Notes and exercises from *Topology*

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Introduction

This document contains notes and exercises from [1].

Chapter I

Section 8

Remark. Let X be a set, $\mathscr{B}$ a filter base on X, and Y a separated topological space. If $f : X \to Y$ and $\lim_{\mathscr{B}} f = b$, then b is the only adherent value of f along $\mathscr{B}$.

Proof. First, b is an adherent value of f along $\mathscr{B}$. If V is a neighborhood of b, then by assumption there is $B \in \mathscr{B}$ with $f(B) \subset V$. For any $B' \in \mathscr{B}$, there is $B'' \in \mathscr{B}$ with $B'' \subset B \cap B'$ and hence

$$f(B'') \subset f(B \cap B') \subset f(B) \cap f(B') \subset V \cap f(B')$$

Since $B'' \neq \emptyset$, it follows that $V \cap f(B') \neq \emptyset$.

Second, if $b' \neq b$, there are neighborhoods V', V of b', b with $V' \cap V = \emptyset$. By assumption, there is $B \in \mathscr{B}$ with $f(B) \subset V$, so $V' \cap f(B) = \emptyset$ and $b' \notin \overline{f(B)}$. It follows that b' is not an adherent value of f along $\mathscr{B}$. □

Section 10

Remark. Let E be a set, $\mathscr{B}$ a filter base on E, and $F = F_1 \times \cdots \times F_n$ a topological product space. If $f = (f_i) : E \to F$ and $l = (l_i) \in F$, then $\lim_{\mathscr{B}} f = l$ if and only if $\lim_{\mathscr{B}} f_i = l_i$ for all i.

Proof. If $\lim_{\mathscr{B}} f = l$ and ω_i is an open neighborhood of l_i in F_i, then

$$\omega = F_1 \times \cdots \times F_{i-1} \times \omega_i \times F_{i+1} \times \cdots \times F_n$$

is an open neighborhood of l in F, so there is $B \in \mathscr{B}$ with $f(B) \subset \omega$ and hence $f_i(B) \subset \omega_i$. It follows that $\lim_{\mathscr{B}} f_i = l_i$.

Conversely, suppose $\lim_{\mathscr{B}} f_i = l_i$ for all i. If V is a neighborhood of l in F, then there is an elementary open neighborhood $\omega = \omega_1 \times \cdots \times \omega_n$ of l in V. Now ω_i is an open neighborhood of l_i in F_i, so there is $B_i \in \mathscr{B}$ with $f_i(B_i) \subset \omega_i$ for all i. Finally, there is $B \in \mathscr{B}$ with $B \subset B_1 \cap \cdots \cap B_n$, so $f(B) \subset \omega \subset V$. It follows that $\lim_{\mathscr{B}} f = l$. $\square$

Section 11

Remark. Let X be a set, $\mathscr{B}$ a filter base on X, and E a compact topological space. If $f : X \to E$, then f has an adherent value along $\mathscr{B}$. Moreover, if l is the only adherent value, then $\lim_{\mathscr{B}} f = l$.

Proof. The family $\{\overline{f(B)} \mid B \in \mathscr{B}\}$ has the finite intersection property, so has nonempty intersection by compactness of E.

If V is an open neighborhood of l, then the family

$$\{\overline{f(B)} \cap \complement V \mid B \in \mathscr{B}\}$$

has empty intersection. By compactness of E, there is a finite subfamily with empty intersection. It follows that there is $B \in \mathscr{B}$ with $f(B) \cap \complement V = \emptyset$, that is $f(B) \subset V$. Therefore $\lim_{\mathscr{B}} f = l$. $\square$

References

[1] Choquet, G. *Topology*. Academic Press, 1966.