Notes and exercises from *Set Theory*

John Peloquin

Introduction

This document contains notes and exercises from [1].

Chapter 1

Exercise (2)**.** There is no set X such that $P(X) \subset X$.

Proof. By the axiom of regularity (1.8), X is $\in$-minimal in $\{X\}$, so $X \notin X$ and hence $P(X) \not\subset X$. $\square$

Exercise (3)**.** If X is inductive, then the set $\{x \in X \mid x \subset X\}$ is inductive. Hence $\boldsymbol{N}$ is transitive and for each $n \in \boldsymbol{N}$, $n = \{m \in \boldsymbol{N} \mid m < n\}$.

Proof. Let $S = \{x \in X \mid x \subset X\}$. By inductivity of X, $\emptyset \in S$, and if $x \in S$, then $x \cup \{x\} \in S$, so S is inductive. Taking $X = \boldsymbol{N}$, it follows that $S = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set. Hence $n \in \boldsymbol{N}$ implies $n \subset \boldsymbol{N}$, so $\boldsymbol{N}$ is transitive and $n = \{m \in \boldsymbol{N} \mid m < n\}$. $\square$

Remark. We proved transitivity of $\boldsymbol{N}$ "by induction" on $\boldsymbol{N}$: $0 \subset \boldsymbol{N}$ and if $n \subset \boldsymbol{N}$ then $n+1 \subset \boldsymbol{N}$, so $n \subset \boldsymbol{N}$ for all $n \in \boldsymbol{N}$. The following exercises are similar.

Exercise (4)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive}\}$ is inductive. Hence every $n \in \boldsymbol{N}$ is transitive.

Proof. The class C of transitive sets is inductive. Indeed, $\emptyset$ is transitive, and if x is transitive then $x \cup \{x\}$ is transitive since $y \in x \cup \{x\}$ implies $y \subset x \subset x \cup \{x\}$. It follows that $\{x \in X \mid x \text{ is transitive}\} = X \cap C$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \boldsymbol{N}$, it follows as above that every $n \in \boldsymbol{N}$ is transitive. $\square$

Exercise (5)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and } x \notin x\}$ is inductive. Hence $n \notin n$ and $n \neq n+1$ for all $n \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x \text{ is transitive and } x \notin x\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive (by inductivity of the class of transitive sets). Also $x \cup \{x\} \notin x$, lest $x \cup \{x\} \subset x$ by transitivity of x and hence $x \in x$—contradicting $x \notin x$. Similarly $x \cup \{x\} \neq x$. Therefore $x \cup \{x\} \notin x \cup \{x\}$. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive, and taking $X = \boldsymbol{N}$ that $n \notin n$ and hence $n \neq n+1$ for all $n \in \boldsymbol{N}$. □

Remark. In order to prove that $n \notin n$ for all $n \in \boldsymbol{N}$ by induction on $\boldsymbol{N}$, we "loaded the induction hypothesis" with transitivity.

Exercise (6)**.** If X is inductive, then the set $\{x \in X \mid x \text{ is transitive and regular}\}$ is inductive, where a set x is called *regular* if every nonempty subset of x has an $\in$-minimal element.

Proof. The class $C = \{x \mid x \text{ is transitive and regular}\}$ is inductive. Indeed, $\emptyset \in C$. If $x \in C$, then $x \cup \{x\}$ is transitive. If $y \subset x \cup \{x\}$ is nonempty, let $z = y - \{x\} \subset x$. If $z = \emptyset$, then $y = \{x\}$, and x is $\in$-minimal in y by regularity of x. If $z \neq \emptyset$ and t is $\in$-minimal in z, then $x \notin t$ by transitivity and regularity of x, so t is $\in$-minimal in y. Therefore $x \cup \{x\}$ is regular. So $x \cup \{x\} \in C$, and C is inductive. It follows as above that $X \cap C$ is inductive. □

Remark. Taking $X = \boldsymbol{N}$ it follows as above that every $n \in \boldsymbol{N}$ is regular.

Exercise (7)**.** Every nonempty $X \subset \boldsymbol{N}$ has an $\in$-minimal element.

