , then  $x \cup \{x\} \in S$ , so  $S$  is inductive. Taking  $X = N$ , it follows that  $S = N$  since  $N$  is the smallest inductive set. Hence  $n \in N$  implies  $n \subset N$ , so  $N$  is transitive and  $n = \{m \in N \mid m < n\}$ .  $\square$

*Remark.* We proved transitivity of  $N$  “by induction” on  $N$ :  $0 \subset N$  and if  $n \subset N$  then  $n + 1 \subset N$ , so  $n \subset N$  for all  $n \in N$ . The following exercises are similar.

**Exercise (4).** If  $X$  is inductive, then the set  $\{x \in X \mid x \text{ is transitive}\}$  is inductive. Hence every  $n \in N$  is transitive.

*Proof.* The class  $C$  of transitive sets is inductive. Indeed,  $\emptyset$  is transitive, and if  $x$  is transitive then  $x \cup \{x\}$  is transitive since  $y \in x \cup \{x\}$  implies  $y \subset x \subset x \cup \{x\}$ . It follows that  $\{x \in X \mid x \text{ is transitive}\} = X \cap C$  is inductive since the intersection of two inductive classes is inductive. Taking  $X = N$ , it follows as above that every  $n \in N$  is transitive.  $\square$

**Exercise (5).** If  $X$  is inductive, then the set  $\{x \in X \mid x \text{ is transitive and } x \not\in x\}$  is inductive. Hence  $n \notin n$  and  $n \neq n+1$  for all  $n \in N$ .

*Proof.* The class  $C = \{x \mid x \text{ is transitive and } x \not\in x\}$  is inductive. Indeed,  $\emptyset \in C$ . If  $x \in C$ , then  $x \cup \{x\}$  is transitive (by inductivity of the class of transitive sets). Also  $x \cup \{x\} \not\in x$ , lest  $x \cup \{x\} \subset x$  by transitivity of  $x$  and hence  $x \in x$ —contradicting  $x \not\in x$ . Similarly  $x \cup \{x\} \neq x$ . Therefore  $x \cup \{x\} \not\in x \cup \{x\}$ . So  $x \cup \{x\} \in C$ , and  $C$  is inductive. It follows as above that  $X \cap C$  is inductive, and taking  $X = N$  that  $n \notin n$  and hence  $n \neq n+1$  for all  $n \in N$ .  $\square$

*Remark.* In order to prove that  $n \notin n$  for all  $n \in N$  by induction on  $N$ , we “loaded the induction hypothesis” with transitivity.

**Exercise (6).** If  $X$  is inductive, then the set  $\{x \in X \mid x \text{ is transitive and regular}\}$  is inductive, where a set  $x$  is called *regular* if every nonempty subset of  $x$  has an  $\in$ -minimal element.

*Proof.* The class  $C = \{x \mid x \text{ is transitive and regular}\}$  is inductive. Indeed,  $\emptyset \in C$ . If  $x \in C$ , then  $x \cup \{x\}$  is transitive. If  $y \subset x \cup \{x\}$  is nonempty, let  $z = y - \{x\} \subset x$ . If  $z = \emptyset$ , then  $y = \{x\}$ , and  $x$  is  $\in$ -minimal in  $y$  by regularity of  $x$ . If  $z \neq \emptyset$  and  $t$  is  $\in$ -minimal in  $z$ , then  $x \not\in t$  by transitivity and regularity of  $x$ , so  $t$  is  $\in$ -minimal in  $y$ . Therefore  $x \cup \{x\}$  is regular. So  $x \cup \{x\} \in C$ , and  $C$  is inductive. It follows as above that  $X \cap C$  is inductive.  $\square$

*Remark.* Taking  $X = N$  it follows as above that every  $n \in N$  is regular.

**Exercise (7).** Every nonempty  $X \subset N$  has an  $\in$ -minimal element.

*Proof.* Choose  $n \in X$ . If  $n$  is not  $\in$ -minimal in  $X$ , then  $n \cap X$  is a nonempty subset of  $n$ , which has an  $\in$ -minimal element  $m$  (Exercise 6). By transitivity of  $n$  (Exercise 4),  $m$  is  $\in$ -minimal in  $X$ .  $\square$

**Exercise (8).** Let  $A \subset N$  be such that  $0 \in A$ , and if  $n \in A$  then  $n+1 \in A$ . Then  $A = N$ .

*Proof.*  $A$  is inductive, so  $A = N$  since  $N$  is the smallest inductive set.  $\square$

## References

[1] Jech, Thomas. *Set Theory*, 3rd ed. Springer, 2002.