uc3m | Universidad Carlos III de Madrid

Network Theory for the Social Sciences in Python

Methods Workshop: Social Sciences PhD Program (2025/2026)

4. Dynamics on Networks

https://github.com/blas-ko/uc3m_networks_workshop_2025

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Summary

1. Introduction
2. Models of dynamics on networks
    - Random walker
    - Independent Cascades
    - Epidemiological models (SIR)
3. Maximizing/minimizing spreading on networks
    - Immunization strategies
    - Seeding strategies
4. Complex contagion
5. Summary

Summary

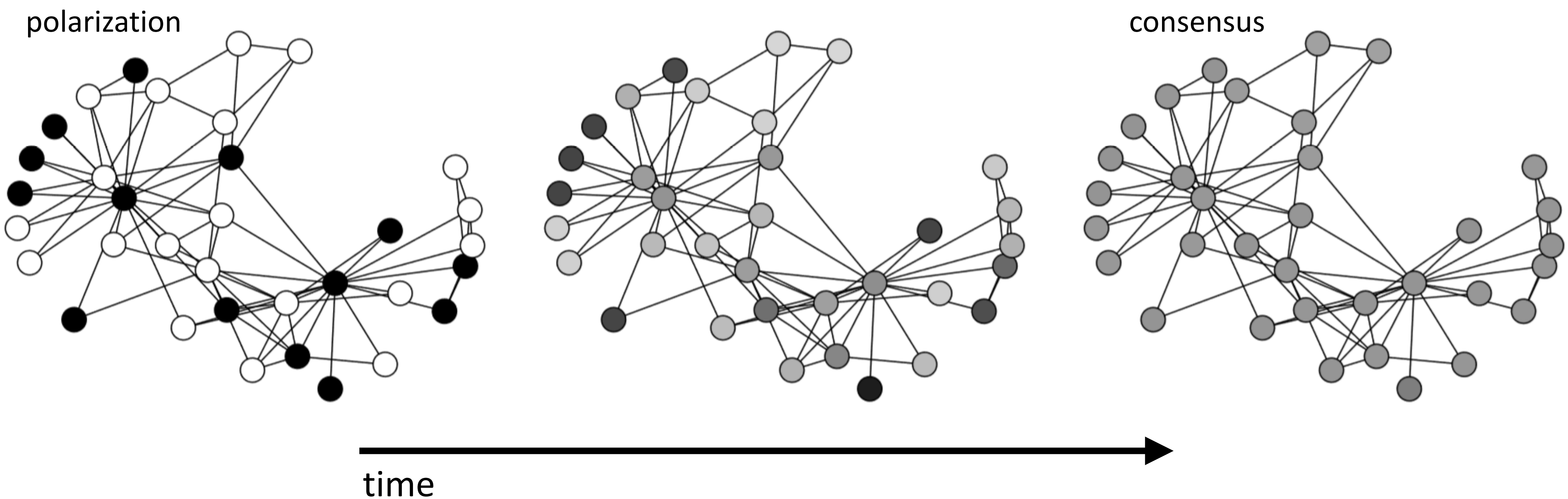
1. Introduction
2. Models of dynamics of networks
    - SI model
    - SIR model
3. Spreading on networks
4. Maximizing/Minimizing spreading
    - Immunization strategies in epidemic spreading
    - Seeding strategies in information spreading
5. Complex contagion

1 Introduction

Introduction

We have seen how networks may change over time.

However, even for a fixed a network structure
dynamical processes might unfold in them



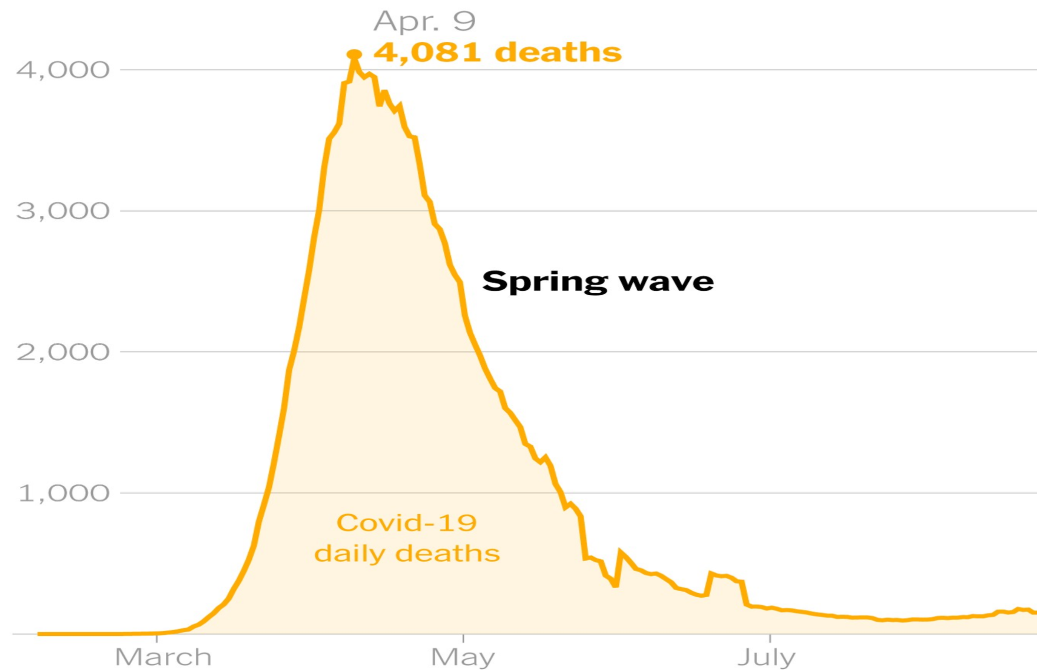
Sayama, H. (2015). Introduction to the modeling and analysis of complex systems. Open SUNY Textbooks.

Introduction

A large range of important problems in our society have to do with **spreading:** ideas, viruses, behavior, information, etc.

- What spreads and how it spreads is different in each case.

