Notes and exercises from *Set Theory*

John Peloquin

Introduction

This document contains notes and exercises from [2].

Chapter 1

Exercise (1.2)**.** There is no set X such that $P(X) \subset X$.

Proof. By the axiom of regularity (1.8), X is $\in$-minimal in $\{X\}$, so $X \notin X$ and hence $P(X) \not\subset X$. $\square$

Exercise (1.3)**.** If X is inductive, then the set $\{x \in X \mid x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive and for each $n \in \boldsymbol{N}$, $n = \{m \in \boldsymbol{N} \mid m < n\}$.

Proof. Let $S = \{x \in X \mid x \subset X\}$. By inductivity of X, $\emptyset \in S$, and if $x \in S$, then $x \cup \{x\} \in S$, so S is inductive. Taking $X = \boldsymbol{N}$, it follows that $S = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set. Hence $n \in \boldsymbol{N}$ implies $n \subset \boldsymbol{N}$, so $\boldsymbol{N}$ is transitive and $n = \{m \in \boldsymbol{N} \mid m < n\}$. $\square$

Remark. We proved transitivity of $\boldsymbol{N}$ "by induction" on $\boldsymbol{N}$: $0 \subset \boldsymbol{N}$ and if $n \subset \boldsymbol{N}$ then $n+1 \subset \boldsymbol{N}$, so $n \subset \boldsymbol{N}$ for all $n \in \boldsymbol{N}$. The following exercises are similar.

Exercise (1.4)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive}\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

Proof. The class C of transitive sets is inductive. Indeed, $\emptyset$ is transitive, and if x is transitive then $x \cup \{x\}$ is transitive since $y \in x \cup \{x\}$ implies $y \subset x \subset x \cup \{x\}$. It follows that $\{x \in X \mid x \text{ is transitive}\} = X \cap C$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \boldsymbol{N}$, it follows as above that every $n \in \boldsymbol{N}$ is transitive. $\square$

Exercise (1.5)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n+1$ for all $n \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x \text{ is transitive and } x \notin x\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive (by inductivity of the class of transitive sets). Also $x \cup \{x\} \notin x$, lest $x \cup \{x\} \subset x$ by transitivity of x and hence $x \in x$—contradicting $x \notin x$. Similarly $x \cup \{x\} \neq x$. Therefore $x \cup \{x\} \notin x \cup \{x\}$. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive, and taking $X = \boldsymbol{N}$ that $n \notin n$ and hence $n \neq n+1$ for all $n \in \boldsymbol{N}$. □

Remark. In order to prove that $n \notin n$ for all $n \in \boldsymbol{N}$ by induction on $\boldsymbol{N}$, we "loaded the induction hypothesis" with transitivity.

Exercise (1.6)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and regular}\}$ is inductive, where a set x is called *regular* if every nonempty subset of x has an $\in$-minimal element.

Proof. The class $C = \{x \mid x \text{ is transitive and regular}\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive. If $y \subset x \cup \{x\}$ is nonempty, let $z = y - \{x\} \subset x$. If $z = \emptyset$, then $y = \{x\}$, and x is $\in$-minimal in y by regularity of x. If $z \neq \emptyset$ and t is $\in$-minimal in z, then $x \notin t$ by transitivity and regularity of x, so t is $\in$-minimal in y. Therefore $x \cup \{x\}$ is regular. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive. □

Remark. Taking $X = \boldsymbol{N}$ it follows as above that every $n \in \boldsymbol{N}$ is regular.

Exercise (1.7)**.** Every nonempty $X \subset \boldsymbol{N}$ has an $\in$-minimal element.

Proof. Choose $n \in X$. If n is not $\in$-minimal in X, then $n \cap X$ is a nonempty subset of n, which has an $\in$-minimal element m (Exercise 1.6). By transitivity of n (Exercise 1.4), m is $\in$-minimal in X. □

Exercise (1.8)**.** If X is inductive, then the set $\{x \in X \mid x = \emptyset \text{ or } \exists y (x = y \cup \{y\})\}$ is inductive. Hence for all $n \in \boldsymbol{N}$, $n = 0$ or $n = m+1$ for some $m \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x = \emptyset \text{ or } \exists y (x = y \cup \{y\})\}$ is obviously inductive, so $X \cap C$ is inductive. Taking $X = \boldsymbol{N}$, the rest follows, with $m \in \boldsymbol{N}$ by transitivity of $\boldsymbol{N}$ (Exercise 1.3). □

Exercise (1.9)**.** Let $A \subset \boldsymbol{N}$ be such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = \boldsymbol{N}$.

Proof. A is inductive, so $A = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set.

Alternately, if $A \neq \boldsymbol{N}$, let m be $\in$-minimal in $\boldsymbol{N} - A$ (Exercise 1.7). Then $m \neq 0$, and $m \neq n+1$ for any $n \in \boldsymbol{N}$, which is impossible (Exercise 1.8). $\square$

Exercise (1.10)**.** Each $n \in \boldsymbol{N}$ is T-finite.

Proof. By induction on $\boldsymbol{N}$. First, $n = 0$ is T-finite since $P(\emptyset) = \{\emptyset\}$ and $\emptyset$ is $\subset$-maximal in $\{\emptyset\}$. If n is T-finite, suppose $X \subset P(n \cup \{n\})$ is nonempty. Let

$$X' = \{\, u - \{n\} \mid u \in X \,\}$$

Then $X' \subset P(n)$ is nonempty and has a $\subset$-maximal element $u - \{n\}$ for some $u \in X$ by T-finiteness of n, where we may assume $n \in u$ if $u \cup \{n\} \in X$. We claim u is $\subset$-maximal in X. Indeed, if $v \in X$ and $u \subset v$, then $u - \{n\} \subset v - \{n\}$, so $u - \{n\} = v - \{n\}$ by $\subset$-maximality of $u - \{n\}$ in X', so $u = v$ by the assumption about u. Therefore $n+1$ is T-finite. $\square$

Exercise (1.11)**.** $\boldsymbol{N}$ is T-infinite. In fact, $\boldsymbol{N} \subset P(\boldsymbol{N})$ has no $\subset$-maximal element.

Proof. We know $\boldsymbol{N} \subset P(\boldsymbol{N})$ by transitivity of $\boldsymbol{N}$ (Exercise 1.3), and for all $n \in \boldsymbol{N}$ we have $n \subset n+1 \in \boldsymbol{N}$ but $n \neq n+1$ (Exercise 1.5). $\square$

Remark. If A is T-finite and $\pi : A \to B$ is surjective (onto), then B is T-finite.

Proof. By pullback. If $X \subset P(B)$ is nonempty, let

$$X_{-1} = \pi_{-1}(X) = \{\, u_{-1} = \pi_{-1}(u) \mid u \in X \,\}$$

Then $X_{-1} \subset P(A)$ is nonempty and has a $\subset$-maximal element u_{-1} for some $u \in X$ by T-finiteness of A. We claim u is $\subset$-maximal in X. Indeed, if $v \in X$ and $u \subset v$, then $u_{-1} \subset v_{-1}$, so $u_{-1} = v_{-1}$ by $\subset$-maximality of u_{-1} in X_{-1}, so $u = v$ by surjectivity of π. Therefore B is T-finite. $\square$

Exercise (1.12)**.** Every finite set is T-finite.

Proof. By definition, every finite set is the image of a T-finite natural number (Exercise 1.10), so is T-finite by the previous remark.

Alternately, if S is T-infinite, choose $X \subset P(S)$ nonempty with no $\subset$-maximal element. By induction on $\boldsymbol{N}$, for each $n \in \boldsymbol{N}$ there is a properly ascending chain $u_0 \subset \cdots \subset u_n \subset S$ of length $n+1$ in X, and hence an injection $n \to S$ sending $m \in n$ into $u_{m+1} - u_m$. If S had k elements for some $k \in \boldsymbol{N}$, there would then be an injection $k+1 \to k$, which is impossible by an easy induction on k. Therefore S is infinite. $\square$

Exercise (1.13)**.** Every infinite set is T-infinite.

