Notes and exercises from *Set Theory*

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Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (1.2)**.** There is no set X such that $P(X) \subset X$.

Proof. By the axiom of regularity (1.8), X is $\in$-minimal in $\{X\}$, so $X \notin X$ and hence $P(X) \not\subset X$. $\square$

Exercise (1.3)**.** If X is inductive, then the set $\{x \in X \mid x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive and for each $n \in \boldsymbol{N}$, $n = \{m \in \boldsymbol{N} \mid m < n\}$.

Proof. Let $S = \{x \in X \mid x \subset X\}$. By inductivity of X, $\emptyset \in S$, and if $x \in S$, then $x \cup \{x\} \in S$, so S is inductive. Taking $X = \boldsymbol{N}$, it follows that $S = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set. Hence $n \in \boldsymbol{N}$ implies $n \subset \boldsymbol{N}$, so $\boldsymbol{N}$ is transitive and $n = \{m \in \boldsymbol{N} \mid m < n\}$. $\square$

Remark. We proved transitivity of $\boldsymbol{N}$ "by induction" on $\boldsymbol{N}$: $0 \subset \boldsymbol{N}$ and if $n \subset \boldsymbol{N}$ then $n+1 \subset \boldsymbol{N}$, so $n \subset \boldsymbol{N}$ for all $n \in \boldsymbol{N}$. The following exercises are similar.

Exercise (1.4)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive}\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

Proof. The class C of transitive sets is inductive. Indeed, $\emptyset$ is transitive, and if x is transitive then $x \cup \{x\}$ is transitive since $y \in x \cup \{x\}$ implies $y \subset x \subset x \cup \{x\}$. It follows that $\{x \in X \mid x \text{ is transitive}\} = X \cap C$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \boldsymbol{N}$, it follows as above that every $n \in \boldsymbol{N}$ is transitive. $\square$

Exercise (1.5)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n+1$ for all $n \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x \text{ is transitive and } x \notin x\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive (by inductivity of the class of transitive sets). Also $x \cup \{x\} \notin x$, lest $x \cup \{x\} \subset x$ by transitivity of x and hence $x \in x$—contradicting $x \notin x$. Similarly $x \cup \{x\} \neq x$. Therefore $x \cup \{x\} \notin x \cup \{x\}$. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive, and taking $X = \boldsymbol{N}$ that $n \notin n$ and hence $n \neq n+1$ for all $n \in \boldsymbol{N}$. □

Remark. In order to prove that $n \notin n$ for all $n \in \boldsymbol{N}$ by induction on $\boldsymbol{N}$, we "loaded the induction hypothesis" with transitivity.

Exercise (1.6)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and regular}\}$ is inductive, where a set x is called *regular* if every nonempty subset of x has an $\in$-minimal element.

Proof. The class $C = \{x \mid x \text{ is transitive and regular}\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive. If $y \subset x \cup \{x\}$ is nonempty, let $z = y - \{x\} \subset x$. If $z = \emptyset$, then $y = \{x\}$, and x is $\in$-minimal in y by regularity of x. If $z \neq \emptyset$ and t is $\in$-minimal in z, then $x \notin t$ by transitivity and regularity of x, so t is $\in$-minimal in y. Therefore $x \cup \{x\}$ is regular. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive. □

Remark. Taking $X = \boldsymbol{N}$ it follows as above that every $n \in \boldsymbol{N}$ is regular.

Exercise (1.7)**.** Every nonempty $X \subset \boldsymbol{N}$ has an $\in$-minimal element.

Proof. Choose $n \in X$. If n is not $\in$-minimal in X, then $n \cap X$ is a nonempty subset of n, which has an $\in$-minimal element m (Exercise 1.6). By transitivity of n (Exercise 1.4), m is $\in$-minimal in X. □

Exercise (1.8)**.** If X is inductive, then the set $\{x \in X \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is inductive. Hence for all $n \in \boldsymbol{N}$, $n = 0$ or $n = m+1$ for some $m \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is obviously inductive, so $X \cap C$ is inductive. Taking $X = \boldsymbol{N}$, the rest follows, with $m \in \boldsymbol{N}$ by transitivity of $\boldsymbol{N}$ (Exercise 1.3). □

Exercise (1.9)**.** Let $A \subset \boldsymbol{N}$ be such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = \boldsymbol{N}$.