Proof. Choose $n \in X$. If n is not $\in$-minimal in X, then $n \cap X$ is a nonempty subset of n, which has an $\in$-minimal element m (Exercise 6). By transitivity of n (Exercise 4), m is $\in$-minimal in X. □

Exercise (8)**.** If X is inductive, then the set $\{x \in X \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is inductive. Hence for all $n \in \boldsymbol{N}$, $n = 0$ or $n = m+1$ for some $m \in \boldsymbol{N}$.

Proof. The class $C = \{x \mid x = \emptyset \text{ or } \exists y(x = y \cup \{y\})\}$ is obviously inductive, so $X \cap C$ is inductive. Taking $X = \boldsymbol{N}$, the rest follows, with $m \in \boldsymbol{N}$ by transitivity of $\boldsymbol{N}$ (Exercise 3). □

Exercise (9)**.** Let $A \subset \boldsymbol{N}$ be such that $0 \in A$, and if $n \in A$ then $n+1 \in A$. Then $A = \boldsymbol{N}$.

Proof. A is inductive, so $A = \boldsymbol{N}$ since $\boldsymbol{N}$ is the smallest inductive set.

Alternately, if $A \neq \boldsymbol{N}$, let m be $\in$-minimal in $\boldsymbol{N} - A$ (Exercise 7). Then $m \neq 0$, and $m \neq n+1$ for any $n \in \boldsymbol{N}$, which is impossible (Exercise 8). □

Exercise (10)**.** Each $n \in \boldsymbol{N}$ is T-finite.

Proof. By induction on $\boldsymbol{N}$. First, $n = 0$ is T-finite since $P(\emptyset) = \{\emptyset\}$ and $\emptyset$ is $\subset$-maximal in $\{\emptyset\}$. If n is T-finite, suppose $X \subset P(n \cup \{n\})$ is nonempty. Let

$$X' = \{\, u - \{n\} \mid u \in X \,\}$$

Then $X' \subset P(n)$ is nonempty and has a $\subset$-maximal element $u - \{n\}$ for some $u \in X$ by T-finiteness of n, where we may assume $n \in u$ if $u \cup \{n\} \in X$. We claim u is $\subset$-maximal in X. Indeed, if $v \in X$ and $u \subset v$, then $u - \{n\} \subset v - \{n\}$, so $u - \{n\} = v - \{n\}$ by $\subset$-maximality of $u - \{n\}$ in X', so $u = v$ by the assumption about u. Therefore $n+1$ is T-finite. □

Exercise (11)**.** $\boldsymbol{N}$ is T-infinite. In fact, $\boldsymbol{N} \subset P(\boldsymbol{N})$ has no $\subset$-maximal element.

Proof. We know $\boldsymbol{N} \subset P(\boldsymbol{N})$ by transitivity of $\boldsymbol{N}$ (Exercise 3), and for all $n \in \boldsymbol{N}$ we have $n \subset n+1 \in \boldsymbol{N}$ but $n \neq n+1$ (Exercise 5). □

Remark. If A is T-finite and $\pi : A \to B$ is surjective (onto), then B is T-finite.

Proof. By pullback. If $X \subset P(B)$ is nonempty, let

$$X_{-1} = \pi_{-1}(X) = \{\, u_{-1} = \pi_{-1}(u) \mid u \in X \,\}$$

Then $X_{-1} \subset P(A)$ is nonempty and has a $\subset$-maximal element u_{-1} for some $u \in X$ by T-finiteness of A. We claim u is $\subset$-maximal in X. Indeed, if $v \in X$ and $u \subset v$, then $u_{-1} \subset v_{-1}$, so $u_{-1} = v_{-1}$ by $\subset$-maximality of u_{-1} in X_{-1}, so $u = v$ by surjectivity of π. Therefore B is T-finite. □

Exercise (12)**.** Every finite set is T-finite.

Proof. By definition, every finite set is the image of a T-finite natural number (Exercise 10), so is T-finite by the previous remark.