Proof. If S is infinite, let X be the set of finite subsets of S. Clearly X is nonempty since $\emptyset \in X$. Also X has no $\subset$-maximal element. Indeed, if $u \in X$ then there is $s \in S-u$ since S is infinite, and $u \cup \{s\}$ is finite (if u has k elements, then $u \cup \{s\}$ has $k+1$ elements), so $u \subset u \cup \{s\} \in X$ where the inclusion is proper. Therefore S is T-infinite. □

Remark. The previous two exercises show that (Cantor) finiteness is equivalent to Tarski finiteness in ZF.

Chapter 2

Remark. In (2.1), $<$ is actually a *well-ordering* on *Ord*. Indeed, if $C \subset Ord$ is a nonempty class of ordinals and $\alpha \in C$ is not $\in$-minimal in C, then $\alpha \cap C$ is a nonempty subset of α which has an $\in$-minimal element β. By transitivity of α, β is $\in$-minimal in C.

This observation provides an alternative proof that *Ord* is a proper class: if *Ord* were a set, then because it is transitive and strictly well-ordered by $\in$, it would be an ordinal, and hence $Ord \in Ord$—contradicting strictness.

Remark. In Definition 2.13, an ordinal is "finite" if and only if it is a "finite ordinal". In fact, if α is not a "finite ordinal" then $\omega \subset \alpha$, and it follows by induction on n that there is no surjection $n \to \omega$ (every function $n \to \omega$ is bounded), hence there is no surjection $n \to \alpha$, so α is not "finite". The converse is trivial.

Remark. In Theorem 2.27, the height of a well-ordering is just its order-type (ordinal), and the rank of an element in a well-ordering is just the order-type of the initial segment given by that element.

Proof. If P is a well-ordering and $P(x) = \{\, y \in P \mid y < x \,\}$, then

$$\operatorname{type} P = \sup_{x \in P}\{\operatorname{type} P(x) + 1\,\} = \{\operatorname{type} P(x) \mid x \in P\,\}$$

Indeed, if $x \in P$ then $\operatorname{type} P(x) < \operatorname{type} P$ (Theorem 2.8), so $\operatorname{type} P(x) + 1 \leq \operatorname{type} P$. Conversely, if $\alpha < \operatorname{type} P$, then $\alpha = \operatorname{type} P(x)$ for some $x \in P$, so $\alpha < \operatorname{type} P(x) + 1$. Taking $P = P(x)$ yields

$$\operatorname{type} P(x) = \sup_{y < x}\{\operatorname{type} P(y) + 1\,\}$$

The result now follows by uniqueness of rank. □

Remark. If P is a well-ordering and $S \subset P$, then $\operatorname{type} S \le \operatorname{type} P$.

Proof. By induction using the previous remark,

$$\operatorname{type} S = \sup_{x \in S}\{\operatorname{type} S(x) + 1\} \le \sup_{x \in P}\{\operatorname{type} P(x) + 1\} = \operatorname{type} P \qquad \square$$

Exercise (2.2)**.** α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$ for all β.

Proof. If α is a limit ordinal and $\beta < \alpha$, then $\beta + 1 \le \alpha$ (2.5) and $\beta + 1 \ne \alpha$, so $\beta + 1 < \alpha$. If α is not a limit ordinal, then $\beta + 1 = \alpha$ for some $\beta < \alpha$. $\square$

Remark. It follows that α is a nonzero limit ordinal if and only if it is inductive.

Exercise (2.3)**.** If X is inductive, then $X \cap \mathit{Ord}$ is inductive. $\boldsymbol{N}$ is the least nonzero limit ordinal, where $\boldsymbol{N} = \bigcap\{X \mid X \text{ inductive}\}$.

Proof. Clearly Ord is inductive, so $X \cap \mathit{Ord}$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \boldsymbol{N}$, it follows that $\boldsymbol{N} \subset \mathit{Ord}$, and since $\boldsymbol{N}$ is also transitive, $\boldsymbol{N}$ is an ordinal. By the previous remark, $\boldsymbol{N}$ is the least nonzero limit ordinal. $\square$

Exercise (2.4)**.** (Without the axiom of infinity.) Let ω be the least nonzero limit ordinal, if it exists, or Ord otherwise. The following are equivalent:

(i) There exists an inductive set.

(ii) There exists an infinite[1] set.

(iii) ω is a set.

Proof. (i) $\iff$ (iii): The smallest inductive set is the least nonzero limit ordinal (Exercise 2.3).

(iii) $\implies$ (ii): ω is infinite (Exercises 1.11–2).

(ii) $\implies$ (i): For any finite set A, let $|A|$ denote the least $n \in \omega$ in bijective correspondence with A (the "number of elements" in A). If X is infinite, let

$$S = \{|A| \mid A \subset X \text{ finite}\}$$

Note S is a set by replacement, and S is inductive. Indeed, $0 \in S$ since $\varnothing \subset X$ and $|\varnothing| = 0$. If $n \in S$ and $A \subset X$ with $|A| = n$, then there must exist $x \in X - A$ since X is infinite, and $|A \cup \{x\}| = n + 1 \in S$. $\square$

[1] In this exercise, a set is *finite* if it is in bijective correspondence with some $n \in \omega$ and *infinite* otherwise, even if $\omega = \mathit{Ord}$.

Remark. This exercise shows that (i)–(iii) are equivalent forms of the axiom of infinity in ZF.

Exercise (2.5)**.** If W is a well-ordered set, then there is no sequence $\langle a_n \mid n \in \boldsymbol{N} \rangle$ in W such that $a_0 > a_1 > \cdots$.

Proof. If there were such a sequence, then $\{a_n \mid n \in \boldsymbol{N}\}$ would be a nonempty subset of W with no least element, contradicting well-ordering. □

Exercise (2.6)**.** There are arbitrarily large limit ordinals.

Proof. Given α, let $\beta = \alpha + \omega$. Clearly $\beta > \alpha$. If $\gamma < \beta$, then either $\gamma < \alpha$, in which case $\gamma + 1 < \alpha + 1 < \beta$, or $\gamma \geq \alpha$, in which case $\gamma = \alpha + n$ for some $n \in \omega$ (Lemma 2.25), so $\gamma + 1 = \alpha + n + 1 < \beta$. Thus β is a limit ordinal (Exercise 2.2). □

Remark (Chain rule). If $f, g : Ord \to Ord$ are nondecreasing and continuous, then so is $f \circ g$. If f and g are normal, then so is $f \circ g$.

Proof. If $\alpha < \beta$, then $g(\alpha) \leq g(\beta)$, so $f(g(\alpha)) \leq f(g(\beta))$. Let α be a nonzero limit ordinal. Clearly $f(g(\alpha)) \geq \lim_{\xi \to \alpha} f(g(\xi))$. If $g(\alpha) = g(\xi')$ for some $\xi' < \alpha$, then $f(g(\alpha)) = f(g(\xi')) \leq \lim_{\xi \to \alpha} f(g(\xi))$. Otherwise, $g(\alpha)$ is a limit ordinal. Indeed, $g(\alpha) = \lim_{\xi \to \alpha} g(\xi)$ by continuity of g, so if $\beta < g(\alpha)$, then $\beta < g(\xi)$ for some $\xi < \alpha$, and hence $\beta + 1 \leq g(\xi) < g(\alpha)$. But then $f(g(\alpha)) = \lim_{\zeta \to g(\alpha)} f(\zeta)$ by continuity of f. If $\zeta < g(\alpha)$, then $\zeta < g(\xi)$ for some $\xi < \alpha$, so $f(\zeta) \leq f(g(\xi))$, and hence $\lim_{\zeta \to g(\alpha)} f(\zeta) \leq \lim_{\xi \to \alpha} f(g(\xi))$. Therefore again $f(g(\alpha)) = \lim_{\xi \to \alpha} f(g(\xi))$ and $f \circ g$ is continuous.

If f and g are also increasing, then $f \circ g$ is increasing and hence normal. □

Exercise (2.7)**.** Every normal sequence $\langle \gamma_\alpha \mid \alpha \in Ord \rangle$ has arbitrarily large fixed points (that is, β such that $\gamma_\beta = \beta$).

