Notes and exercises from *Topology*

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Introduction

This document contains notes and exercises from [1].

Chapter I

Section 8

Remark. Let X be a set, $\mathscr{B}$ a filter base on X, and Y a separated topological space. If $f : X \to Y$ and $\lim_{\mathscr{B}} f = b$, then b is the only adherent value of f along $\mathscr{B}$.

Proof. First, b is an adherent value of f along $\mathscr{B}$. If V is a neighborhood of b, then by assumption there is $B \in \mathscr{B}$ with $f(B) \subset V$. For any $B' \in \mathscr{B}$, there is $B'' \in \mathscr{B}$ with $B'' \subset B \cap B'$ and hence

$$f(B'') \subset f(B \cap B') \subset f(B) \cap f(B') \subset V \cap f(B')$$

Since $B'' \neq \emptyset$, it follows that $V \cap f(B') \neq \emptyset$.

Second, if $b' \neq b$, there are neighborhoods V', V of b', b with $V' \cap V = \emptyset$. By assumption, there is $B \in \mathscr{B}$ with $f(B) \subset V$, so $V' \cap f(B) = \emptyset$ and $b' \notin \overline{f(B)}$. It follows that b' is not an adherent value of f along $\mathscr{B}$. □

Section 10

Remark. Let E be a set, $\mathscr{B}$ a filter base on E, and $F = F_1 \times \cdots \times F_n$ a topological product space. If $f = (f_i) : E \to F$ and $l = (l_i) \in F$, then $\lim_{\mathscr{B}} f = l$ if and only if $\lim_{\mathscr{B}} f_i = l_i$ for all i.

Proof. If $\lim_{\mathscr{B}} f = l$ and ω_i is an open neighborhood of l_i in F_i, then

$$\omega = F_1 \times \cdots \times F_{i-1} \times \omega_i \times F_{i+1} \times \cdots \times F_n$$

is an open neighborhood of l in F, so there is $B \in \mathscr{B}$ with $f(B) \subset \omega$ and hence $f_i(B) \subset \omega_i$. It follows that $\lim_{\mathscr{B}} f_i = l_i$.

Conversely, suppose $\lim_{\mathscr{B}} f_i = l_i$ for all i. If V is a neighborhood of l in F, then there is an elementary open neighborhood $\omega = \omega_1 \times \cdots \times \omega_n$ of l in V. Now ω_i is an open neighborhood of l_i in F_i, so there is $B_i \in \mathscr{B}$ with $f_i(B_i) \subset \omega_i$ for all i. Finally, there is $B \in \mathscr{B}$ with $B \subset B_1 \cap \cdots \cap B_n$, so $f(B) \subset \omega \subset V$. It follows that $\lim_{\mathscr{B}} f = l$. □

Section 11

Remark. Let X be a set, $\mathscr{B}$ a filter base on X, and E a compact topological space. If $f : X \to E$, then f has an adherent value along $\mathscr{B}$. Moreover, if l is the only adherent value, then $\lim_{\mathscr{B}} f = l$.

Proof. The family $\{\overline{f(B)} \mid B \in \mathscr{B}\}$ has the finite intersection property, so has nonempty intersection by compactness of E.

If V is an open neighborhood of l, then the family

$$\{\overline{f(B)} \cap \complement V \mid B \in \mathscr{B}\}$$

has empty intersection. By compactness of E, there is a finite subfamily with empty intersection. It follows that there is $B \in \mathscr{B}$ with $f(B) \cap \complement V = \emptyset$, that is $f(B) \subset V$. Therefore $\lim_{\mathscr{B}} f = l$. □

Section 12

Remark. Let E be a separated topological space which is not compact. Then the family

$$\mathscr{B} = \{B \subset E \mid \complement B \text{ is compact}\}$$

is a filter base on E.

Proof. If $B_1, B_2 \in \mathscr{B}$, then

$$\complement(B_1 \cap B_2) = (\complement B_1) \cup (\complement B_2)$$

is compact (11.14), so $B_1 \cap B_2 \in \mathscr{B}$. Finally, $\emptyset \notin \mathscr{B}$ since E is not compact. □

Remark. Since compact sets are topologically "finite", this filter base allows us to take limits as "$x \to \infty$" in E.

Remark. Let E be a locally compact topological space which is not compact. If $f : E \to \mathbb{R}$ is continuous and $\lim_{x\to\infty} f(x) = l \in \mathbb{R}$, then f is bounded. Moreover, f attains its bounds which are distinct from l.

Proof. By assumption, there is $K \subset E$ compact with $l - 1 < f(x) < l + 1$ for $x \notin K$. On the other hand, f is continuous on K (9.5), and hence bounded on K (11.17), so f is bounded on E.

Let $s = \sup_E f$. If $l \neq s$ then $l < s$, and by assumption there is $K \subset E$ compact with $f(x) < (l + s)/2$ for $x \notin K$. It follows that $s = \sup_K f$, so $f(x) = s$ for some $x \in K$ (11.17). Similarly for $\inf_E f$. □

Section 14

Remark. If G is a topological group and V is a neighborhood of the unit $e \in G$, then $A \subset G$ is small of order V if and only if $AA^{-1} \subset V$.

Remark. If G is an additive (commutative) topological group, to see that $x \mapsto -x$ is uniformly continuous it suffices to observe that if V is a neighborhood of $0 \in G$ and $A \subset G$ is small of order V, then $-A$ is also small of order V since

$$-A - (-A) = -A + A = A - A \subset V$$

This does not require V to be symmetric.

References

[1] Choquet, G. *Topology*. Academic Press, 1966.