Notes and exercises from *Set Theory*

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Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (2)**.** There is no set X such that $P(X) \subset X$.

Proof. By the axiom of regularity (1.8), X is $\in$-minimal in $\{X\}$, so $X \notin X$ and hence $P(X) \not\subset X$. $\square$

Exercise (3)**.** If X is inductive, then the set $\{x \in X \mid x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive and for each $n \in \boldsymbol{N}$, $n = \{m \in \boldsymbol{N} \mid m < n\}$.

Proof. Let $S = \{x \in X \mid x \subset X\}$. By inductivity of X, $\emptyset \in S$, and if $x \in S$, then $x \cup \{x\} \in S$, so S is inductive. Taking $X = \boldsymbol{N}$, it follows that $S = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set. Hence $n \in \boldsymbol{N}$ implies $n \subset \boldsymbol{N}$, so $\boldsymbol{N}$ is transitive and $n = \{m \in \boldsymbol{N} \mid m < n\}$. $\square$

Remark. We proved transitivity of $\boldsymbol{N}$ "by induction" on $\boldsymbol{N}$: $0 \subset \boldsymbol{N}$ and if $n \subset \boldsymbol{N}$ then $n+1 \subset \boldsymbol{N}$, so $n \subset \boldsymbol{N}$ for all $n \in \boldsymbol{N}$. The following exercises are similar.

Exercise (4)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive}\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

Proof. The class C of transitive sets is inductive. Indeed, $\emptyset$ is transitive, and if x is transitive then $x \cup \{x\}$ is transitive since $y \in x \cup \{x\}$ implies $y \subset x \subset x \cup \{x\}$. It follows that $\{x \in X \mid x \text{ is transitive}\} = X \cap C$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \boldsymbol{N}$, it follows as above that every $n \in \boldsymbol{N}$ is transitive. $\square$

Exercise (5)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n+1$ for all $n \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x \text{ is transitive and } x \notin x\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive (by inductivity of the class of transitive sets). Also $x \cup \{x\} \notin x$, lest $x \cup \{x\} \subset x$ by transitivity of x and hence $x \in x$—contradicting $x \notin x$. Similarly $x \cup \{x\} \neq x$. Therefore $x \cup \{x\} \notin x \cup \{x\}$. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive, and taking $X = \boldsymbol{N}$ that $n \notin n$ and hence $n \neq n+1$ for all $n \in \boldsymbol{N}$. □

Remark. In order to prove that $n \notin n$ for all $n \in \boldsymbol{N}$ by induction on $\boldsymbol{N}$, we "loaded the induction hypothesis" with transitivity.

References

[1] Jech, Thomas. *Set Theory*, 3rd ed. Springer, 2002.