Proof. Given α, let $\beta_0 = \gamma_\alpha$ and $\beta_{n+1} = \gamma_{\beta_n}$ for all $n \in \omega$. Note $\beta_{n+1} \geq \beta_n \geq \alpha$ for all $n \in \omega$ since γ is increasing (Lemma 2.4). Let $\beta = \lim_{n \to \omega} \beta_n$. Then

$$\gamma_\beta = \lim_{n \to \omega} \gamma_{\beta_n} = \lim_{n \to \omega} \beta_{n+1} = \beta$$

by the chain rule above (taking $\beta_\alpha = \beta$ for all $\alpha \geq \omega$). □

Exercise (2.8)**.** For all α, β, γ:

(i) $\alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma$

(ii) $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$

(iii) $(\alpha^\beta)^\gamma = \alpha^{\beta\cdot\gamma}$

Proof. By induction on γ.

(i) If $\gamma = 0$,

$$\alpha\cdot(\beta+\gamma) = \alpha\cdot(\beta+0) = \alpha\cdot\beta = \alpha\cdot\beta+0 = \alpha\cdot\beta+\alpha\cdot 0 = \alpha\cdot\beta+\alpha\cdot\gamma$$

If the result holds for γ, then

$$\begin{aligned}
\alpha\cdot(\beta+(\gamma+1)) &= \alpha\cdot((\beta+\gamma)+1) && \text{by associativity of } +\\
&= \alpha\cdot(\beta+\gamma)+\alpha && \text{by definition of } \cdot\\
&= (\alpha\cdot\beta+\alpha\cdot\gamma)+\alpha && \text{by hypothesis}\\
&= \alpha\cdot\beta+(\alpha\cdot\gamma+\alpha) && \text{by associativity of } +\\
&= \alpha\cdot\beta+\alpha\cdot(\gamma+1) && \text{by definition of } \cdot
\end{aligned}$$

so the result holds for $\gamma+1$. If γ is a nonzero limit ordinal and the result holds for all $\xi<\gamma$, then

$$\begin{aligned}
\alpha\cdot(\beta+\gamma) &= \lim_{\xi\to\gamma}\alpha\cdot(\beta+\xi) && \text{by continuity of } \xi\mapsto\alpha\cdot(\beta+\xi)\\
&= \lim_{\xi\to\gamma}(\alpha\cdot\beta+\alpha\cdot\xi) && \text{by hypothesis}\\
&= \alpha\cdot\beta+\alpha\cdot\gamma && \text{by continuity of } \xi\mapsto\alpha\cdot\beta+\alpha\cdot\xi
\end{aligned}$$

so the result holds for γ. Note that continuity of the composite mappings involved follows from continuity of addition and multiplication and the chain rule above.

(ii) Similar.

(iii) Similar, using (i) and (ii) in the successor step. □

Exercise (2.9)**.**

(i) $(\omega+1)\cdot 2 = \omega+1+\omega+1 = \omega+\omega+1 = \omega\cdot 2+1 < \omega\cdot 2+2 = \omega\cdot 2+1\cdot 2$

(ii) $(\omega\cdot 2)^2 = \omega\cdot 2\cdot\omega\cdot 2 = \omega\cdot\omega\cdot 2 = \omega^2\cdot 2 < \omega^2\cdot 4 = \omega^2\cdot 2^2$

Remark. This result shows that $(\alpha+\beta)\cdot\gamma$ does not in general equal $\alpha\cdot\gamma+\beta\cdot\gamma$, and $(\alpha\cdot\beta)^\gamma$ does not in general equal $\alpha^\gamma\cdot\beta^\gamma$.

Exercise (2.10)**.** If $\alpha < \beta$, then $\alpha + \gamma \le \beta + \gamma$, $\alpha \cdot \gamma \le \beta \cdot \gamma$, and $\alpha^\gamma \le \beta^\gamma$ for all γ.

Proof. By induction on γ. □

Exercise (2.11)**.** $2 < 3$ but

(i) $2 + \omega = \omega = 3 + \omega$

(ii) $2 \cdot \omega = \omega = 3 \cdot \omega$

(iii) $2^\omega = \omega = 3^\omega$

Exercise (2.12)**.** Let $\epsilon_0 = \lim_{n\to\omega} \alpha_n$ where $\alpha_0 = \omega$ and $\alpha_{n+1} = \omega^{\alpha_n}$. Then ϵ_0 is the least ordinal ϵ such that $\omega^\epsilon = \epsilon$.

Proof. By continuity of exponentiation and the chain rule above (taking $\alpha_\beta = \epsilon_0$ for all $\beta \ge \omega$),

$$\omega^{\epsilon_0} = \lim_{n\to\omega} \omega^{\alpha_n} = \lim_{n\to\omega} \alpha_{n+1} = \epsilon_0$$

If $\omega^\epsilon = \epsilon$, we prove by induction that $\alpha_n \le \epsilon$ for all $n \in \omega$, from which it follows that $\epsilon_0 \le \epsilon$. Indeed, since $\omega^0 = 1 \ne 0$, we have $\epsilon \ne 0$ and hence $\alpha_0 = \omega \le \omega^\epsilon = \epsilon$. If $\alpha_n \le \epsilon$, then $\alpha_{n+1} = \omega^{\alpha_n} \le \omega^\epsilon = \epsilon$. □

Exercise (2.13)**.** A limit ordinal $\gamma > 0$ is indecomposable if and only if $\alpha + \gamma = \gamma$ for all $\alpha < \gamma$ if and only if $\gamma = \omega^\alpha$ for some $\alpha > 0$.

Proof. If $\gamma = \alpha + \beta$ with $\alpha, \beta < \gamma$, then $\alpha + \gamma > \alpha + \beta = \gamma$. Conversely, if $\alpha < \gamma$ and $\alpha + \gamma > \gamma$, fix β such that $\alpha + \beta = \gamma$ (Lemma 2.25). If $\beta \ge \gamma$, then $\gamma = \alpha + \beta \ge \alpha + \gamma > \gamma$, which is impossible, so $\beta < \gamma$.

The forward direction of the second equivalence follows from the Cantor normal form (Theorem 2.26). For the reverse direction, we prove by induction on $\alpha > 0$ that ω^α is indecomposable. The result holds for $\alpha = 1$ since $n + \omega = \omega$ for all $n \in \omega$. If ω^α is indecomposable and $\beta < \omega^{\alpha+1} = \omega^\alpha \cdot \omega$, then $\beta < \omega^\alpha \cdot n$ for some $n \in \omega$, so (Exercises 2.10 and 2.8)

$$\beta + \omega^{\alpha+1} \le \omega^\alpha \cdot n + \omega^\alpha \cdot \omega = \omega^\alpha \cdot (n + \omega) = \omega^\alpha \cdot \omega = \omega^{\alpha+1}$$

and hence $\omega^{\alpha+1}$ is indecomposable. Finally if $\alpha > 0$ is a limit ordinal, ω^ξ is indecomposable for all $\xi < \alpha$, and $\beta < \omega^\alpha$, then by continuity

$$\beta + \omega^\alpha = \lim_{\xi\to\alpha} (\beta + \omega^\xi) = \lim_{\xi\to\alpha} \omega^\xi = \omega^\alpha$$

and hence ω^α is indecomposable. □

Chapter 3

Remark. If κ and λ are ordinals which are cardinals, then $\kappa \le \lambda$ in the *ordinal* ordering if and only if $\kappa \le \lambda$ in the *cardinal* ordering (3.2). Indeed, if $\kappa \le \lambda$ in the ordinals, then $\kappa \subset \lambda$, so $\kappa \le \lambda$ in the cardinals. Conversely, if $\kappa \le \lambda$ in the cardinals, then we cannot have $\lambda < \kappa$ in the ordinals since κ is a cardinal.

Remark. An ordinal is a "finite" cardinal if and only if it is a "finite cardinal". In fact, if an ordinal is a "finite" cardinal, then it is a "finite ordinal" by the remark above, and hence it is a "finite cardinal". The converse is just the *pigeonhole principle*, which is proved by induction on ω.

Remark. The arithmetic operations for finite cardinals in (3.3) agree with the corresponding operations for finite ordinals.

Exercise (3.1)**.**

(i) A subset of a finite set is finite.