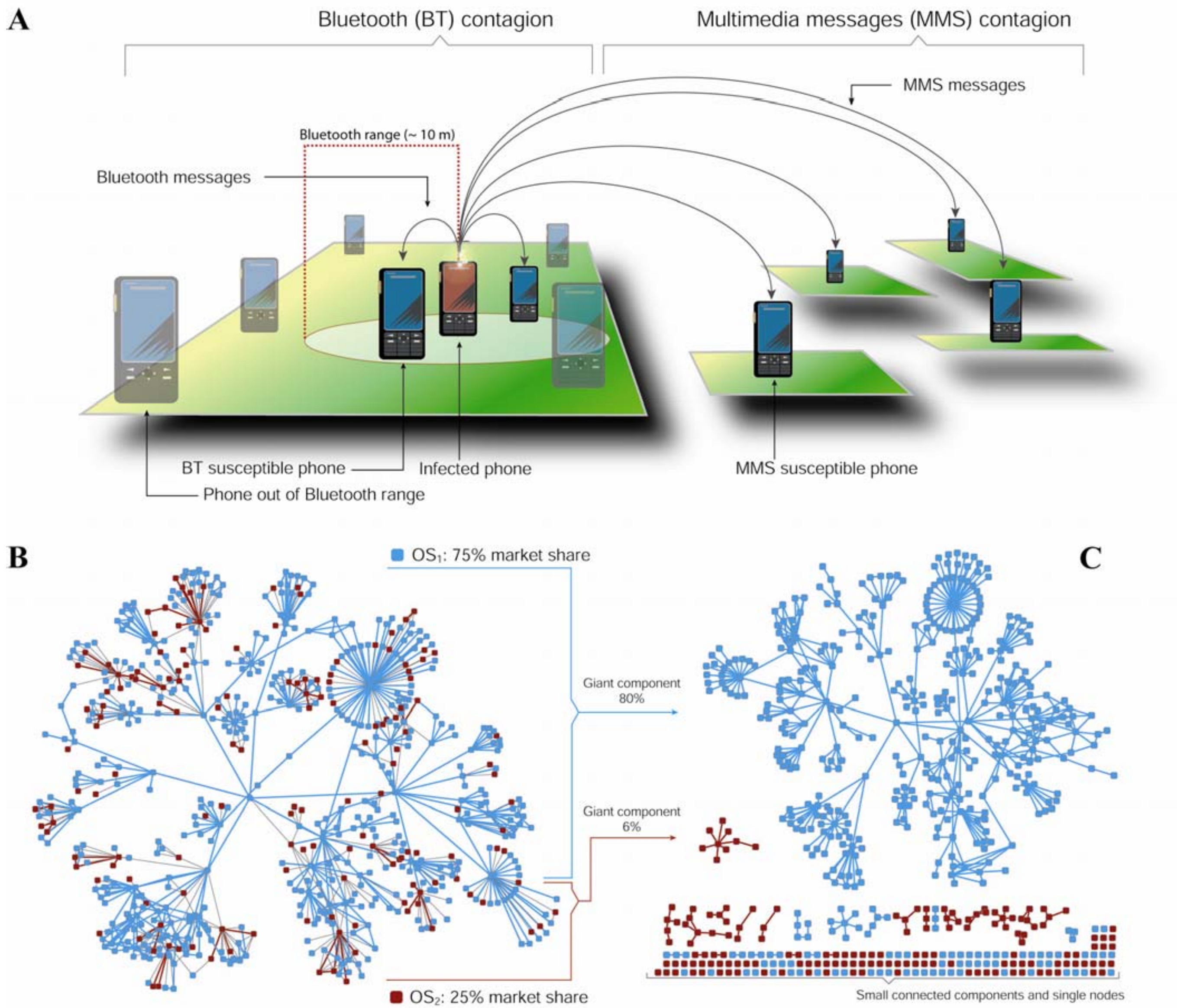
- **Example: Pathogens in diseases** like Influenza, SARS (COVID), tuberculosis, Ebola, AIDS, etc. They have different transmission mechanisms.



Jang, S., Han, S. H. & Rhee, J.-Y. Cluster of Coronavirus Disease Associated with Fitness Dance Classes, South Korea. *Emerg Infect Dis* **26**, 1917–1920 (2020).

Introduction

Computer: viruses/malware



Wang, P., González, M. C., Hidalgo, C. A. & Barabási, A.-L. Understanding the Spreading Patterns of Mobile Phone Viruses. Science 324, 1071–1076 (2009).

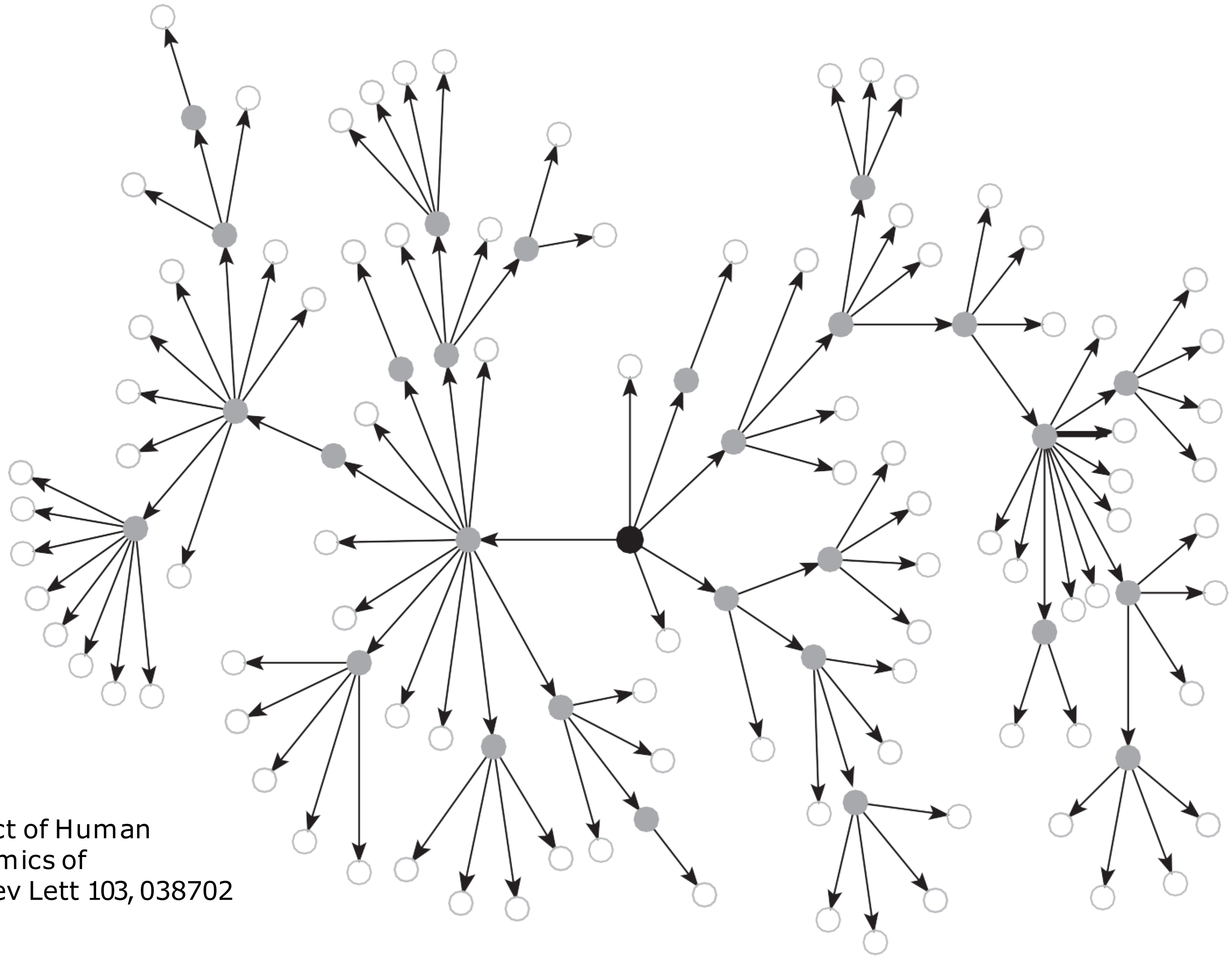
Introduction

Information: fake news

Vosoughi, S., Roy, D. & Aral, S. The spread of true and false news online. Science 359, 1146–1151 (2018).

