

Notes and exercises from *Set Theory*

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Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (2). There is no set X such that $P(X) \subset X$.

Proof. By the axiom of regularity (1.8), X is \in -minimal in $\{X\}$, so $X \notin X$ and hence $P(X) \not\subset X$. \square

Exercise (3). If X is inductive, then the set $\{x \in X \mid x \subset X\}$ is inductive. Hence \mathbf{N} is transitive and for each $n \in \mathbf{N}$, $n = \{m \in \mathbf{N} \mid m < n\}$.

Proof. Let $S = \{x \in X \mid x \subset X\}$. By inductivity of X , $\emptyset \in S$, and if $x \in S$, then $x \cup \{x\} \in S$, so S is inductive. Taking $X = \mathbf{N}$, it follows that $S = \mathbf{N}$ since \mathbf{N} is the smallest inductive set. Hence $n \in \mathbf{N}$ implies $n \subset \mathbf{N}$, so \mathbf{N} is transitive and $n = \{m \in \mathbf{N} \mid m < n\}$. \square

Remark. We proved transitivity of \mathbf{N} “by induction” on \mathbf{N} : $0 \subset \mathbf{N}$ and if $n \subset \mathbf{N}$ then $n + 1 \subset \mathbf{N}$, so $n \subset \mathbf{N}$ for all $n \in \mathbf{N}$. The following exercises are similar.

Exercise (4). If X is inductive, then the set $\{x \in X \mid x \text{ is transitive}\}$ is inductive. Hence every $n \in \mathbf{N}$ is transitive.

Proof. The class C of transitive sets is inductive. Indeed, \emptyset is transitive, and if x is transitive then $x \cup \{x\}$ is transitive since $y \in x \cup \{x\}$ implies $y \subset x \subset x \cup \{x\}$. It follows that $\{x \in X \mid x \text{ is transitive}\} = X \cap C$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \mathbf{N}$, it follows as above that every $n \in \mathbf{N}$ is transitive. \square

Exercise (5). If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and } x \not\in x\}$ is inductive. Hence $n \not\in n$ and $n \neq n + 1$ for all $n \in \mathbf{N}$.

Proof. The class $C = \{x \mid x \text{ is transitive and } x \not\in x\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive (by inductivity of the class of transitive sets). Also $x \cup \{x\} \not\in x$, lest $x \cup \{x\} \subset x$ by transitivity of x and hence $x \in x$ —contradicting $x \not\in x$. Similarly $x \cup \{x\} \neq x$. Therefore $x \cup \{x\} \notin x \cup \{x\}$. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive, and taking $X = \mathbf{N}$ that $n \not\in n$ and hence $n \neq n + 1$ for all $n \in \mathbf{N}$. \square

Remark. In order to prove that $n \not\in n$ for all $n \in \mathbf{N}$ by induction on \mathbf{N} , we “loaded the induction hypothesis” with transitivity.

Exercise (6). If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and regular}\}$ is inductive, where a set x is called *regular* if every nonempty subset of x has an \in -minimal element.

Proof. The class $C = \{x \mid x \text{ is transitive and regular}\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive. If $y \subset x \cup \{x\}$ is nonempty, let $z = y - \{x\} \subset x$. If $z = \emptyset$, then $y = \{x\}$, and x is \in -minimal in y by regularity of x . If $z \neq \emptyset$ and t is \in -minimal in z , then $x \not\in t$ by transitivity and regularity of x , so t is \in -minimal in y . Therefore $x \cup \{x\}$ is regular. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive. \square

Remark. Taking $X = \mathbf{N}$ it follows as above that every $n \in \mathbf{N}$ is regular.

Exercise (7). Every nonempty $X \subset \mathbf{N}$ has an \in -minimal element.

Proof. Choose $n \in X$. If n is not \in -minimal in X , then $n \cap X$ is a nonempty subset of n , which has an \in -minimal element m (Exercise 6). By transitivity of n (Exercise 4), m is \in -minimal in X . \square

Exercise (8). If X is inductive, then the set $\{x \in X \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is inductive. Hence for all $n \in \mathbf{N}$, $n = 0$ or $n = m + 1$ for some $m \in \mathbf{N}$.