Proof. A is inductive, so $A = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set.

Alternately, if $A \neq \boldsymbol{N}$, let m be $\in$-minimal in $\boldsymbol{N} - A$ (Exercise 1.7). Then $m \neq 0$, and $m \neq n + 1$ for any $n \in \boldsymbol{N}$, which is impossible (Exercise 1.8). □

Exercise (1.10)**.** Each $n \in \boldsymbol{N}$ is T-finite.

Proof. By induction on $\boldsymbol{N}$. First, $n = 0$ is T-finite since $P(\emptyset) = \{\emptyset\}$ and $\emptyset$ is $\subset$-maximal in $\{\emptyset\}$. If n is T-finite, suppose $X \subset P(n \cup \{n\})$ is nonempty. Let

$$X' = \{\, u - \{n\} \mid u \in X \,\}$$

Then $X' \subset P(n)$ is nonempty and has a $\subset$-maximal element $u - \{n\}$ for some $u \in X$ by T-finiteness of n, where we may assume $n \in u$ if $u \cup \{n\} \in X$. We claim u is $\subset$-maximal in X. Indeed, if $v \in X$ and $u \subset v$, then $u - \{n\} \subset v - \{n\}$, so $u - \{n\} = v - \{n\}$ by $\subset$-maximality of $u - \{n\}$ in X', so $u = v$ by the assumption about u. Therefore $n + 1$ is T-finite. □

Exercise (1.11)**.** $\boldsymbol{N}$ is T-infinite. In fact, $\boldsymbol{N} \subset P(\boldsymbol{N})$ has no $\subset$-maximal element.

Proof. We know $\boldsymbol{N} \subset P(\boldsymbol{N})$ by transitivity of $\boldsymbol{N}$ (Exercise 1.3), and for all $n \in \boldsymbol{N}$ we have $n \subset n + 1 \in \boldsymbol{N}$ but $n \neq n + 1$ (Exercise 1.5). □

Remark. If A is T-finite and $\pi : A \to B$ is surjective (onto), then B is T-finite.

Proof. By pullback. If $X \subset P(B)$ is nonempty, let

$$X_{-1} = \pi_{-1}(X) = \{\, u_{-1} = \pi_{-1}(u) \mid u \in X \,\}$$

Then $X_{-1} \subset P(A)$ is nonempty and has a $\subset$-maximal element u_{-1} for some $u \in X$ by T-finiteness of A. We claim u is $\subset$-maximal in X. Indeed, if $v \in X$ and $u \subset v$, then $u_{-1} \subset v_{-1}$, so $u_{-1} = v_{-1}$ by $\subset$-maximality of u_{-1} in X_{-1}, so $u = v$ by surjectivity of π. Therefore B is T-finite. □

Exercise (1.12)**.** Every finite set is T-finite.

Proof. By definition, every finite set is the image of a T-finite natural number (Exercise 1.10), so is T-finite by the previous remark.

Alternately, if S is T-infinite, choose $X \subset P(S)$ nonempty with no $\subset$-maximal element. By induction on $\boldsymbol{N}$, for each $n \in \boldsymbol{N}$ there is a properly ascending chain $u_0 \subset \cdots \subset u_n \subset S$ of length $n + 1$ in X, and hence an injection $n \to S$ sending $m \in n$ into $u_{m+1} - u_m$. If S had k elements for some $k \in \boldsymbol{N}$, there would then be an injection $k+1 \to k$, which is impossible by an easy induction on k. Therefore S is infinite. □

Exercise (1.13)**.** Every infinite set is T-infinite.

Proof. If S is infinite, let X be the set of finite subsets of S. Clearly X is nonempty since $\emptyset \in X$. Also X has no $\subset$-maximal element. Indeed, if $u \in X$ then there is $s \in S - u$ since S is infinite, and $u \cup \{s\}$ is finite (if u has k elements, then $u \cup \{s\}$ has $k+1$ elements), so $u \subset u \cup \{s\} \in X$ where the inclusion is proper. Therefore S is T-infinite. □

Remark. The previous two exercises show that (Cantor) finiteness is equivalent to Tarski finiteness in ZF.

Chapter 2

Remark. In (2.1), $<$ is actually a *well-ordering* on *Ord*. Indeed, if $C \subset \mathit{Ord}$ is a nonempty class of ordinals and $\alpha \in C$ is not $\in$-minimal in C, then $\alpha \cap C$ is a nonempty subset of α which has an $\in$-minimal element β. By transitivity of α, β is $\in$-minimal in C.