Introduction

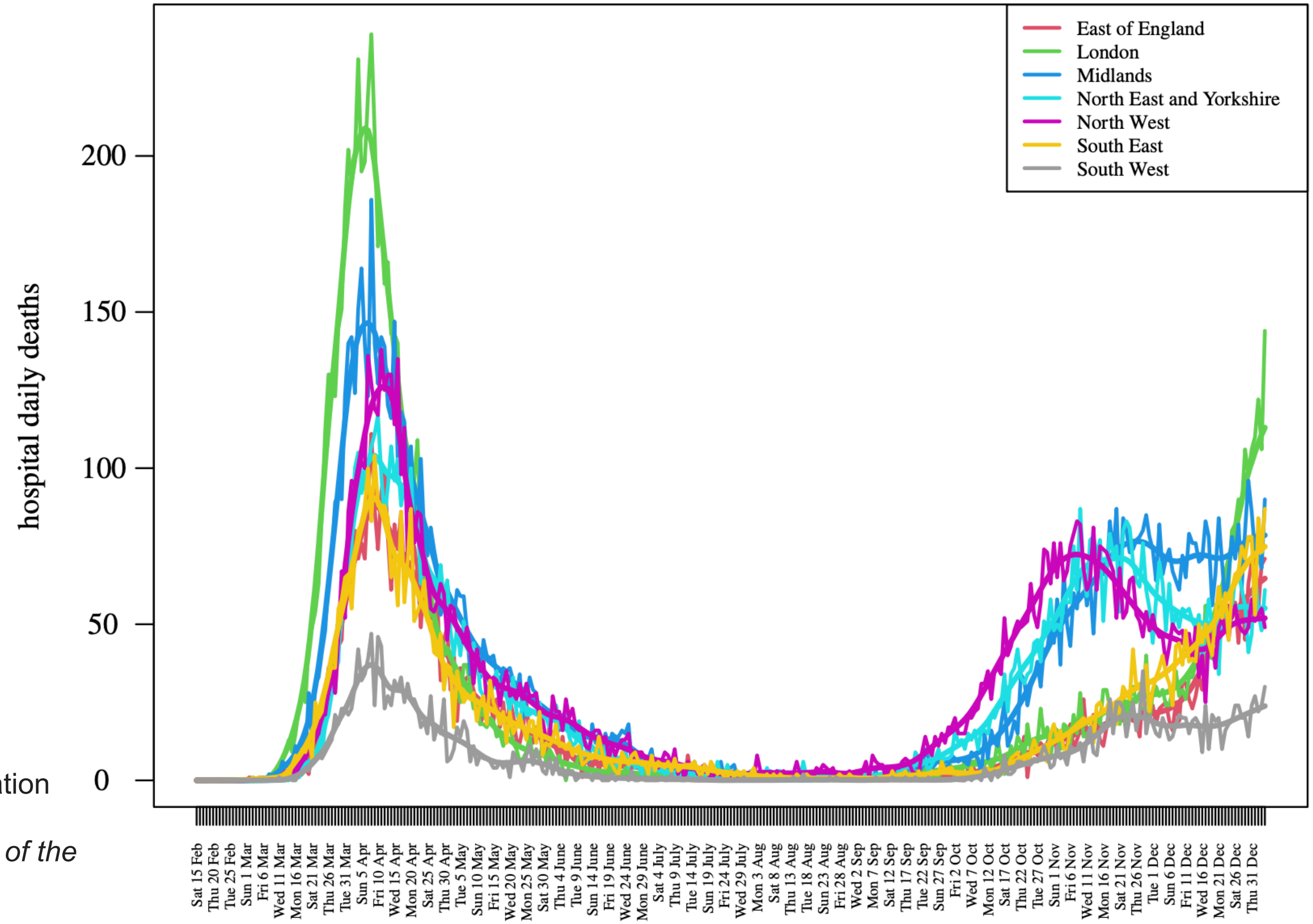
Information:
Viral marketing



Iribarren, J. L. & Moro, E. Impact of Human Activity Patterns on the Dynamics of Information Diffusion. Phys Rev Lett 103, 038702 (2009).

Introduction

Diseases: Viruses and epidemics



Vernon, Ian, et al. "Bayesian emulation and history matching of JUNE." *Philosophical Transactions of the Royal Society A* 380.2233 (2022): 20220039.

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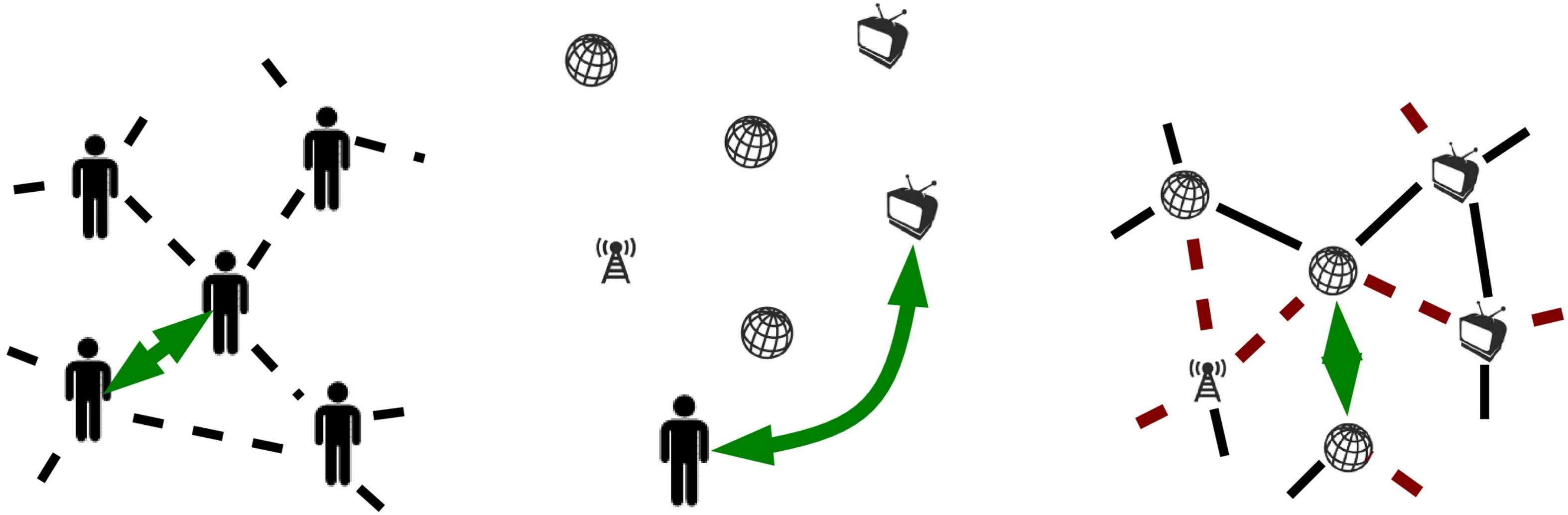
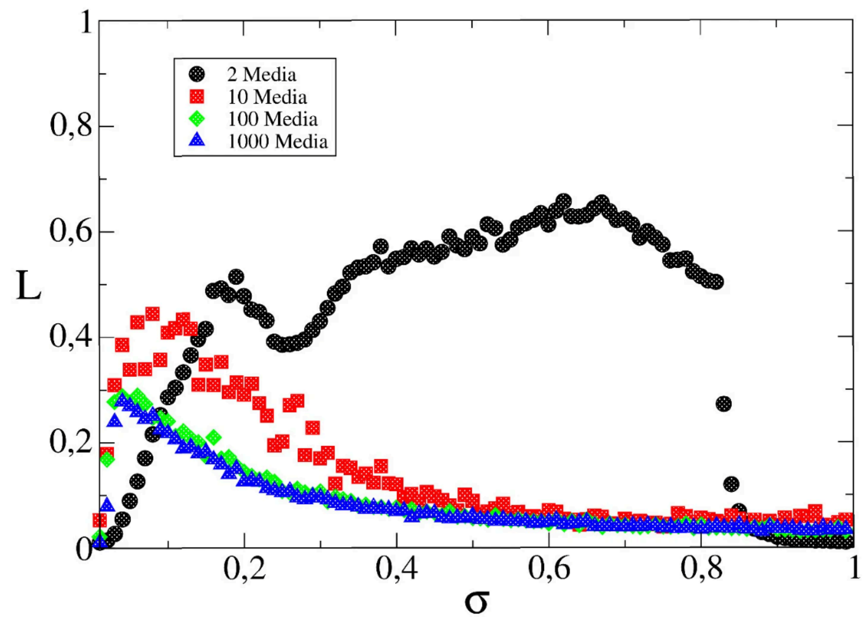
Introduction