(ii) A finite union of finite sets is finite.

(iii) The power set of a finite set is finite.

(iv) An image (projection) of a finite set is finite.

Proof.

(i) By an easy induction on $n \in \boldsymbol{N}$, every subset of n has m elements for some $m \in \boldsymbol{N}$ with $m \le n$, from which the result follows.

Alternately, if $B \subset A$, then $P(B) \subset P(A)$, so $X \subset P(B)$ implies $X \subset P(A)$. If A is finite, then A is T-finite (Exercise 1.12), so B is T-finite and hence B is finite (Exercise 1.13).

(ii) If $|A| = m$ and $|B| = n$ and $A \cap B = \emptyset$, then $|A \cup B| = m + n$. If $A \cap B \ne \emptyset$, then $A \cup B$ is a subset of the disjoint union,[2] so $|A \cup B| \le m + n$ by (i). Therefore the union of two finite sets is finite, and the union of any finite set of finite sets is finite by induction.

(iii) If $|A| = n$, then $|P(A)| = 2^n$ (Lemma 3.3).

(iv) If $f : n \to B$ is surjective, define $g : B \to n$ by letting $g(b)$ be the least $m \in n$ such that $f(m) = b$. Then g is injective, so B has the same cardinality as a subset of n, and hence $|B| \le n$ by (i). The result follows. □

[2] Technically, $A \cup B$ has the same cardinality as a subset of $(A \times \{0\}) \cup (B \times \{1\})$.

Exercise (3.2)**.**

(i) A subset of a countable set is at most countable.

(ii) A finite union of countable sets is countable.

(iii) An image (projection) of a countable set is at most countable.

Proof.

(i) If $B \subset \boldsymbol{N}$ is infinite, let

$$\begin{aligned} b_0 &= \text{least in } B \\ b_{n+1} &= \text{least in } B - \{b_0, \ldots, b_n\} \text{ (nonempty since } B \text{ is infinite)} \end{aligned}$$

Let $C = \{b_n \mid n \in \boldsymbol{N}\}$, which is countable since $n \mapsto b_n$ is a bijection. If $B \neq C$, let b be least in $B - C$. There are only finitely many elements of B less than b (Exercise 3.1(i)), which must be $b_0, \ldots, b_k$ for some $k \in \boldsymbol{N}$ by hypothesis. But then b is least in $B - \{b_0, \ldots, b_k\}$, so $b = b_{k+1} \in C$, which contradicts $b \notin C$. Therefore $B = C$. The result follows.

(ii) Similar to the proof of Exercise 3.1(ii), replacing m and n with $\aleph_0$ and using the fact that $\aleph_0 \le |A \cup B| \le \aleph_0 + \aleph_0 = \aleph_0$ (Theorem 3.5).

(iii) Similar to the proof of Exercise 3.1(iv), replacing n with $\boldsymbol{N}$. □

Exercise (3.3)**.** $\boldsymbol{N} \times \boldsymbol{N}$ is countable.

Proof. The mapping $(m, n) \mapsto 2^m(2n+1) - 1$ is a bijection from $\boldsymbol{N} \times \boldsymbol{N}$ to $\boldsymbol{N}$. In fact, it is injective by uniqueness of prime factorizations. If $k \in \boldsymbol{N}$, let $m \in \boldsymbol{N}$ be the highest power of 2 dividing $k+1$. Then $k+1 = 2^m(2n+1)$ for some $n \in \boldsymbol{N}$, so k is the image of (m, n). □

Exercise (3.4)**.**

(i) The set of all finite sequences in $\boldsymbol{N}$ is countable.

(ii) The set of all finite subsets of a countable set is countable.

Proof.

(i) The set is at least countable since there are countably many sequences of length one, and it is at most countable since the mapping

$$(m_1, \ldots, m_n) \mapsto p_1^{m_1+1} \cdots p_n^{m_n+1}$$

is injective, where p_k is the k-th prime.

(ii) The set of all finite subsets of $\boldsymbol{N}$ is at least countable since there are countably many singletons, and it is at most countable since it is the image of the mapping which takes each finite sequence in $\boldsymbol{N}$ to its underlying set (part (i) and Exercise 3.2(iii)). The result follows. □

Exercise (3.7)**.** If B is a projection of ω_α, then $|B| \leq \aleph_\alpha$.

Proof. If $f : \omega_\alpha \to B$ is surjective, define $g : B \to \omega_\alpha$ by letting $g(b)$ be the least $\beta \in \omega_\alpha$ such that $f(\beta) = b$. Then g is injective, so $|B| \leq \aleph_\alpha$. □

Exercise (3.9)**.** If B is a projection of A, then $|P(B)| \leq |P(A)|$.

Proof. If $f : A \to B$ is surjective, define $g : P(B) \to P(A)$ by $g(X) = f_{-1}(X)$. If $g(X) = g(Y)$, then

$$X = f(g(X)) = f(g(Y)) = Y$$

by surjectivity of f, so g is injective. □

Exercise (3.10)**.** $\omega_{\alpha+1}$ is a projection of $P(\omega_\alpha)$.

Proof. If $X \subset \omega_\alpha$, then type $X \leq \omega_\alpha$ by a remark about well-orderings above, so we can define $f : P(\omega_\alpha) \to \omega_{\alpha+1}$ by $f(X) = \operatorname{type} X$. If $\beta \in \omega_{\alpha+1}$, then $\beta \subset \omega_\alpha$ with $f(\beta) = \beta$, so f is surjective. □

Exercise (3.11)**.** $\aleph_{\alpha+1} < 2^{2^{\aleph_\alpha}}$

Proof. $\aleph_{\alpha+1} < 2^{\aleph_{\alpha+1}} \leq 2^{2^{\aleph_\alpha}}$, where the first inequality follows from Theorem 3.1 and the second inequality follows from Exercises 3.9–10. □

Exercise (3.12)**.** If $\aleph_\alpha$ is an uncountable limit cardinal, then $\operatorname{cf}\omega_\alpha = \operatorname{cf}\alpha$; also, ω_α is the limit of a cofinal sequence $\langle \omega_{\alpha_\xi} \mid \xi < \operatorname{cf}\alpha \rangle$ of cardinals.[3]

Proof. By Lemma 3.7(ii) and its proof, since $\omega_\alpha = \lim_{\xi \to \alpha} \omega_\xi$. □

Exercise (3.13)**.** (ZF) ω_2 is not a countable union of countable sets.

Proof. If $\omega_2 = \bigcup_{n<\omega} S_n$ where each S_n is countable, let $\alpha_n = \operatorname{type} S_n < \omega_1$. Then $\alpha = \sup_n \alpha_n \leq \omega_1$. For $(n, \xi) \in \omega \times \alpha$, let $f(n, \xi)$ be the ξ-th element of S_n if $\xi \in \alpha_n$ or the first element of S_n if $\xi \notin \alpha_n$. Then $f : \omega \times \alpha \to \omega_2$ is surjective. But $0 < |\omega \times \alpha| \leq \aleph_0 \cdot \aleph_1 = \aleph_1$, so ω_2 is a projection of ω_1 and $\aleph_2 \leq \aleph_1$—contradicting $\aleph_1 < \aleph_2$. Therefore ω_2 is not a countable union of countable sets. □

[3] The second part of the exercise in the book is incorrect. For example, $\omega_{\omega+\omega}$ is not the limit of a cofinal sequence $\langle \omega_n \mid n < \omega \rangle$.

Exercise (3.14)**.** S is D-infinite if and only if S has a countable subset.

Proof. If S is D-infinite, let $f : S \to X$ be a bijection with $X \subset S$ and $X \neq S$. Let $s_0 \in S - X$ and $s_{n+1} = f(s_n)$ for all $n \in \boldsymbol{N}$. By induction on n, $s_n \neq s_m$ for all $m < n$, so $n \mapsto s_n$ is injective and $\{ s_n \mid n \in \boldsymbol{N} \}$ is a countable subset of S.