This observation provides an alternative proof that *Ord* is a proper class: if *Ord* were a set, then because it is transitive and strictly well-ordered by $\in$, it would be an ordinal, and hence $\mathit{Ord} \in \mathit{Ord}$—contradicting strictness.

Exercise (2.2)**.** α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$ for all β.

Proof. If α is a limit ordinal and $\beta < \alpha$, then $\beta + 1 \leq \alpha$ (2.5) and $\beta + 1 \neq \alpha$, so $\beta + 1 < \alpha$. If α is not a limit ordinal, then $\beta + 1 = \alpha$ for some $\beta < \alpha$. □

Remark. It follows that α is a nonzero limit ordinal if and only if it is inductive.

Exercise (2.3)**.** If X is inductive, then $X \cap \mathit{Ord}$ is inductive. $\boldsymbol{N}$ is the least nonzero limit ordinal, where $\boldsymbol{N} = \bigcap\{X \mid X \text{ inductive}\}$.

Proof. Clearly *Ord* is inductive, so $X \cap \mathit{Ord}$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \boldsymbol{N}$, it follows that $\boldsymbol{N} \subset \mathit{Ord}$, and since $\boldsymbol{N}$ is also transitive, $\boldsymbol{N}$ is an ordinal. By the previous remark, $\boldsymbol{N}$ is the least nonzero limit ordinal. □

Exercise (2.4)**.** (Without the axiom of infinity.) Let ω be the least nonzero limit ordinal, if it exists, or *Ord* otherwise. The following are equivalent:

(i) There exists an inductive set.

(ii) There exists an infinite[1] set.

(iii) ω is a set.

Proof. (i) $\iff$ (iii): The smallest inductive set is the least nonzero limit ordinal (Exercise 2.3).

(iii) $\implies$ (ii): ω is infinite (Exercises 1.11–2).

(ii) $\implies$ (i): For any finite set A, let $|A|$ denote the least $n \in \omega$ in bijective correspondence with A (the "number of elements" in A). If X is infinite, let

$$S = \{|A| \mid A \subset X \text{ finite}\}$$

Note S is a set by replacement, and S is inductive. Indeed, $0 \in S$ since $\emptyset \subset X$ and $|\emptyset| = 0$. If $n \in S$ and $A \subset X$ with $|A| = n$, then there must exist $x \in X - A$ since X is infinite, and $|A \cup \{x\}| = n + 1 \in S$. $\square$

Remark. This exercise shows that (i)–(iii) are equivalent forms of the axiom of infinity in ZF.

Exercise (2.5)**.** If W is a well-ordered set, then there is no sequence $\langle a_n \mid n \in \boldsymbol{N} \rangle$ in W such that $a_0 > a_1 > \cdots$.

Proof. If there were such a sequence, then $\{a_n \mid n \in \boldsymbol{N}\}$ would be a nonempty subset of W with no least element, contradicting well-ordering. $\square$

Exercise (2.6)**.** There are arbitrarily large limit ordinals.

Proof. Given α, let $\beta = \alpha + \omega$. Clearly $\beta > \alpha$. If $\gamma < \beta$, then either $\gamma < \alpha$, in which case $\gamma + 1 < \alpha + 1 < \beta$, or $\gamma \geq \alpha$, in which case $\gamma = \alpha + n$ for some $n \in \omega$ (Lemma 2.25), so $\gamma + 1 = \alpha + n + 1 < \beta$. Thus β is a limit ordinal (Exercise 2.2). $\square$

Remark (Chain rule). If $f, g : Ord \to Ord$ are nondecreasing and continuous, then so is $f \circ g$. If f and g are normal, then so is $f \circ g$.