Proof. The class $C = \{x \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is obviously inductive, so $X \cap C$ is inductive. Taking $X = \mathbf{N}$, the rest follows, with $m \in \mathbf{N}$ by transitivity of \mathbf{N} (Exercise 3). \square

Exercise (9). Let $A \subset \mathbf{N}$ be such that $0 \in A$, and if $n \in A$ then $n + 1 \in A$. Then $A = \mathbf{N}$.

Proof. A is inductive, so $A = N$ since N is the smallest inductive set.

Alternately, if $A \neq N$, let m be \in -minimal in $N - A$ (Exercise 7). Then $m \neq 0$, and $m \neq n + 1$ for any $n \in N$, which is impossible (Exercise 8). \square

Exercise (10). Each $n \in N$ is T-finite.

Proof. By induction on N . First, $n = 0$ is T-finite since $P(\emptyset) = \{\emptyset\}$ and \emptyset is \subset -maximal in $\{\emptyset\}$. If n is T-finite, suppose $X \subset P(n \cup \{n\})$ is nonempty. Let

$$X' = \{u - \{n\} \mid u \in X\}$$

Then $X' \subset P(n)$ is nonempty and has a \subset -maximal element $u - \{n\}$ for some $u \in X$ by T-finiteness of n , where we may assume $n \in u$ if $u \cup \{n\} \in X$. We claim u is \subset -maximal in X . Indeed, if $v \in X$ and $u \subset v$, then $u - \{n\} \subset v - \{n\}$, so $u - \{n\} = v - \{n\}$ by \subset -maximality of $u - \{n\}$ in X' , so $u = v$ by the assumption about u . Therefore $n + 1$ is T-finite. \square

Exercise (11). N is T-infinite. In fact, $N \subset P(N)$ has no \subset -maximal element.

Proof. We know $N \subset P(N)$ by transitivity of N (Exercise 3), and for all $n \in N$ we have $n \subset n + 1 \in N$ but $n \neq n + 1$ (Exercise 5). \square

Remark. If A is T-finite and $\pi : A \rightarrow B$ is surjective (onto), then B is T-finite.

Proof. By pullback. If $X \subset P(B)$ is nonempty, let

$$X_{-1} = \pi_{-1}(X) = \{u_{-1} = \pi_{-1}(u) \mid u \in X\}$$

Then $X_{-1} \subset P(A)$ is nonempty and has a \subset -maximal element u_{-1} for some $u \in X$ by T-finiteness of A . We claim u is \subset -maximal in X . Indeed, if $v \in X$ and $u \subset v$, then $u_{-1} \subset v_{-1}$, so $u_{-1} = v_{-1}$ by \subset -maximality of u_{-1} in X_{-1} , so $u = v$ by surjectivity of π . Therefore B is T-finite. \square

Exercise (12). Every finite set is T-finite.

Proof. By definition, every finite set is the image of a T-finite natural number (Exercise 10), so is T-finite by the previous remark.

Alternately, if S is T-infinite, choose $X \subset P(S)$ nonempty with no \subset -maximal element. By induction on N , for each $n \in N$ there is a properly ascending chain $u_0 \subset \cdots \subset u_n \subset S$ of length $n + 1$ in X , and hence an injection $n \rightarrow S$ sending $m \in n$ into $u_{m+1} - u_m$. If S had k elements for some $k \in N$, there would then be an injection $k + 1 \rightarrow k$, which is impossible by an easy induction on k . Therefore S is infinite. \square

Exercise (13). Every infinite set is T-infinite.

Proof. If S is infinite, let X be the set of finite subsets of S . Clearly X is nonempty since $\emptyset \in X$. Also X has no \subset -maximal element. Indeed, if $u \in X$ then there is $s \in S - u$ since S is infinite, and $u \cup \{s\}$ is finite (if u has k elements, then $u \cup \{s\}$ has $k + 1$ elements), so $u \subset u \cup \{s\} \in X$ where the inclusion is proper. Therefore S is T-infinite. \square

Remark. The previous two exercises show that (Cantor) finiteness is equivalent to Tarski finiteness in ZF.