Conversely, if $n \mapsto s_n \in S$ is injective, define $f : S \to S$ by

$$f(s) = \begin{cases} s_{n+1} & \text{if } s = s_n \\ s & \text{if } s \neq s_n \text{ for all } n \end{cases}$$

Then f is a bijection from S to $S - \{s_0\}$, so S is D-infinite. □

Remark. It follows that a D-finite subset of a countable set must be finite, since an infinite subset of a countable set is countable (Exercise 3.2(i)).

Exercise (3.15)**.**

(i) If A and B are D-finite, then $A \cup B$ and $A \times B$ are D-finite.

(ii) The set of all finite injective sequences in a D-finite set is D-finite.

(iii) The union of a disjoint D-finite family of D-finite sets is D-finite.

Proof. In all cases we use the fact that a set is D-infinite if and only if it has a countable subset (Exercise 3.14).

(i) If $A \cup B$ is not D-finite, let $n \mapsto s_n \in A \cup B$ be injective. If A is D-finite, there must be N such that for all $n \geq N$, $s_n \notin A$, so $s_n \in B$. But then $n \mapsto s_{N+n} \in B$ is injective, so B is not D-finite.

If $A \times B$ is not D-finite, let $R \subset A \times B$ be countable. If $\operatorname{dom} R$ is infinite, then we can enumerate a countable subset of A using an enumeration of R, so A is not D-finite. If $\operatorname{dom} R$ is finite, then $\operatorname{ran} R$ must be infinite, so we can enumerate a countable subset of B using an enumeration of R, and hence B is not D-finite.

(ii) Let S be a set. If $n \mapsto f_n$ is injective with $f_n : l_n \to S$ injective and $l_n \in \boldsymbol{N}$ for all $n \in \boldsymbol{N}$, choose k_0 least with $l_{k_0} \neq 0$ and define

$$\begin{aligned} s_0 &= f_{k_0}(0) \\ s_{n+1} &= f_k(m) \qquad && k \text{ least such that } \operatorname{ran} f_k \not\subset \{s_0, \dots, s_n\} \\ & && m \text{ least such that } f_k(m) \notin \{s_0, \dots, s_n\} \end{aligned}$$

Note k must exist since there are only finitely many injective sequences in a finite set. Clearly $n \mapsto s_n$ is injective, so S is D-infinite.

(iii) Let S be a set of disjoint D-finite sets. If $n \mapsto s_n \in \bigcup S$ is injective, define

$$\begin{aligned} x_0 &= \text{the unique set in } S \text{ containing } s_0 \\ x_{n+1} &= \text{the unique set in } S \text{ containing } s_k \\ &\quad k \text{ least such that } s_k \notin \bigcup\{x_0, \ldots, x_n\} \end{aligned}$$

Note k must exist since for each $x \in S$ there is N with $s_n \notin x$ for all $n \geq N$. Clearly $n \mapsto x_n$ is injective, so S is D-infinite. □

Exercise (3.16)**.** If A is infinite, then $P(P(A))$ is D-infinite.

Proof. The mapping $n \mapsto \{X \subset A \mid |X| = n\}$ is injective (Exercise 3.14). □

Chapter 4

Remark. In Definition 4.2, a "dense subset" of a linear ordering is a subset that is "dense" (as a linear ordering itself), but the converse is not true. For example, the positive rationals form a subset of the reals that is "dense" (between any two positive rationals is another positive rational) but is not a "dense subset" (between two negative reals there is no positive rational).

Remark. In Theorem 4.4, property (iii) of a Dedekind cut is not needed for this purely order-theoretic construction.[4]

Exercise (4.1)**.** The set of all continuous functions $f : \boldsymbol{R} \to \boldsymbol{R}$ has cardinality $\mathfrak{c}$, while $|\boldsymbol{R}^{\boldsymbol{R}}| = 2^{\mathfrak{c}}$.

Proof. By definition $|\boldsymbol{R}^{\boldsymbol{R}}| = \mathfrak{c}^{\mathfrak{c}}$, and $2^{\mathfrak{c}} \leq \mathfrak{c}^{\mathfrak{c}} \leq (2^{\mathfrak{c}})^{\mathfrak{c}} = 2^{\mathfrak{c}\cdot\mathfrak{c}} = 2^{\mathfrak{c}}$ since $\mathfrak{c} \cdot \mathfrak{c} = 2^{\aleph_0} \cdot 2^{\aleph_0} = 2^{\aleph_0+\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$.

There are at least $\mathfrak{c}$ continuous functions since the constant functions are continuous. Conversely, there are at most $\mathfrak{c}$ since a continuous function on $\boldsymbol{R}$ is determined by its values on the dense subset $\boldsymbol{Q}$, so the mapping $f \mapsto f \restriction \boldsymbol{Q}$ is injective, and $|\boldsymbol{R}^{\boldsymbol{Q}}| = \mathfrak{c}^{\aleph_0} = (2^{\aleph_0})^{\aleph_0} = 2^{\aleph_0\cdot\aleph_0} = 2^{\aleph_0} = \mathfrak{c}$. □

Exercise (4.3)**.** The set of all algebraic reals is countable.

Proof. The set is at least countable since the rationals are algebraic (m/n is a root of the polynomial $nx - m$). Conversely, it is at most countable since there

[4] See [3], Chapter 1, Exercise 20.

are only countably many nonzero polynomials with integer coefficients (Exercise 3.4(i)) each of which has only finitely many real roots, and a countable union of finite ordered sets is countable. (If S is a countable family of finite ordered sets, then it is possible to enumerate a countable subset of $\bigcup S$ and to define an injection $\bigcup S \to \omega \times \omega$.) □

Exercise (4.4)**.** If $S \subset \boldsymbol{R}$ is countable, then $|\boldsymbol{R} - S| = \mathfrak{c}$.

Proof. It is sufficient to prove that if $S \subset \boldsymbol{R} \times \boldsymbol{R}$ is countable, then $|\boldsymbol{R} \times \boldsymbol{R} - S| = \mathfrak{c}$, since $|\boldsymbol{R} \times \boldsymbol{R}| = |\boldsymbol{R}|$. It cannot be that for every $x \in \boldsymbol{R}$ there is $y \in \boldsymbol{R}$ with $(x, y) \in S$, lest S would be uncountable. Hence there is $x \in \boldsymbol{R}$ with $\{x\} \times \boldsymbol{R} \cap S = \emptyset$, that is, $\{x\} \times \boldsymbol{R} \subset \boldsymbol{R} \times \boldsymbol{R} - S$, so $\mathfrak{c} = |\{x\} \times \boldsymbol{R}| \leq |\boldsymbol{R} \times \boldsymbol{R} - S| \leq \mathfrak{c}$. □

Exercise (4.5)**.**

(i) The set of all irrational numbers has cardinality $\mathfrak{c}$.

(ii) The set of all transcendental numbers has cardinality $\mathfrak{c}$.

Proof. By Exercises 4.3 and 4.4. □

Exercise (4.6)**.** The set of all open sets of reals has cardinality $\mathfrak{c}$.

Proof. The set has cardinality at least $\mathfrak{c}$ since $r \mapsto (r, r+1)$ is injective. The set has cardinality at most $\mathfrak{c}$ since the mapping

$$O \mapsto \{(p, q) \in \boldsymbol{Q} \times \boldsymbol{Q} \mid \text{the interval } (p, q) \subset O\}$$

for open sets O is injective by density of $\boldsymbol{Q}$ in $\boldsymbol{R}$, and $|P(\boldsymbol{Q} \times \boldsymbol{Q})| = 2^{\aleph_0} = \mathfrak{c}$. □

Exercise (4.7)**.** The Cantor set $\boldsymbol{C}$ is perfect.

Proof. $\boldsymbol{C} \neq \emptyset$ since $0 \in \boldsymbol{C}$, $\boldsymbol{C}$ is closed since it is the complement in $[0, 1]$ of the union of the open intervals removed in the construction, and every point in $\boldsymbol{C}$ is the limit of a set of endpoints of those intervals, all of which are in $\boldsymbol{C}$. □

Exercise (4.8)**.** If P is perfect and $P \cap (a, b) \neq \emptyset$, then $|P \cap (a, b)| = \mathfrak{c}$.

