Exercises and Notes from *Topology*

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Fall 2008

Exercises and notes from *Topology* (2nd ed.) by James R. Munkres.

Chapter 1

Section 3

EXERCISE 15. Assume the least upper bound property for $\mathbb{R}$.

(a) The sets $[0,1]$ and $[0,1)$ have the least upper bound property.

Proof. Let A be a nonempty subset of $[0,1)$ which is bounded above in $[0,1)$. Then there exists $0 \le b < 1$ such that for all $a \in A$, $a \le b$. Let β be the least upper bound of A in $\mathbb{R}$. Then $\beta \in [0,1)$ since $0 \le a \le \beta \le b < 1$ for any fixed $a \in A$. This shows that $[0,1)$ has the least upper bound property.

The proof is nearly identical for $[0,1]$. $\square$

(b) Under the dictionary order, the sets $[0,1]\times[0,1]$ and $[0,1)\times[0,1]$ have the least upper bound property, but the sets $[0,1]\times[0,1)$ and $[0,1)\times[0,1)$ do not.

Proof. To prove the latter claim, let

$$A = \{(0, 1-1/2^n) \mid n = 0,1,\ldots\} \subseteq [0,1)\times[0,1) \subseteq [0,1]\times[0,1)$$

Now A is bounded above in both sets, since for example $(1/2,0)$ is an upper bound. Note that there is no upper bound with first coordinate 0. But also there is no *least* upper bound with nonzero first coordinate, since for example

$$B = \{(1/2^n, 0) \mid n = 1,2,\ldots\}$$

is a set of upper bounds with arbitrarily small nonzero first coordinates. Thus it follows that there is no least upper bound for A in either set, so neither set has the least upper bound property.

To prove the first claim, suppose A is a nonempty subset of $[0,1)\times[0,1]$ which is bounded above (in either set). Let α be the least upper bound of the set of left coordinates of elements in A. If the set $A \cap (\{\alpha\}\times[0,1])$ is nonempty, let β be the least upper bound of right coordinates in this set; otherwise let $\beta = 0$.

We claim that (α, β) is a least upper bound for A. Indeed, it is an upper bound since for all $(a, b) \in A$, $a \leq \alpha$, and $b \leq \beta$ if $a = \alpha$. Suppose now $(\gamma, \delta) < (\alpha, \beta)$. If $\gamma < \alpha$, then there exists $(a, b) \in A$ with $\gamma < a \leq \alpha$, so $(\gamma, \delta) < (a, b)$ and hence (γ, δ) is not an upper bound for A. If $\gamma = \alpha$, then we must have $0 \leq \delta < \beta$, which means (by definition of β) there exists $(a, b) \in A$ with $a = \alpha$ and $\delta < b \leq \beta$. But then $(\gamma, \delta) < (a, b)$ again, so (γ, δ) is not an upper bound for A. These cases are exhaustive, hence (α, β) is the least upper bound for A. □

Note that the latter argument fails for the second pair of sets in the problem, since for those sets β is *not* guaranteed to exist in $[0, 1)$.

Chapter 2

Section 13

EXERCISE 1. Let X be a topological space. Suppose $A \subseteq X$ and for all $x \in A$ there exists an open set U with $x \in U \subseteq A$. Then A is open.

Proof. For each $x \in A$ choose an open set U_x with $x \in U_x \subseteq A$. Then $A = \bigcup_{x \in A} U_x$ is a union of open sets, and hence is open. □

EXERCISE 3. Let X be a set. Define the following collections:

$$\mathcal{T}_c = \{U \subseteq X \mid X - U \text{ is countable or all of } X\}$$
$$\mathcal{T}_\infty = \{U \subseteq X \mid X - U \text{ is infinite or empty or all of } X\}$$

Then $\mathcal{T}_c$ is a topology on X, but $\mathcal{T}_\infty$ is not in general.