Behavior: Opinion dynamics





Quattrociocchi, Walter, Guido Caldarelli, and Antonio Scala. "Opinion dynamics on interacting networks: media competition and social influence." *Scientific reports* 4.1 (2014): 4938.

2 | Models of dynamics on networks

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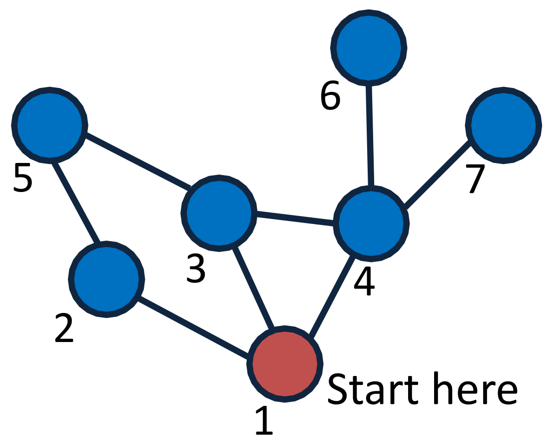
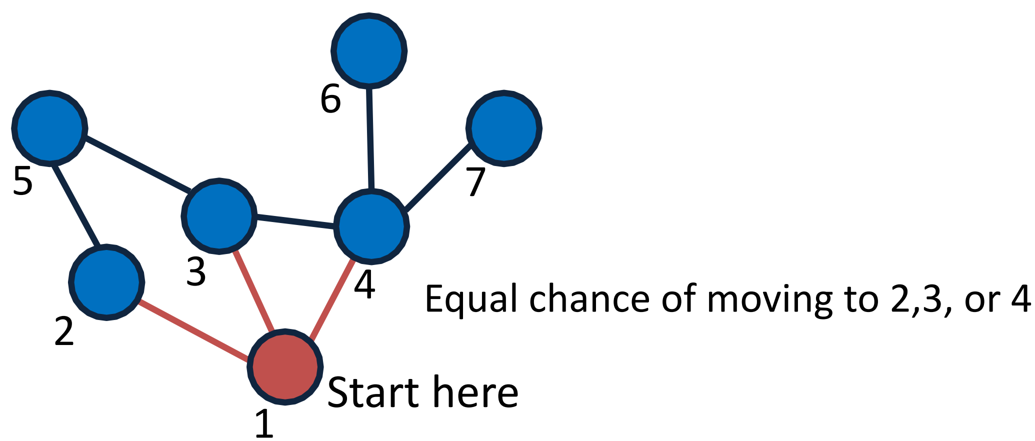
Dynamics on network

What's the simplest way to move in a network?



Dynamics on network

What's the simplest way to move in a network?
at random.

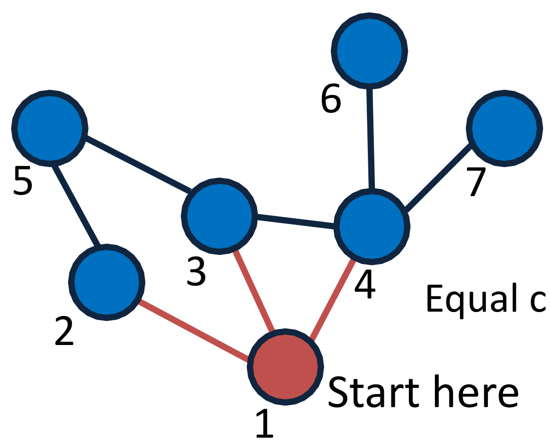


Dynamics on network

What's the simplest way to move in a network?
at random.

Let's keep a small counter

$$N(1) = \begin{cases} 1 : 1 \\ 2 : 0 \\ 3 : 0 \\ 4 : 0 \\ 5 : 0 \\ 6 : 0 \\ 7 : 0 \end{cases}$$

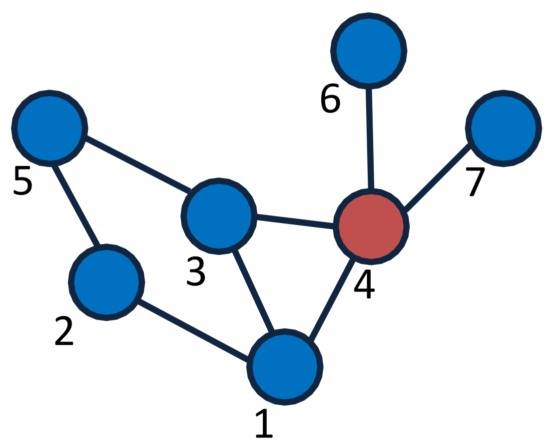


The random walker

What's the simplest way to move in a network?
at random.

Let's keep a small counter

$$N(2) = \begin{cases} 1 : 1 \\ 2 : 0 \\ 3 : 0 \\ 4 : 1 \\ 5 : 0 \\ 6 : 0 \\ 7 : 0 \end{cases}$$



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The random walker

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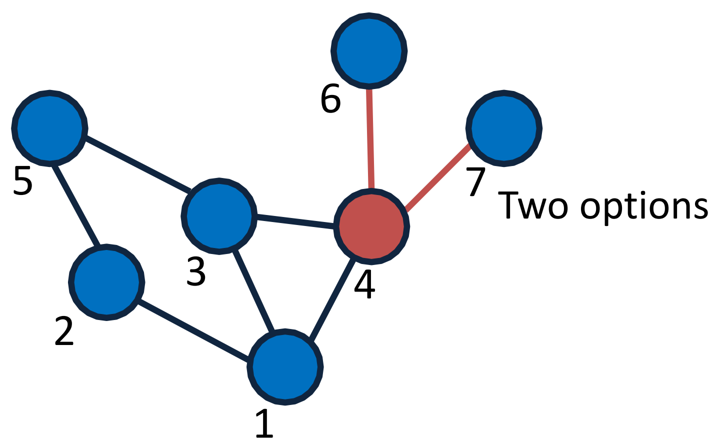

The random walker

What's the simplest way to move in a network?
at random.

