Notes and exercises from *Set Theory*

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Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (2)**.** There is no set X such that $P(X) \subset X$.

Proof. By the axiom of regularity (1.8), X is $\in$-minimal in $\{X\}$, so $X \notin X$ and hence $P(X) \not\subset X$. $\square$

Exercise (3)**.** If X is inductive, then the set $\{x \in X \mid x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive and for each $n \in \boldsymbol{N}$, $n = \{m \in \boldsymbol{N} \mid m < n\}$.

Proof. Let $S = \{x \in X \mid x \subset X\}$. By inductivity of X, $\emptyset \in S$, and if $x \in S$, then $x \cup \{x\} \in S$, so S is inductive. Taking $X = \boldsymbol{N}$, it follows that $S = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set. Hence $n \in \boldsymbol{N}$ implies $n \subset \boldsymbol{N}$, so $\boldsymbol{N}$ is transitive and $n = \{m \in \boldsymbol{N} \mid m < n\}$. $\square$

Remark. We proved transitivity of $\boldsymbol{N}$ “by induction” on $\boldsymbol{N}$: $0 \subset \boldsymbol{N}$ and if $n \subset \boldsymbol{N}$ then $n+1 \subset \boldsymbol{N}$, so $n \subset \boldsymbol{N}$ for all $n \in \boldsymbol{N}$. The following exercises are similar.

Exercise (4)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive}\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

Proof. The class C of transitive sets is inductive. Indeed, $\emptyset$ is transitive, and if x is transitive then $x \cup \{x\}$ is transitive since $y \in x \cup \{x\}$ implies $y \subset x \subset x \cup \{x\}$. It follows that $\{x \in X \mid x \text{ is transitive}\} = X \cap C$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \boldsymbol{N}$, it follows as above that every $n \in \boldsymbol{N}$ is transitive. $\square$

Exercise (5)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n+1$ for all $n \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x \text{ is transitive and } x \notin x\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive (by inductivity of the class of transitive sets). Also $x \cup \{x\} \notin x$, lest $x \cup \{x\} \subset x$ by transitivity of x and hence $x \in x$—contradicting $x \notin x$. Similarly $x \cup \{x\} \neq x$. Therefore $x \cup \{x\} \notin x \cup \{x\}$. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive, and taking $X = \boldsymbol{N}$ that $n \notin n$ and hence $n \neq n+1$ for all $n \in \boldsymbol{N}$. $\square$

Remark. In order to prove that $n \notin n$ for all $n \in \boldsymbol{N}$ by induction on $\boldsymbol{N}$, we "loaded the induction hypothesis" with transitivity.

Exercise (6)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and regular}\}$ is inductive, where a set x is called *regular* if every nonempty subset of x has an $\in$-minimal element.

Proof. The class $C = \{x \mid x \text{ is transitive and regular}\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive. If $y \subset x \cup \{x\}$ is nonempty, let $z = y - \{x\} \subset x$. If $z = \emptyset$, then $y = \{x\}$, and x is $\in$-minimal in y by regularity of x. If $z \neq \emptyset$ and t is $\in$-minimal in z, then $x \notin t$ by transitivity and regularity of x, so t is $\in$-minimal in y. Therefore $x \cup \{x\}$ is regular. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive. $\square$

Remark. Taking $X = \boldsymbol{N}$ it follows as above that every $n \in \boldsymbol{N}$ is regular.

Exercise (7)**.** Every nonempty $X \subset \boldsymbol{N}$ has an $\in$-minimal element.

Proof. Choose $n \in X$. If n is not $\in$-minimal in X, then $n \cap X$ is a nonempty subset of n, which has an $\in$-minimal element m (Exercise 6). By transitivity of n (Exercise 4), m is $\in$-minimal in X. $\square$

Exercise (8)**.** If X is inductive, then the set $\{x \in X \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is inductive. Hence for all $n \in \boldsymbol{N}$, $n = 0$ or $n = m+1$ for some $m \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is obviously inductive, so $X \cap C$ is inductive. Taking $X = \boldsymbol{N}$, the rest follows, with $m \in \boldsymbol{N}$ by transitivity of $\boldsymbol{N}$. $\square$

Exercise (9)**.** Let $A \subset \boldsymbol{N}$ be such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = \boldsymbol{N}$.

Proof. A is inductive, so $A = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set.

Alternately, if $A \neq \boldsymbol{N}$, let m be $\in$-minimal in $\boldsymbol{N} - A$ (Exercise 7). Then $m \neq 0$, and $m \neq n+1$ for any $n \in \boldsymbol{N}$, which is impossible (Exercise 8). $\square$

References

[1] Jech, Thomas. *Set Theory*, 3rd ed. Springer, 2002.