Proof. Define $\alpha = \inf(P \cap (a, b))$ and $\beta = \sup(P \cap (a, b))$. Then $a \leq \alpha < \beta \leq b$ and $\alpha, \beta \in P$. Moreover, $P \cap [\alpha, \beta]$ is perfect, since in particular α and β are not isolated points. So $\mathfrak{c} = |P \cap [\alpha, \beta]| \leq |P \cap [a, b]|$ (Theorem 4.5), from which the result follows. □

Exercise (4.9)**.** If F is closed and $P \not\subset F$ is perfect, then $|P - F| = \mathfrak{c}$.

Proof. If $x \in P - F$, then by closure of F there is (a, b) with $x \in (a, b)$ and $F \cap (a, b) = \emptyset$. Now $P \cap (a, b) \neq \emptyset$ and $P \cap (a, b) \subset P - F$, so $\mathfrak{c} = |P \cap (a, b)| \le |P - F|$ (Exercise 4.8). $\square$

Exercise (4.10)**.** If P is perfect, then $P^* = P$.

Proof. If $x \in P^*$, then x is a limit point of P, so $x \in P$ by closure of P. Conversely if $x \in P \cap (a, b)$, then $P \cap (a, b)$ is uncountable (Exercise 4.8), so $x \in P^*$. $\square$

Exercise (4.11)**.** If $P \subset A$ is perfect, then $P \subset A^*$.

Proof. $P = P^* \subset A^*$ (Exercise 4.10). $\square$

Exercise (4.12)**.** If F is an uncountable closed set and P is the perfect subset of F constructed in the Cantor-Bendixon theorem (Theorem 4.6), then $P = F^*$.

Proof. We know $P \subset F^*$ (Exercise 4.11). If $x \in F^*$, then every neighborhood of x contains uncountably many points of F, and hence a point of P since $F - P$ is at most countable. Therefore $x \in P$ by closure of P. $\square$

Exercise (4.13)**.** If F is an uncountable closed set, then $F = F^* \cup (F - F^*)$ is the unique partition[5] of F into a perfect subset and an at most countable subset.

Proof. We know that $F^* \cup (F - F^*)$ is such a partition (Exercise 4.12). If $P \cup S$ is another such partition and $P \neq F^*$, then $P \not\subset F^*$ or $F^* \not\subset P$, so S is uncountable or $F - F^*$ is uncountable (Exercise 4.9)—a contradiction. $\square$

Exercise (4.14)**.** $\boldsymbol{Q}$ is not a countable intersection of open sets.

Proof. If $\boldsymbol{Q} = \bigcap O_n$ with O_n open, then $\boldsymbol{Q} \subset O_n$ and hence O_n is dense in $\boldsymbol{R}$ for all n. It follows that $\boldsymbol{I} = \bigcup(\boldsymbol{R} - O_n)$ is a countable union of closed nowhere dense sets, and since $\boldsymbol{Q}$ is a countable union of nowhere dense singletons, $\boldsymbol{R} = \boldsymbol{Q} \cup \boldsymbol{I}$ is a countable union of nowhere dense sets—contradicting the Baire category theorem (Theorem 4.8). $\square$

Exercise (4.15)**.** If B is Borel and $f : \boldsymbol{R} \to \boldsymbol{R}$ is continuous, then $f_{-1}(B)$ is Borel.

[5] We allow that $F - F^*$ may be empty.

Proof. By induction on the Borel sets $\mathscr{B}$. Let

$$\mathscr{C} = \{X \subset \boldsymbol{R} \mid f_{-1}(X) \in \mathscr{B}\}$$

Then $\boldsymbol{R} \in \mathscr{C}$ since $f_{-1}(\boldsymbol{R}) = \boldsymbol{R} \in \mathscr{B}$. If $X \in \mathscr{C}$, then $f_{-1}(\boldsymbol{R} - X) = \boldsymbol{R} - f_{-1}(X) \in \mathscr{B}$, so $\boldsymbol{R} - X \in \mathscr{C}$. If $X_n \in \mathscr{C}$, then $f_{-1}(\bigcup X_n) = \bigcup f_{-1}(X_n) \in \mathscr{B}$, so $\bigcup X_n \in \mathscr{C}$. Finally, if O is open, then $f_{-1}(O)$ is open by continuity of f, so $f_{-1}(O) \in \mathscr{B}$ and $O \in \mathscr{C}$. Therefore $\mathscr{C}$ is a σ-algebra containing the open sets, so $\mathscr{B} \subset \mathscr{C}$. □

Exercise (4.16)**.** If $f : \boldsymbol{R} \to \boldsymbol{R}$, the set of points where f is continuous is G_δ.

Proof. For $\alpha > 0$, f is *α-continuous at x* if there is $\delta > 0$ such that $|f(y) - f(z)| < \alpha$ for all $y, z \in (x - \delta, x + \delta)$.[6] If C is the set of points where f is continuous and C_n is the set of points where f is $1/n$-continuous, then C_n is open for all n and $C = \bigcap C_n$. □

Chapter 5

Remark. If κ is infinite and $|A| \geq \kappa$, then $|[A]^{<\kappa}| = |A|^{<\kappa}$.

Proof. Since $[A]^{<\kappa} = \bigcup_{\lambda<\kappa} [A]^\lambda$ is a partition, we have (Lemmas 5.7–8)

$$|[A]^{<\kappa}| = \sum_{\lambda<\kappa} |A|^\lambda = \kappa \cdot \sup_{\lambda<\kappa} |A|^\lambda = \kappa \cdot |A|^{<\kappa} = |A|^{<\kappa}$$ □

Remark. An infinite cardinal κ is singular if and only if $\kappa = \sum_{i<\lambda} \kappa_i$ for cardinals $\lambda < \kappa$ and $\kappa_i < \kappa$ for all $i < \lambda$.

Proof. If κ is singular, let $\lambda = \operatorname{cf} \kappa < \kappa$ and choose $S_i \subset \kappa$ with $\kappa_i = |S_i| < \kappa$ and $\kappa = \bigcup_{i<\lambda} S_i$ (Lemma 3.10). Then

$$\kappa = \Big|\bigcup_{i<\lambda} S_i\Big| \leq \sum_{i<\lambda} \kappa_i \leq \lambda \cdot \kappa = \kappa$$

The converse follows from Lemma 3.10. □

Remark. If $\kappa_i \geq 2$ for $1 \leq i \leq n$, then $\sum \kappa_i \leq \prod \kappa_i$.

Proof.

$$2^{n-1} \sum_{i=1}^{n} \kappa_i \leq n \prod_{i=1}^{n} \kappa_i \leq 2^{n-1} \prod_{i=1}^{n} \kappa_i$$ □

[6] See [1], Definition 4.6.5.

Remark. Properties of singular cardinals may be determined by properties of regular cardinals below. For example, the continuum function is continuous at singular cardinals below which it is eventually constant (Corollary 5.17). This idea is extended by SCH (Theorem 5.22).

Exercise (5.2)**.** If X is infinite and S is the set of finite subsets of X, then $|S| = |X|$.

Proof. $|X| \le |S|$ since $x \mapsto \{x\}$ is injective. Conversely, S is the projection of the set $X^{<\omega} = \bigcup_{n<\omega} X^n$ of finite sequences in X under the mapping which takes each sequence to its underlying set. Therefore (by (5.2))

$$|S| \le |X^{<\omega}| = \sum_{n<\omega} |X|^n = \aleph_0 \cdot |X| = |X| \qquad \square$$

Remark. This generalizes Exercise 3.4(ii).

Exercise (5.3)**.** If P is a linear ordering such that every initial segment of P has cardinality $< \kappa$, then $|P| \le \kappa$.

Proof. Call $S \subset P$ *cofinal* in P if for all $x \in P$ there is $y \in S$ with $x \le y$. If S is not cofinal in P, choose $F(S) \in P$ with $F(S) > y$ for all $y \in S$. By recursion on the ordinals, define

$$x_\alpha = F(\{x_\xi \mid \xi < \alpha\}) \quad \text{if } \{x_\xi \mid \xi < \alpha\} \text{ is not cofinal in } P$$

Let θ be the least ordinal at which the recursion cannot continue. If $\theta = 0$, then the result is trivial. If $\theta = \alpha + 1$, then x_α is greatest in P, so $P = P(x_\alpha) \cup \{x_\alpha\}$ and $|P| \le \kappa$. If $\theta > 0$ is a limit ordinal, then

$$P = \bigcup_{\alpha<\theta} P(x_\alpha)$$

Also $\aleph_0 \le |\theta| \le \kappa$, lest $|P(x_{\kappa+1})| \ge \kappa$. Therefore $|P| \le \kappa \cdot \kappa = \kappa$. $\square$

Exercise (5.4)**.** If A can be well-ordered, then $P(A)$ can be linearly ordered.