Proof. If $\alpha < \beta$, then $g(\alpha) \leq g(\beta)$, so $f(g(\alpha)) \leq f(g(\beta))$. Let α be a nonzero limit ordinal. Clearly $f(g(\alpha)) \geq \lim_{\xi \to \alpha} f(g(\xi))$. If $g(\alpha) = g(\xi')$ for some $\xi' < \alpha$, then $f(g(\alpha)) = f(g(\xi')) \leq \lim_{\xi \to \alpha} f(g(\xi))$. Otherwise, $g(\alpha)$ is a limit ordinal. Indeed, $g(\alpha) = \lim_{\xi \to \alpha} g(\xi)$ by continuity of g, so if $\beta < g(\alpha)$, then $\beta < g(\xi)$ for some $\xi < \alpha$, and hence $\beta + 1 \leq g(\xi) < g(\alpha)$. But then $f(g(\alpha)) = \lim_{\zeta \to g(\alpha)} f(\zeta)$ by continuity

[1] In this exercise, a set is *finite* if it is in bijective correspondence with some $n \in \omega$ and *infinite* otherwise, even if $\omega = Ord$.

of f. If $\zeta < g(\alpha)$, then $\zeta < g(\xi)$ for some $\xi < \alpha$, so $f(\zeta) \le f(g(\xi))$, and hence $\lim_{\zeta\to g(\alpha)} f(\zeta) \le \lim_{\xi\to\alpha} f(g(\xi))$. Therefore again $f(g(\alpha)) = \lim_{\xi\to\alpha} f(g(\xi))$ and $f \circ g$ is continuous.

If f and g are also increasing, then $f \circ g$ is increasing and hence normal. □

Exercise (2.7)**.** Every normal sequence $\langle \gamma_\alpha \mid \alpha \in Ord \rangle$ has arbitrarily large fixed points (that is, β such that $\gamma_\beta = \beta$).

Proof. Given α, let $\beta_0 = \gamma_\alpha$ and $\beta_{n+1} = \gamma_{\beta_n}$ for all $n \in \omega$. Note $\beta_{n+1} \ge \beta_n \ge \alpha$ for all $n \in \omega$ since γ is increasing (Lemma 2.4). Let $\beta = \lim_{n\to\omega} \beta_n$. Then

$$\gamma_\beta = \lim_{n\to\omega} \gamma_{\beta_n} = \lim_{n\to\omega} \beta_{n+1} = \beta$$

by the chain rule above (taking $\beta_\alpha = \beta$ for all $\alpha \ge \omega$). □

Exercise (2.8)**.** For all α, β, γ:

(i) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

(ii) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$

(iii) $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$

Proof. By induction on γ.

(i) If $\gamma = 0$,

$$\alpha \cdot (\beta + \gamma) = \alpha \cdot (\beta + 0) = \alpha \cdot \beta = \alpha \cdot \beta + 0 = \alpha \cdot \beta + \alpha \cdot 0 = \alpha \cdot \beta + \alpha \cdot \gamma$$

If the result holds for γ, then

$$\begin{aligned}
\alpha \cdot (\beta + (\gamma + 1)) &= \alpha \cdot ((\beta + \gamma) + 1) && \text{by associativity of } + \\
&= \alpha \cdot (\beta + \gamma) + \alpha && \text{by definition of } \cdot \\
&= (\alpha \cdot \beta + \alpha \cdot \gamma) + \alpha && \text{by hypothesis} \\
&= \alpha \cdot \beta + (\alpha \cdot \gamma + \alpha) && \text{by associativity of } + \\
&= \alpha \cdot \beta + \alpha \cdot (\gamma + 1) && \text{by definition of } \cdot
\end{aligned}$$

so the result holds for $\gamma + 1$. If γ is a nonzero limit ordinal and the result holds for all $\xi < \gamma$, then

$$\begin{aligned}
\alpha \cdot (\beta + \gamma) &= \lim_{\xi\to\gamma} \alpha \cdot (\beta + \xi) && \text{by continuity of } \xi \mapsto \alpha \cdot (\beta + \xi) \\
&= \lim_{\xi\to\gamma} (\alpha \cdot \beta + \alpha \cdot \xi) && \text{by hypothesis} \\
&= \alpha \cdot \beta + \alpha \cdot \gamma && \text{by continuity of } \xi \mapsto \alpha \cdot \beta + \alpha \cdot \xi
\end{aligned}$$

so the result holds for γ. Note that continuity of the composite mappings involved follows from continuity of addition and multiplication and the chain rule above.

(ii) Similar.