Proof. Let $\{U_\alpha\}$ be an indexed subcollection of $\mathcal{T}_c$. Set $U = \bigcup U_\alpha$. If the subcollection is empty, so is U, so $X - U = X$ and $U \in \mathcal{T}_c$. Otherwise fix α and note that

$$X - U = X - \bigcup U_\alpha = \bigcap (X - U_\alpha) \subseteq X - U_\alpha$$

which is countable since $X - U_\alpha$ is countable. Thus $U \in \mathcal{T}_c$.

Now let $\{U_\alpha\}$ be finite and set $U = \bigcap U_\alpha$. Then $X - U = X - \bigcap U_\alpha = \bigcup (X - U_\alpha)$ is a finite union of countable sets, which is countable, so $U \in \mathcal{T}_c$.

Set $X = \mathbb{R}$, $U_{-1} = (-\infty, 0)$ and $U_1 = (0, \infty)$. Then $U_{-1} \in \mathcal{T}_\infty$ and $U_1 \in T_\infty$ but

$$U_{-1} \cup U_1 = (-\infty, 0) \cup (0, \infty) = \mathbb{R} - \{0\} \notin \mathcal{T}_\infty$$

Therefore $\mathcal{T}_\infty$ is not a topology on X in general. □

EXERCISE 4.

(a) Let $\{\mathscr{T}_\alpha\}$ be a family of topologies on X. Then $\bigcap \mathscr{T}_\alpha$ is also a topology on X, but $\bigcup \mathscr{T}_\alpha$ is not in general.

Proof. The first claim is immediate from the fact that each $\mathscr{T}_\alpha$ is a topology. (Note if the family is empty, the intersection gives the discrete topology on X.)

To see that the union is not in general, set $X = \{a, b, c\}$ and define topologies

$$\mathscr{T}_1 = \{\emptyset, \{a\}, X\} \qquad \mathscr{T}_2 = \{\emptyset, \{b\}, X\}$$

Then $\mathscr{T} = \mathscr{T}_1 \cup \mathscr{T}_2$ is not a topology, as $\{a\}, \{b\} \in \mathscr{T}$ but $\{a\} \cup \{b\} = \{a, b\} \notin \mathscr{T}$. □

(b) Let $\{T_\alpha\}$ be a family of topologies on X. Then there exists a unique smallest topology on X containing all $\mathscr{T}_\alpha$, and in addition a unique largest topology on X contained in all $\mathscr{T}_\alpha$.

Proof. Let $\mathscr{T}$ be the topology generated by the subbasis $S = \bigcup \mathscr{T}_\alpha$. Then $\mathscr{T}$ is a topology on X containing all $\mathscr{T}_\alpha$, and any topology on X containing all $\mathscr{T}_\alpha$ must contain $\mathscr{T}$ (since such a topology will be closed under unions of finite intersections over S), so $\mathscr{T}$ is smallest and unique.

It is immediate that $\bigcap T_\alpha$ is the largest topology on X contained in all $\mathscr{T}_\alpha$. □

(c) Set $X = \{a, b, c\}$ and define the topologies

$$\mathscr{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X\} \qquad \mathscr{T}_2 = \{\emptyset, \{a\}, \{b, c\}, X\}$$

The smallest topology containing both is $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, X\}$. The largest topology contained in both is $\{\emptyset, \{a\}, X\}$.

EXERCISE 5. Let $\mathscr{A}$ be a basis for a topology on X. Then the topology generated by $\mathscr{A}$ is equal to the intersection of all topologies on X containing $\mathscr{A}$.

Proof. Both equal the unique smallest topology of X containing $\mathscr{A}$. □

EXERCISE 6. The topologies $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable.

Proof. Note $0 \in [0, 1)$, but there are no $a < b$ with $0 \in (a, b) - K \subseteq [0, 1)$. It follows that $\mathbb{R}_l \not\subseteq \mathbb{R}_K$ (Lemma 13.3). Conversely, note that $0 \in (-1, 1) - K$, but there are no $a < b$ with $0 \in [a, b) \subseteq (-1, 1) - K$, so $\mathbb{R}_K \not\subseteq \mathbb{R}_l$. □

EXERCISE 8.