Proof. Identify $P(A)$ with 2^A and order 2^A with the lexicographic order: $f < g$ if $f(x) < g(x)$ at the least $x \in A$ where $f(x) \ne g(x)$. $\square$

Remark. The lexicographic order is not in general a well-ordering. For example, consider $2^{\boldsymbol{N}}$ and the subset $2^{\boldsymbol{N}} - \{0\}$.

Exercise (5.5)**.** ZF + ZL implies AC.

Proof. If C is a set of nonempty sets, let P be the set of "partial" choice functions for C partially ordered by inclusion: $f : S \to \bigcup C$ with $S \subset C$ and $f(X) \in X$ for all $X \in S$. Then $\emptyset \in P$ and every chain in P has its union as an upper bound. By ZL, there is a maximal element $F \in P$. Now $\operatorname{dom} F = X$, lest F can be properly extended and is not maximal, so F is a choice function for C. $\square$

Exercise (5.6)**.** ZF + CC implies that every infinite set has a countable subset.

Proof. If X is infinite, choose an injection $f_n : n \to X$ for all $n \in \boldsymbol{N}$ by CC and recursively define $g : \boldsymbol{N} \to X$ by $g(n) = f_{2^n}(k)$ where k is least such that $f_{2^n}(k) \neq g(m)$ for all $m < n$. Then g is injective. $\square$

Exercise (5.7)**.** ZF + DC implies CC.

Proof. If C is a countable set of nonempty sets, let $n \mapsto C_n \in C$ be a bijection and assume without loss of generality that the C_n are pairwise disjoint. (If they are not disjoint, replace them with $C_n \times \{n\}$.)

Define a relation E on $A = \bigcup C$ by

$$b \mathrel{E} a \iff (\exists n \in \boldsymbol{N})(b \in C_{n+1} \wedge a \in C_n)$$

Note A is nonempty and for every $a \in A$ there is $b \in A$ with $b \mathrel{E} a$ since the C_n are nonempty. By DC, there is a sequence $a_0, a_1, \ldots \in A$ with $a_{n+1} \mathrel{E} a_n$ for all $n \in \boldsymbol{N}$. By definition of E and disjointness of the C_n, there is $m \in \boldsymbol{N}$ such that $a_n \in C_{n+m}$ for all $n \in \boldsymbol{N}$. For $0 \le n < m$, choose $a_{n-m} \in C_n$.[7] Then $C_n \mapsto a_{n-m}$ $(n \in \boldsymbol{N})$ is a choice function for C. $\square$

Exercise (5.11)**.** $\prod_{0<n<\omega} n = 2^{\aleph_0}$.

Proof. By Lemmas 5.9 and 5.6,

$$\prod_{0<n<\omega} n = \Big(\sup_{0<n<\omega} n\Big)^{\aleph_0} = \aleph_0^{\aleph_0} = 2^{\aleph_0} \qquad \square$$

Exercise (5.13)**.** $\prod_{\alpha<\omega+\omega} \aleph_\alpha = \aleph_{\omega+\omega}^{\aleph_0}$

Proof. By (5.16) and Lemma 5.9,

$$\prod_{\alpha<\omega+\omega} \aleph_\alpha = \prod_{n<\omega} \aleph_n \cdot \prod_{n<\omega} \aleph_{\omega+n} = \aleph_\omega^{\aleph_0} \cdot \aleph_{\omega+\omega}^{\aleph_0} = \aleph_{\omega+\omega}^{\aleph_0} \qquad \square$$

[7] This does not require an axiom of choice since it involves only finitely many choices.

Exercise (5.14)**.** If GCH holds, then (i) $2^{<\kappa} = \kappa$ for infinite κ and (ii) $\kappa^{<\kappa} = \kappa$ for regular κ.

Proof. For (i),
$$2^{<\kappa} = \sup\{2^{\mu} \mid \mu < \kappa\} = \sup\{\mu^{+} \mid \mu < \kappa\} = \kappa$$
For (ii), by Theorem 5.15,
$$\kappa^{<\kappa} = \sup\{\kappa^{\mu} \mid \mu < \kappa\} = \sup\{\kappa \mid \mu < \kappa\} = \kappa \qquad \square$$

Exercise (5.17)**.** If κ is infinite and $0 < \lambda < \mathrm{cf}\,\kappa$, then $\kappa^{\lambda} = \sum_{\alpha<\kappa} |\alpha|^{\lambda}$.

Proof. Since $\kappa^{\lambda} = \bigcup_{\alpha<\kappa} \alpha^{\lambda}$ (Lemma 3.9(ii)),
$$\kappa^{\lambda} = \Big|\bigcup_{\alpha<\kappa} \alpha^{\lambda}\Big| \le \sum_{\alpha<\kappa} |\alpha|^{\lambda} \le \kappa \cdot \kappa^{\lambda} = \kappa^{\lambda} \qquad \square$$

Remark. The next few exercises show how κ^{λ} can be computed recursively in λ sometimes (compare with Theorem 5.20).

Exercise (5.18)**.** $\aleph_{\omega}^{\aleph_1} = \aleph_{\omega}^{\aleph_0} \cdot 2^{\aleph_1}$.

Proof. By Exercise 5.19, taking $\alpha = \omega$. $\square$

Exercise (5.19)**.** If $\alpha < \omega_1$, then $\aleph_{\alpha}^{\aleph_1} = \aleph_{\alpha}^{\aleph_0} \cdot 2^{\aleph_1}$.

Proof. By induction on α. If $\alpha = 0$, then $\aleph_{\alpha}^{\aleph_1} = 2^{\aleph_1} = \aleph_{\alpha}^{\aleph_0} \cdot 2^{\aleph_1}$ (Lemma 5.6). If the result holds for α, then by the Hausdorff formula (5.22),
$$\aleph_{\alpha+1}^{\aleph_1} = \aleph_{\alpha}^{\aleph_1} \cdot \aleph_{\alpha+1} = \aleph_{\alpha}^{\aleph_0} \cdot \aleph_{\alpha+1} \cdot 2^{\aleph_1} = \aleph_{\alpha+1}^{\aleph_0} \cdot 2^{\aleph_1}$$
so the result holds for $\alpha + 1$. If $\alpha > 0$ is a limit ordinal and the result holds for all $\xi < \alpha$, then $\mathrm{cf}\,\aleph_{\alpha} = \mathrm{cf}\,\alpha = \aleph_0$ and (Lemma 5.19)
$$\aleph_{\alpha}^{\aleph_1} = \big(\lim_{\xi\to\alpha} \aleph_{\xi}^{\aleph_1}\big)^{\aleph_0} = \big(\lim_{\xi\to\alpha} (\aleph_{\xi}^{\aleph_0} \cdot 2^{\aleph_1})\big)^{\aleph_0} \le \big(\aleph_{\alpha}^{\aleph_0} \cdot 2^{\aleph_1}\big)^{\aleph_0} = \aleph_{\alpha}^{\aleph_0} \cdot 2^{\aleph_1} \le \aleph_{\alpha}^{\aleph_1}$$
so the result holds for α. $\square$

Remark. The next few exercises explore the behavior of $2^{<\kappa}$ and $\kappa^{<\kappa}$ at limit cardinals κ.

Exercise (5.21)**.** If κ is regular and limit, then $\kappa^{<\kappa} = 2^{<\kappa}$. If κ is regular and strong limit, then $\kappa^{<\kappa} = \kappa$.