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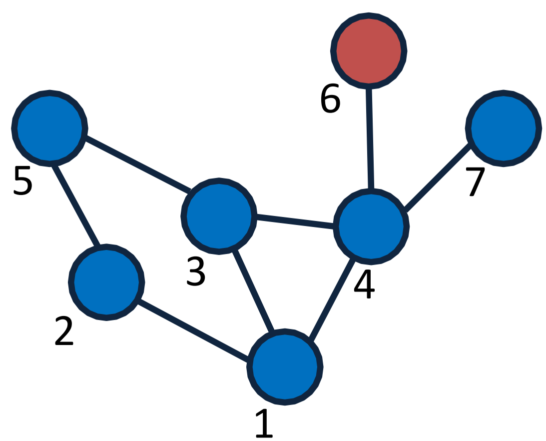
$$N(3) = \begin{cases} 1 : 1 \\ 2 : 0 \\ 3 : 0 \\ 4 : 1 \\ 5 : 0 \\ 6 : 1 \\ 7 : 0 \end{cases}$$

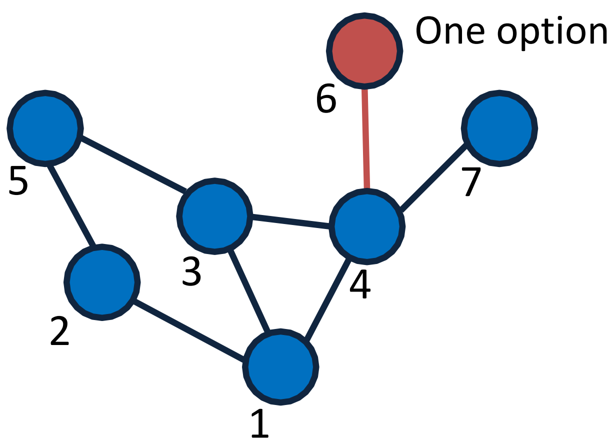


The random walker

What’s the simplest way to move in a network?
at random.

Let’s keep a small counter

$$N(3) = \begin{cases} 1:1 \\ 2:0 \\ 3:0 \\ 4:1 \\ 5:0 \\ 6:1 \\ 7:0 \end{cases}$$

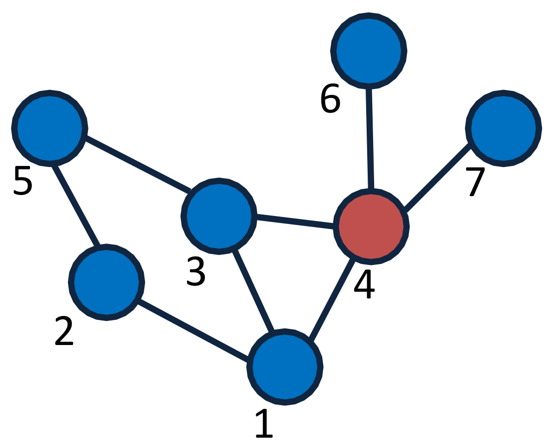


The random walker

What's the simplest way to move in a network?
at random.

Let's keep a small counter

$$N(4) = \begin{cases} 1 : 1 \\ 2 : 0 \\ 3 : 0 \\ 4 : 2 \\ 5 : 0 \\ 6 : 1 \\ 7 : 0 \end{cases}$$



The random walker

What's the simplest way to move in a network?
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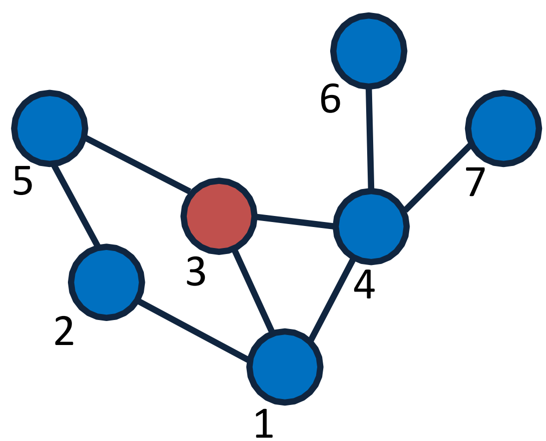
Let's keep a small counter

$$N(5) = \begin{cases} 1 : 1 \\ 2 : 0 \\ 3 : 1 \\ 4 : 2 \\ 5 : 0 \\ 6 : 1 \\ 7 : 0 \end{cases}$$

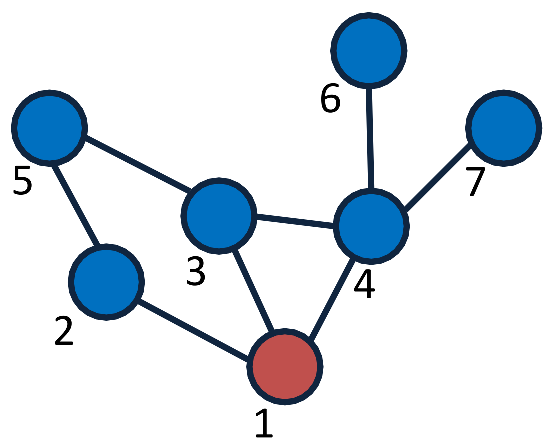


The random walker

What's the simplest way to move in a network?
at random.

Let's keep a small counter

$$N(6) = \begin{cases} 1:2 \\ 2:0 \\ 3:1 \\ 4:2 \\ 5:0 \\ 6:1 \\ 7:0 \end{cases}$$



The random walker

What's the simplest way to move in a network?
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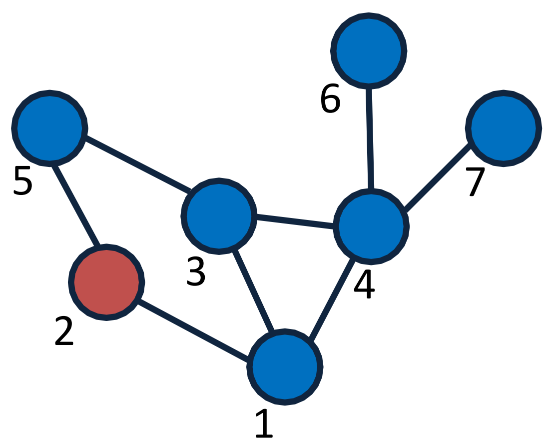
Let's keep a small counter

$$N(7) = \begin{cases} 1 : 2 \\ 2 : 1 \\ 3 : 1 \\ 4 : 2 \\ 5 : 0 \\ 6 : 1 \\ 7 : 0 \end{cases}$$