(a) Let $\mathscr{B} = \{(a,b) \mid a < b \text{ rationals}\}$. Then $\mathscr{B}$ is a countable basis generating the standard topology on $\mathbb{R}$.

Proof. It is immediate that $\mathscr{B}$ is a countable collection of open sets. Note any interval (r,s) with $r,s \in \mathbb{R}$ can be expressed as a union of elements of $\mathscr{B}$ (since the rationals are dense in $\mathbb{R}$), hence any open set can be expressed as a union of elements of $\mathscr{B}$ (Lemma 13.1). Therefore $\mathscr{B}$ is a basis for the standard topology (Lemma 13.2). $\square$

(c) Let $\mathscr{C} = \{[a,b) \mid a < b \text{ rationals}\}$. Then $\mathscr{C}$ is a basis for $\mathbb{R}$ generating a topology different than $\mathbb{R}_l$.

Proof. It is immediate that $\mathscr{C}$ is a basis. Consider basis element $B = [\pi, \pi+1)$ in $\mathbb{R}_l$. Note $\pi \in B$, but there are no $a < b$ rational with $\pi \in [a,b) \subseteq [\pi, \pi+1)$, lest $a = \pi$ be irrational. Thus the topology generated by $\mathscr{C}$ is different from $\mathbb{R}_l$. $\square$

Section 16

EXERCISE 1. Let Y be a subspace of the topological space X and $A \subseteq Y$. Then the topology A inherits as a subspace of Y is the same as the topology A inherits as a subspace of X.

Proof. For the purposes of this proof, let $\mathscr{T}_{A,Z}$ denote the topology A inherits as a subspace of the topological space Z. Let $\mathscr{T}$ denote the topology of X. Then

$$\begin{aligned}
\mathscr{T}_{A,Y} &= \{U \cap A \mid U \in \mathscr{T}_{Y,X}\} && \text{by definition}\\
&= \{(U \cap Y) \cap A \mid U \in \mathscr{T}\} && \text{since } \mathscr{T}_{Y,X} = \{U \cap Y \mid U \in \mathscr{T}\}\\
&= \{U \cap A \mid U \in \mathscr{T}\} && \text{since } A \subseteq Y\\
&= \mathscr{T}_{A,X} && \text{by definition}
\end{aligned}$$

This establishes the result. $\square$

EXERCISE 2. Let $\mathscr{T}$ and $\mathscr{T}'$ be topologies on X with $\mathscr{T} \subset \mathscr{T}'$, and suppose $Y \subseteq X$. Let $\mathscr{T}_Y$ and $\mathscr{T}'_Y$ denote the corresponding subspace topologies on Y. Then $\mathscr{T}_Y \subseteq \mathscr{T}'_Y$, but this inclusion need not be proper.

Proof. It is immediate from definitions that $\mathscr{T}_Y \subseteq \mathscr{T}'_Y$.

To see equality, set $X = \mathbb{R}$ and $Y = \mathbb{Z}$, let $\mathscr{T}$ be the topology generated by intervals (a,b) with $a < b$ integers, and let $\mathscr{T}'$ be the standard topology. Then $\mathscr{T} \subset \mathscr{T}'$, but $\mathscr{T}_Y = \mathscr{T}'_Y$ is just the discrete topology on $\mathbb{Z}$. $\square$

EXERCISE 3. Consider $Y = [-1,1]$ as a subspace of $\mathbb{R}$. Then we have the following:

Subset	Open in Y	Open in $\mathbb{R}$
$A = \{x \mid \frac{1}{2} < \lvert x\rvert < 1\}$	Yes	Yes
$B = \{x \mid \frac{1}{2} < \lvert x\rvert \le 1\}$	Yes	No
$C = \{x \mid \frac{1}{2} \le \lvert x\rvert < 1\}$	No	No
$D = \{x \mid \frac{1}{2} \le \lvert x\rvert \le 1\}$	No	No
$E = \{x \mid 0 < \lvert x\rvert < 1 \text{ and } 1/x \notin \mathbb{Z}_+\}$	Yes	Yes

Note $E = (-1,0) \cup \left(\bigcup_{n\ge1}(\frac{1}{n+1}, \frac{1}{n})\right)$, a union of open sets in both spaces.