(iii) Similar, using (i) and (ii) in the successor step. □

Exercise (2.9)**.**

(i) $(\omega+1)\cdot 2 = \omega+1+\omega+1 = \omega+\omega+1 = \omega\cdot 2+1 < \omega\cdot 2+2 = \omega\cdot 2+1\cdot 2$

(ii) $(\omega\cdot 2)^2 = \omega\cdot 2\cdot\omega\cdot 2 = \omega\cdot\omega\cdot 2 = \omega^2\cdot 2 < \omega^2\cdot 4 = \omega^2\cdot 2^2$

Remark. This result shows that $(\alpha+\beta)\cdot\gamma$ does not in general equal $\alpha\cdot\gamma+\beta\cdot\gamma$, and $(\alpha\cdot\beta)^\gamma$ does not in general equal $\alpha^\gamma\cdot\beta^\gamma$.

Exercise (2.10)**.** If $\alpha<\beta$, then $\alpha+\gamma\leq\beta+\gamma$, $\alpha\cdot\gamma\leq\beta\cdot\gamma$, and $\alpha^\gamma\leq\beta^\gamma$ for all γ.

Proof. By induction on γ. □

Exercise (2.11)**.** $2<3$ but

(i) $2+\omega=\omega=3+\omega$

(ii) $2\cdot\omega=\omega=3\cdot\omega$

(iii) $2^\omega=\omega=3^\omega$

Exercise (2.12)**.** Let $\epsilon_0=\lim_{n\to\omega}\alpha_n$ where $\alpha_0=\omega$ and $\alpha_{n+1}=\omega^{\alpha_n}$. Then ϵ_0 is the least ordinal ϵ such that $\omega^\epsilon=\epsilon$.

Proof. By continuity of exponentiation and the chain rule above (taking $\alpha_\beta=\epsilon_0$ for all $\beta\geq\omega$),

$$\omega^{\epsilon_0}=\lim_{n\to\omega}\omega^{\alpha_n}=\lim_{n\to\omega}\alpha_{n+1}=\epsilon_0$$

If $\omega^\epsilon=\epsilon$, we prove by induction that $\alpha_n\leq\epsilon$ for all $n\in\omega$, from which it follows that $\epsilon_0\leq\epsilon$. Indeed, since $\omega^0=1\neq 0$, we have $\epsilon\neq 0$ and hence $\alpha_0=\omega\leq\omega^\epsilon=\epsilon$. If $\alpha_n\leq\epsilon$, then $\alpha_{n+1}=\omega^{\alpha_n}\leq\omega^\epsilon=\epsilon$. □

Exercise (2.13)**.** A limit ordinal $\gamma>0$ is indecomposable if and only if $\alpha+\gamma=\gamma$ for all $\alpha<\gamma$ if and only if $\gamma=\omega^\alpha$ for some $\alpha>0$.

Proof. If $\gamma = \alpha + \beta$ with $\alpha, \beta < \gamma$, then $\alpha + \gamma > \alpha + \beta = \gamma$. Conversely, if $\alpha < \gamma$ and $\alpha + \gamma > \gamma$, fix β such that $\alpha + \beta = \gamma$ (Lemma 2.25). If $\beta \geq \gamma$, then $\gamma = \alpha + \beta \geq \alpha + \gamma > \gamma$, which is impossible, so $\beta < \gamma$.

The forward direction of the second equivalence follows from the Cantor normal form (Theorem 2.26). For the reverse direction, we prove by induction on $\alpha > 0$ that ω^α is indecomposable. The result holds for $\alpha = 1$ since $n + \omega = \omega$ for all $n \in \omega$. If ω^α is indecomposable and $\beta < \omega^{\alpha+1} = \omega^\alpha \cdot \omega$, then $\beta < \omega^\alpha \cdot n$ for some $n \in \omega$, so (Exercises 2.10 and 2.8)

$$\beta + \omega^{\alpha+1} \leq \omega^\alpha \cdot n + \omega^\alpha \cdot \omega = \omega^\alpha \cdot (n + \omega) = \omega^\alpha \cdot \omega = \omega^{\alpha+1}$$

and hence $\omega^{\alpha+1}$ is indecomposable. Finally if $\alpha > 0$ is a limit ordinal, ω^ξ is indecomposable for all $\xi < \alpha$, and $\beta < \omega^\alpha$, then by continuity

$$\beta + \omega^\alpha = \lim_{\xi \to \alpha} (\beta + \omega^\xi) = \lim_{\xi \to \alpha} \omega^\xi = \omega^\alpha$$

and hence ω^α is indecomposable. □

References

[1] Jech, Thomas. *Set Theory*, 3rd ed. Springer, 2002.