The random walker

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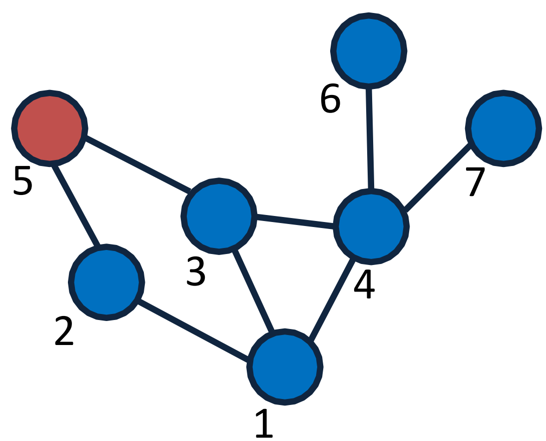
Let's keep a small counter

$$N(8) = \begin{cases} 1 : 2 \\ 2 : 1 \\ 3 : 1 \\ 4 : 2 \\ 5 : 1 \\ 6 : 1 \\ 7 : 0 \end{cases}$$



The random walker

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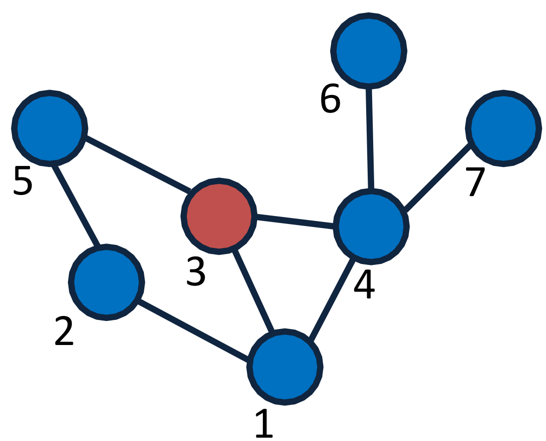
Let's keep a small counter

$$N(9) = \begin{cases} 1 : 2 \\ 2 : 1 \\ 3 : 2 \\ 4 : 2 \\ 5 : 1 \\ 6 : 1 \\ 7 : 0 \end{cases}$$



The random walker

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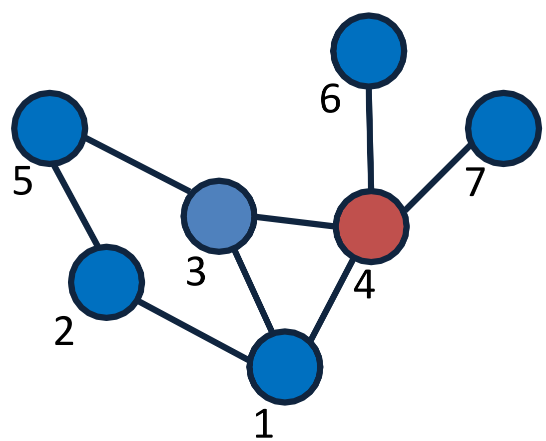
Let's keep a small counter

$$N(10) = \begin{cases} 1 : 2 \\ 2 : 1 \\ 3 : 2 \\ 4 : 3 \\ 5 : 1 \\ 6 : 1 \\ 7 : 0 \end{cases}$$



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The random walker

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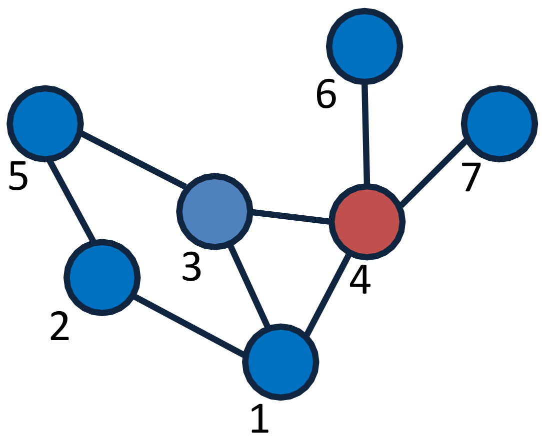
$$N(10) = \begin{cases} 1 : 2 \\ 2 : 1 \\ 3 : 2 \\ 4 : 3 \\ 5 : 1 \\ 6 : 1 \\ 7 : 0 \end{cases}$$



Less connected nodes receive fewer visits

The random walker

Properties:

- **equilibrium**: if undirected, the walker reaches a stationary distribution. As T grows, $N(T)_i \propto k_i$

$$p_i = \frac{k_i}{2L}$$

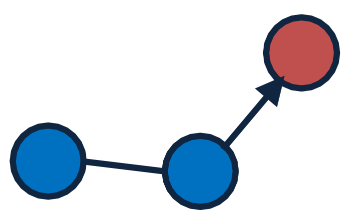
Probability of finding the walker at node i

Number of edges

- **mean return time**: average time to go from node i back to itself.

$$\langle T_{i \to i} \rangle = \frac{1}{p_i}$$

- **sinks**: if directed, the walker may get trapped because of ingoing edges with no outgoing ones



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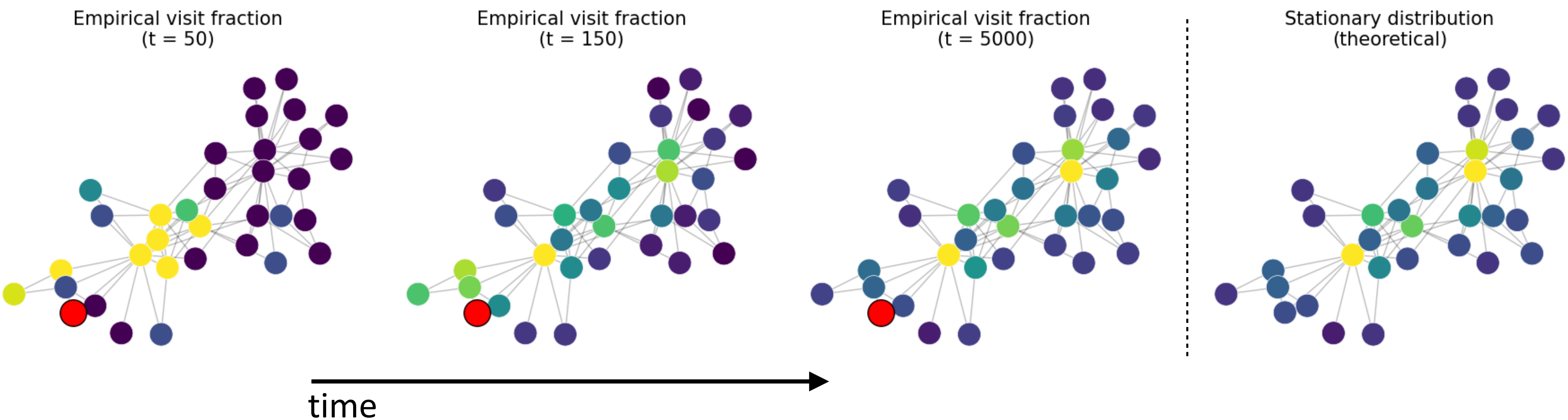
The random walker

- **Timescales**:
  - Short walks reveal *local properties* of the network (initial node memory).
  - Longer walks show *global* features under equilibrium.