Proof. If $\kappa \geq 2$, then $\kappa^{<\kappa} \geq 2^{<\kappa}$. If κ is regular and limit and $\lambda < \kappa$, then

$$\kappa^{\lambda} = \sup\{\mu^{\lambda} \mid \mu < \kappa\}$$

Indeed, note that if $f : \lambda \to \kappa$, then $f : \lambda \to \alpha$ for some $\alpha < \kappa$ (Lemma 3.9(ii)), so $f : \lambda \to |\alpha|^{+} < \kappa$. For $\mu < \kappa$, $\mu^{\lambda} \leq (2^{\mu})^{\lambda} = 2^{\mu \cdot \lambda} \leq 2^{<\kappa}$, so $\kappa^{\lambda} \leq 2^{<\kappa}$, and it follows that $\kappa^{<\kappa} \leq 2^{<\kappa}$. If κ is additionally strong limit, then $2^{<\kappa} \leq \kappa$. $\square$

Exercise (5.22)**.** If κ is singular and not strong limit, then $\kappa^{<\kappa} = 2^{<\kappa} > \kappa$.

Proof. As above, $\kappa^{<\kappa} \geq 2^{<\kappa}$. Choose $\lambda < \kappa$ with $\kappa \leq 2^{\lambda}$. If $\mu < \kappa$, then $\kappa^{\mu} \leq (2^{\lambda})^{\mu} = 2^{\lambda \cdot \mu} \leq 2^{<\kappa}$, so it follows that $\kappa^{<\kappa} \leq 2^{<\kappa}$. Finally, $\kappa < \kappa^{\operatorname{cf}\kappa} \leq \kappa^{<\kappa}$ by König's theorem (Corollary 5.14). $\square$

Exercise (5.23)**.** If κ is singular and strong limit, then $2^{<\kappa} = \kappa$ and $\kappa^{<\kappa} = \kappa^{\operatorname{cf}\kappa}$.

Proof. Clearly $2^{<\kappa} \leq \kappa$. Since $\kappa = \sup\{\lambda \mid \lambda < \kappa\}$ and $\lambda < 2^{\lambda} \leq 2^{<\kappa}$ for $\lambda < \kappa$, it follows that $\kappa \leq 2^{<\kappa}$. Finally (Lemma 5.6, Theorem 5.16(iii)),

$$\kappa^{\operatorname{cf}\kappa} \leq \kappa^{<\kappa} \leq \kappa^{\kappa} = 2^{\kappa} = (2^{<\kappa})^{\operatorname{cf}\kappa} = \kappa^{\operatorname{cf}\kappa} \qquad \square$$

Remark. The next few exercises show how behavior of the continuum function at regular cardinals influences cardinal exponentiation at singular cardinals.

Exercise (5.24)**.** If $2^{\aleph_0} \geq \aleph_{\omega}$, then $\aleph_{\omega}^{\aleph_0} = 2^{\aleph_0}$.

Proof. By Theorem 5.20(ii). $\square$

Remark. $2^{\aleph_0} \neq \aleph_{\omega}$ by König's theorem (Corollary 5.12).

Exercise (5.25)**.** If $2^{\aleph_1} = \aleph_2$ and $\aleph_{\omega}^{\aleph_0} \geq \aleph_{\omega_1}$, then $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega}^{\aleph_0}$.

Proof. By Exercise 5.18,

$$\aleph_{\omega}^{\aleph_0} \leq \aleph_{\omega_1}^{\aleph_1} \leq (\aleph_{\omega}^{\aleph_0})^{\aleph_1} = \aleph_{\omega}^{\aleph_1} = \aleph_{\omega}^{\aleph_0} \cdot 2^{\aleph_1} = \aleph_{\omega}^{\aleph_0} \cdot \aleph_2 = \aleph_{\omega}^{\aleph_0} \qquad \square$$

Exercise (5.26)**.** If $2^{\aleph_0} \geq \aleph_{\omega_1}$, then $\gimel(\aleph_{\omega}) = 2^{\aleph_0}$ and $\gimel(\aleph_{\omega_1}) = 2^{\aleph_1}$.

Proof. By Theorem 5.20(ii). $\square$

Exercise (5.27)**.** If $2^{\aleph_1} = \aleph_2$, then $\aleph_{\omega}^{\aleph_0} \neq \aleph_{\omega_1}$.

Proof. If $\aleph_{\omega}^{\aleph_0} = \aleph_{\omega_1}$, then $\aleph_{\omega_1}^{\aleph_1} = \aleph_{\omega_1}$ (Exercise 5.25), which contradicts König's theorem (Corollary 5.14). $\square$

Remark. The next few exercises explore the behavior of the gimel function at singular cardinals.

Exercise (5.28)**.** If $\kappa \leq \gimel(\lambda)$ and $\operatorname{cf}\kappa \leq \operatorname{cf}\lambda$, then $\gimel(\kappa) \leq \gimel(\lambda)$.

Proof.
$$\gimel(\kappa) = \kappa^{\operatorname{cf}\kappa} \leq \gimel(\lambda)^{\operatorname{cf}\kappa} = (\lambda^{\operatorname{cf}\lambda})^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\lambda\cdot\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\lambda} = \gimel(\lambda)$$
□

Exercise (5.29)**.** If κ is singular and $2^{\operatorname{cf}\kappa} < \kappa \leq \lambda^{\operatorname{cf}\kappa}$ for some $\lambda < \kappa$, then $\gimel(\kappa) = \gimel(\lambda)$ where λ is least such that $\kappa \leq \lambda^{\operatorname{cf}\kappa}$.

Proof. By Theorem 5.20. $\gimel(\kappa) = \kappa^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\kappa}$, so it suffices to prove that $\lambda^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\lambda} = \gimel(\lambda)$. If $\lambda \leq \operatorname{cf}\kappa$, then $\lambda^{\operatorname{cf}\kappa} = 2^{\operatorname{cf}\kappa} < \kappa \leq \lambda^{\operatorname{cf}\kappa}$—a contradiction. If there is $\mu < \lambda \leq \mu^{\operatorname{cf}\kappa}$, then $\kappa \leq \lambda^{\operatorname{cf}\kappa} = \mu^{\operatorname{cf}\kappa}$—a contradiction since λ is least with this property. If $\operatorname{cf}\kappa < \operatorname{cf}\lambda$, then $\lambda^{\operatorname{cf}\kappa} = \lambda < \kappa$—a contradiction. Thus $\lambda^{\operatorname{cf}\kappa} = \lambda^{\operatorname{cf}\lambda}$. □

Chapter 6

Remark. In the proof of Theorem 6.5, to see that F is the unique function on T satisfying (6.6), first observe that if $F(x) = y_1$ and $F(x) = y_2$ are witnessed by f_1 and f_2, respectively, then $D = \operatorname{dom} f_1 \cap \operatorname{dom} f_2$ is transitive and $f_1(z) = f_2(z)$ for all $z \in D$ by $\in$-induction on D, so in particular $y_1 = f_1(x) = f_2(x) = y_2$. It follows that F is a function, and if $x \in \operatorname{dom} F$ then there is a unique witnessing function f_x with smallest domain.

If $x \subset \operatorname{dom} F$, let $f = \bigcup_{w\in x} f_w$.[8] Then f is a function by $\in$-induction as above, $\operatorname{dom} f = \bigcup_{w\in x} \operatorname{dom} f_w$ is transitive with $x \subset \operatorname{dom} f$, and $f(z) = G(f \restriction z)$ for all $z \in \operatorname{dom} f$. Thus $f \cup \{(x, G(f \restriction x))\}$ witnesses $F(x) = G(f \restriction x)$, and $x \in \operatorname{dom} F$. By $\in$-induction, it follows that $\operatorname{dom} F = T$.

Now $F(x) = f_x(x) = G(f_x \restriction x) = G(F \restriction x)$, so F satisfies (6.6). Finally, if F' also satisfies (6.6), then $F = F'$ by $\in$-induction, so F is unique.

References

[1] Abbott, Stephen. *Understanding Analysis.* Springer, 2001.

[2] Jech, Thomas. *Set Theory*, 3rd ed. Springer, 2002.

[8] The set $\{f_w \mid w \in x\}$ exists by replacement.

[3] Rudin, Walter. *Principles of Mathematical Analysis*, 3rd ed. McGraw-Hill, 1976.