EXERCISE 4. The projection maps on the product space are open maps (that is, they map open sets to open sets).

Proof. Let $X_1 \times X_2$ be a product space and suppose $U \subseteq X_1 \times X_2$ is open. If $u \in U$, then by definition there exists some basis element $B_1 \times B_2 \subseteq X_1 \times X_2$, where B_i is open in X_i, and $u \in B_1 \times B_2$. Then $\pi_i(u) \in B_i$. Since u was arbitrary, $\pi_i(U)$ can be expressed as a union of open sets in X_i, and hence is open in X_i as desired. □

EXERCISE 6. The countable collection

$$\{(a,b) \times (c,d) \mid a < b \text{ and } c < d \text{ rationals}\}$$

is a basis for $\mathbb{R}^2$.

Proof. Immediate by the density of $\mathbb{Q}$ in $\mathbb{R}$ (cf. Exercise 13.8(a)). □

EXERCISE 7. If X is an ordered set and Y is a proper, convex subset of X, it need not be the case that Y is an interval or ray in X.

Indeed, consider $X = \mathbb{Q}$ and $Y = \{q \in \mathbb{Q} \mid q^2 < 2\}$. Then it is immediate that Y is proper and convex in X. But Y is not an interval or a ray in X. Note that Y is bounded in X by elements not in Y, so it is not a ray. Also it has no least or greatest element, so it is not a closed or half-open interval. If $Y = (a,b)$ for a,b rational, then $b^2 \not< 2$. But then since $b^2 \neq 2$ (no rational has square equal to 2), we must have $2 < b^2$. Now by computation we can choose $c < b$ with $2 < c^2$. But then since $c < b$, $c \in Y$, so $2 < c^2 < 2$—a contradiction. This shows that Y is not an open interval either.

EXERCISE 10. Let $I = [0,1]$. Let $\mathscr{T}_1$ denote the product topology on $I \times I$, $\mathscr{T}_2$ denote the dictionary order topology on $I \times I$, and $\mathscr{T}_3$ denote the subspace topology $I \times I$ inherits from $\mathbb{R} \times \mathbb{R}$ under the dictionary order topology.

Then $\mathscr{T}_3$ is strictly finer than $\mathscr{T}_1$ and $\mathscr{T}_2$, while $\mathscr{T}_1$ and $\mathscr{T}_2$ are not comparable.

Proof. Recall that basis elements of $\mathscr{T}_1$ are interiors of rectangles (with possible edges touching edges of the square), basis elements of $\mathscr{T}_2$ are open vertical segments in the square (with possible endpoints only at the lower left and upper right corners of the square), and basis elements of $\mathscr{T}_3$ are restrictions of open vertical segments in $\mathbb{R} \times \mathbb{R}$ to the square.

It is then immediate that $\mathcal{T}_3$ is finer than both $\mathcal{T}_1$ and $\mathcal{T}_2$. To see that it is strictly finer, note that $(0 \times \frac{1}{2}, 0 \times 1]$ is a basis element in $\mathcal{T}_3$, but there are no basis elements of $\mathcal{T}_1$ or $\mathcal{T}_2$ containing the point 0×1 and contained in this set.

It is immediate that $\mathcal{T}_1$ is not finer than $\mathcal{T}_2$. Note that the rectangle $[0,1] \times [0, \frac{1}{2})$ is a basis element in $\mathcal{T}_1$ containing the point $\frac{1}{2} \times 0$, but there is no basis element of $\mathcal{T}_2$ containing this point which is contained in the rectangle. Hence $\mathcal{T}_2$ is not finer than $\mathcal{T}_1$. Thus $\mathcal{T}_1$ and $\mathcal{T}_2$ are not comparable. $\square$