Chapter 2

Remark. In (2.1), $<$ is actually a *well-ordering* on Ord . Indeed, if $C \subset Ord$ is a nonempty class of ordinals and $\alpha \in C$ is not \in -minimal in C , then $\alpha \cap C$ is a nonempty subset of α which has an \in -minimal element β . By transitivity of α , β is \in -minimal in C .

This observation provides an alternative proof that Ord is a proper class: if Ord were a set, then because it is transitive and strictly well-ordered by \in , it would be an ordinal, and hence $Ord \in Ord$ —contradicting strictness.

Exercise (2). α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$ for all β .

Proof. If α is a limit ordinal and $\beta < \alpha$, then $\beta + 1 \leq \alpha$ (2.5) and $\beta + 1 \neq \alpha$, so $\beta + 1 < \alpha$. If α is not a limit ordinal, then $\beta + 1 = \alpha$ for some $\beta < \alpha$. \square

Remark. It follows that α is a nonzero limit ordinal if and only if it is inductive.

Exercise (3). If X is inductive, then $X \cap Ord$ is inductive. $N = \bigcap \{X \mid X \text{ inductive}\}$ is the least nonzero limit ordinal.

Proof. Clearly Ord is inductive, so $X \cap Ord$ is inductive since the intersection of two inductive classes is inductive. Taking $X = N$, it follows that $N \subset Ord$, and since N is also transitive, N is an ordinal. By the previous remark, N is the least nonzero limit ordinal. \square

Exercise (4). (Without the axiom of infinity.) Let ω be the least nonzero limit ordinal, if it exists, or Ord otherwise. The following are equivalent:

- (i) There exists an inductive set.

(ii) There exists an infinite¹ set.

(iii) ω is a set.

Proof. (i) \iff (iii): The smallest inductive set is the least nonzero limit ordinal (Exercise 3).

(iii) \implies (ii): ω is infinite (Exercises 1.11–2).

(ii) \implies (i): For any finite set A , let $|A|$ denote the least $n \in \omega$ in bijective correspondence with A (the “number of elements” in A). If X is infinite, let

$$S = \{|A| \mid A \subset X \text{ finite}\}$$

Note S is a set by replacement, and S is inductive. Indeed, $0 \in S$ since $\emptyset \subset X$ and $|\emptyset| = 0$. If $n \in S$ and $A \subset X$ with $|A| = n$, then there must exist $x \in X - A$ since X is infinite, and $|A \cup \{x\}| = n + 1 \in S$. \square

Remark. This exercise shows that (i)–(iii) are equivalent forms of the axiom of infinity in ZF.

Exercise (5). If W is a well-ordered set, then there is no sequence $\langle a_n \mid n \in \mathbf{N} \rangle$ in W such that $a_0 > a_1 > \dots$.

Proof. If there were such a sequence, then $\{a_n \mid n \in \mathbf{N}\}$ would be a nonempty subset of W with no least element, contradicting well-ordering. \square

Exercise (6). There are arbitrarily large limit ordinals.

Proof. Given α , let $\beta = \alpha + \omega$. Clearly $\beta > \alpha$. If $\gamma < \beta$, then either $\gamma < \alpha$, in which case $\gamma + 1 < \alpha + 1 < \beta$, or $\gamma \geq \alpha$, in which case $\gamma = \alpha + n$ for some $n \in \omega$ (2.25), so $\gamma + 1 = \alpha + n + 1 < \beta$. Therefore β is a limit ordinal (Exercise 2). \square

References

[1] Jech, Thomas. *Set Theory*, 3rd ed. Springer, 2002.

¹In this exercise, a set is *finite* if it is in bijective correspondence with some $n \in \omega$ and *infinite* otherwise, even if $\omega = \text{Ord}$.