Alternately, if S is T-infinite, choose $X \subset P(S)$ nonempty with no $\subset$-maximal element. By induction on $\boldsymbol{N}$, for each $n \in \boldsymbol{N}$ there is a properly ascending chain $u_0 \subset \cdots \subset u_n \subset S$ of length $n+1$ in X, and hence an injection $n \to S$ sending $m \in n$ into $u_{m+1} - u_m$. If S had k elements for some $k \in \boldsymbol{N}$, there would then be an injection $k+1 \to k$, which is impossible by an easy induction on k. Therefore S is infinite. □

Exercise (13)**.** Every infinite set is T-infinite.

Proof. If S is infinite, let X be the set of finite subsets of S. Clearly X is nonempty since $\emptyset \in X$. Also X has no $\subset$-maximal element. Indeed, if $u \in X$ then there is $s \in S - u$ since S is infinite, and $u \cup \{s\}$ is finite (if u has k elements, then $u \cup \{s\}$ has $k+1$ elements), so $u \subset u \cup \{s\} \in X$ where the inclusion is proper. Therefore S is T-infinite. □

Remark. The previous two exercises show that (Cantor) finiteness is equivalent to Tarski finiteness in ZF.

Chapter 2

Remark. In (2.1), $<$ is actually a *well-ordering* on *Ord*. Indeed, if $C \subset Ord$ is a nonempty class of ordinals and $\alpha \in C$ is not $\in$-minimal in C, then $\alpha \cap C$ is a nonempty subset of α which has an $\in$-minimal element β. By transitivity of α, β is $\in$-minimal in C.

This observation provides an alternative proof that *Ord* is a proper class: if *Ord* were a set, then because it is transitive and strictly well-ordered by $\in$, it would be an ordinal, and hence $Ord \in Ord$—contradicting strictness.

Exercise (2)**.** α is a limit ordinal if and only if $\beta < \alpha$ implies $\beta + 1 < \alpha$ for all β.

Proof. If α is a limit ordinal and $\beta < \alpha$, then $\beta + 1 \leq \alpha$ (2.5) and $\beta + 1 \neq \alpha$, so $\beta + 1 < \alpha$. If α is not a limit ordinal, then $\beta + 1 = \alpha$ for some $\beta < \alpha$. □

Remark. It follows that α is a nonzero limit ordinal if and only if it is inductive.

Exercise (3)**.** If X is inductive, then $X \cap Ord$ is inductive. $\boldsymbol{N} = \bigcap\{X \mid X \text{ inductive}\}$ is the least nonzero limit ordinal.

Proof. Clearly *Ord* is inductive, so $X \cap Ord$ is inductive since the intersection of two inductive classes is inductive. Taking $X = \boldsymbol{N}$, it follows that $\boldsymbol{N} \subset Ord$, and since $\boldsymbol{N}$ is also transitive, $\boldsymbol{N}$ is an ordinal. By the previous remark, $\boldsymbol{N}$ is the least nonzero limit ordinal. □

Exercise (4)**.** (Without the axiom of infinity.) Let ω be the least nonzero limit ordinal, if it exists, or *Ord* otherwise. The following are equivalent:

(i) There exists an inductive set.

(ii) There exists an infinite[1] set.

(iii) ω is a set.

Proof. (i) $\Longleftrightarrow$ (iii): The smallest inductive set is the least nonzero limit ordinal (Exercise 3).

(iii) $\Longrightarrow$ (ii): ω is infinite (Exercises 1.11–2).

(ii) $\Longrightarrow$ (i): For any finite set A, let $|A|$ denote the least $n \in \omega$ in bijective correspondence with A (the "number of elements" in A). If X is infinite, let

$$S = \{|A| \mid A \subset X \text{ finite}\}$$

Note S is a set by replacement, and S is inductive. Indeed, $0 \in S$ since $\emptyset \subset X$ and $|\emptyset| = 0$. If $n \in S$ and $A \subset X$ with $|A| = n$, then there must exist $x \in X - A$ since X is infinite, and $|A \cup \{x\}| = n + 1 \in S$. □

Remark. This exercise shows that (i)–(iii) are equivalent forms of the axiom of infinity in ZF.

References

[1] Jech, Thomas. *Set Theory*, 3rd ed. Springer, 2002.

[1] In this exercise, a set is *finite* if it is in bijective correspondence with some $n \in \omega$ and *infinite* otherwise, even if $\omega = Ord$.