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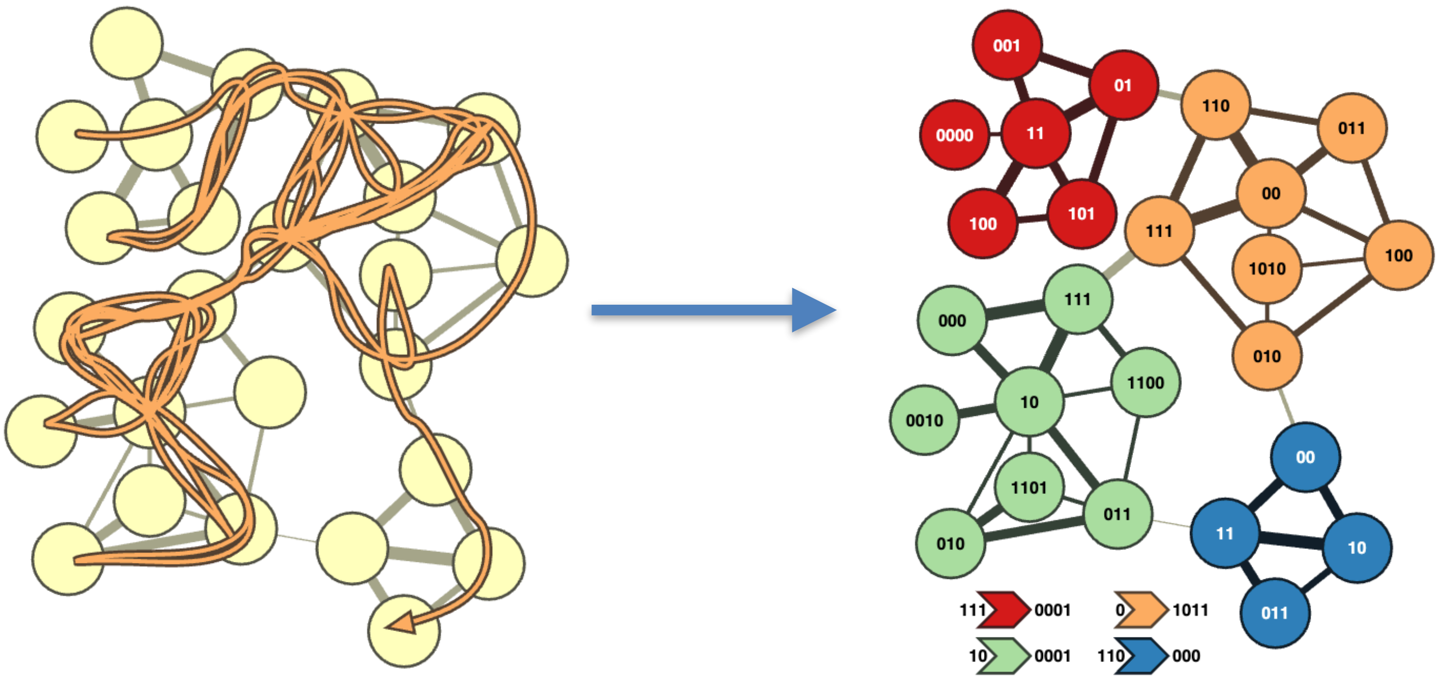
The random walker

Some applications

- **Community detection:** Communities are regions where random walkers get trapped (*Infomap*).



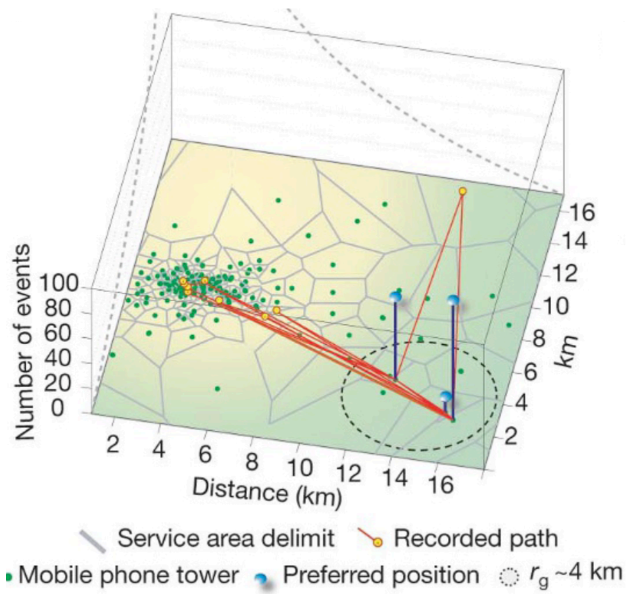
Rosvall, Martin, and Carl T. Bergstrom. "Maps of random walks on complex networks reveal community structure." Proceedings of the national academy of sciences 105.4 (2008): 1118-1123.

The random walker

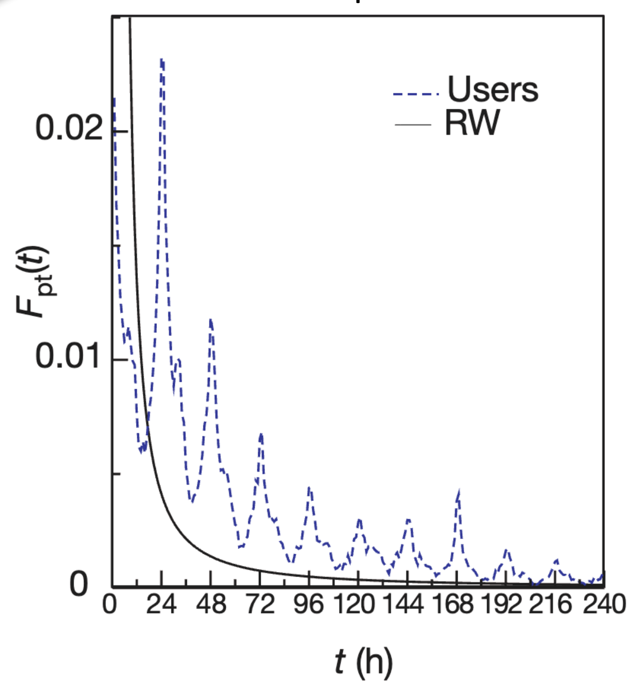
Some applications

- **Human mobility:** Null model for human mobility patterns

human trajectories



return-time probabilities



Gonzalez, M. C., Hidalgo, C. A., & Barabasi, A. L. (2008). Understanding individual human mobility patterns. nature, 453(7196), 779-782.

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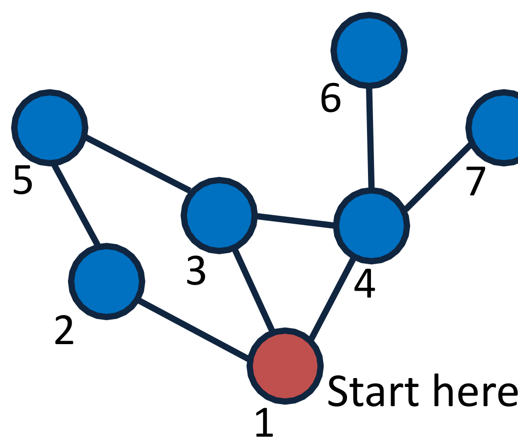
Independent Cascade Model

Instead of *moving*, what if the point now is *spreading?*

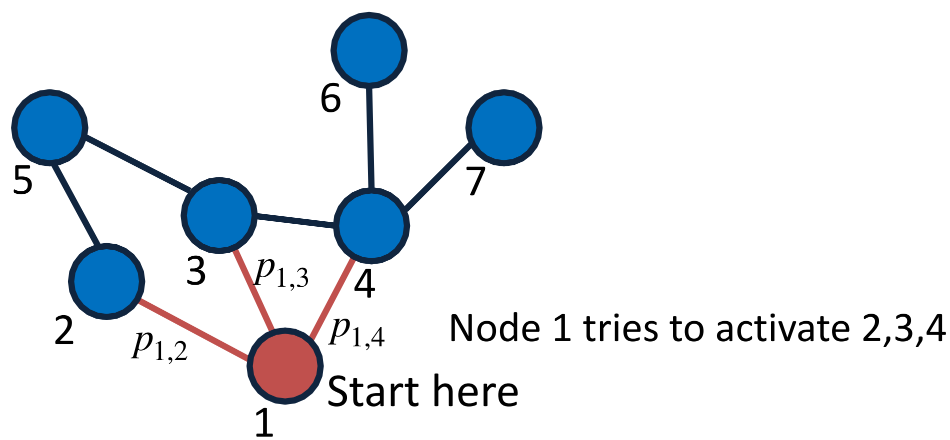


Independent Cascade Model

Instead of *moving*, what if the point now is *spreading?*

Model:

- p_{ij}: probability that node i reaches/activates neighbor j
- When activated, a node has a single chance to activate its neighbors
- The process continues until no one else is activated



Kempe, David, Jon Kleinberg, and Éva Tardos. "Maximizing the spread of influence through a social network." *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*. 2003.

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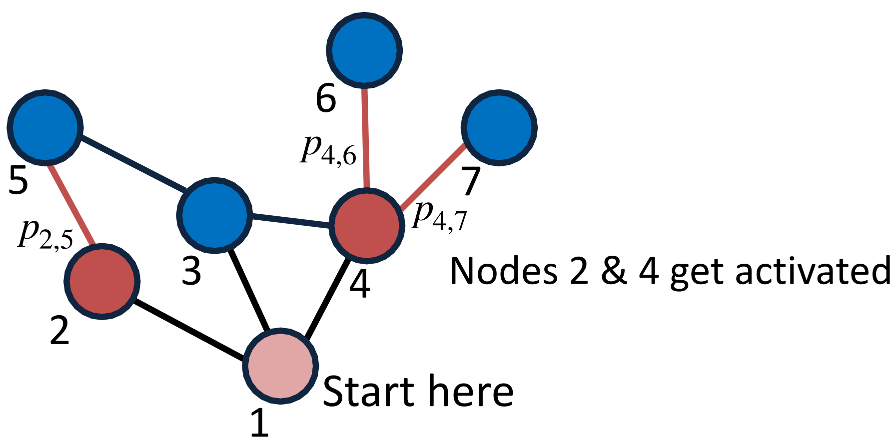

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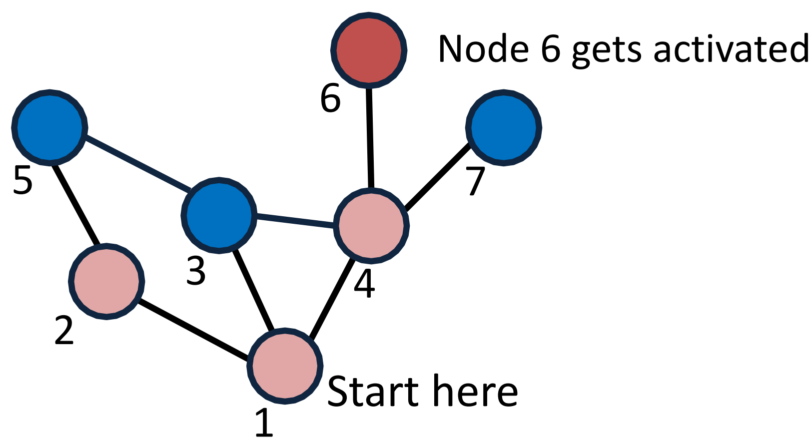

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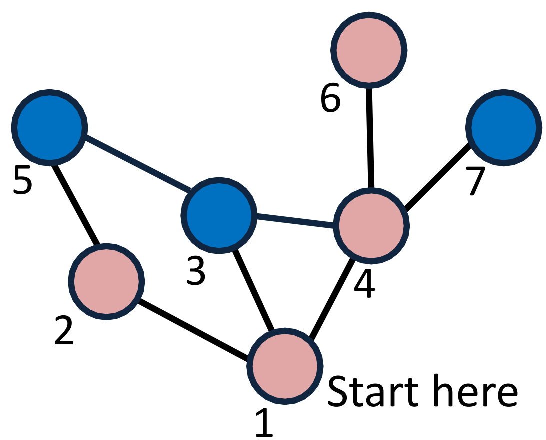

Cascade
length = 3 steps
size = 4 nodes

No new activations → cascade ends

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Kempe, David, Jon Kleinberg, and Éva Tardos. "Maximizing the spread of influence through a social network." *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*. 2003.

Independent Cascade Model

Properties:

- **indicators**: cascade size (# activated nodes),
  cascade length (# steps involved)

- **reproduction number**: at initial stages, approximation of number of activations of node i

$$r_i \sim Bin(k_i, p) \qquad \Longrightarrow \qquad R_0 = \langle r_i \rangle_i = p\langle k \rangle$$

\# of activations from i

Expected number of activations
(Reproduction number)

  - $R < 1$ (subcritical): cascades die quickly
  - $R = 1$ (critical): large fluctuations
  - $R > 1$ (supercritical): large cascades

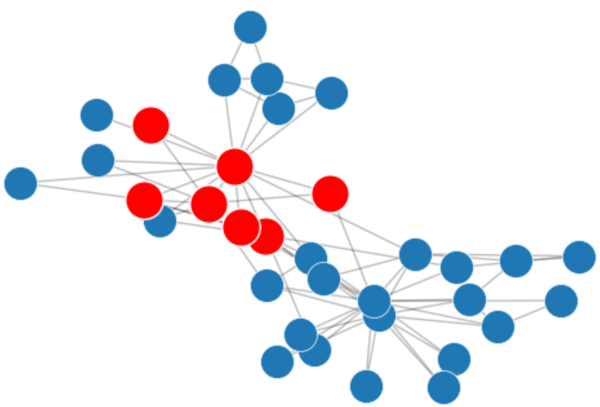
Independent Cascade Model

Reproduction number → critical activation probability

$$p_c = \frac{1}{\langle k \rangle}$$

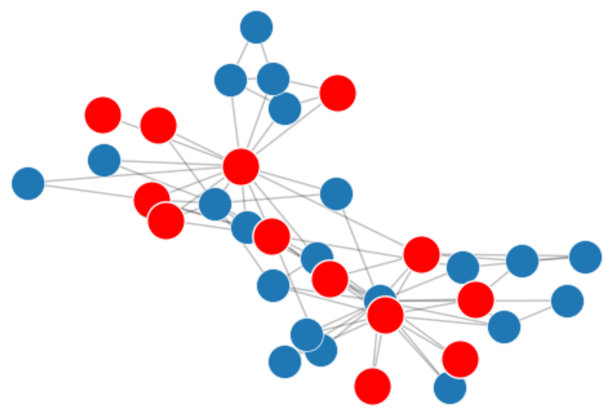
subcritical regime

$p = 0.109$
$(p_c = 0.218)$



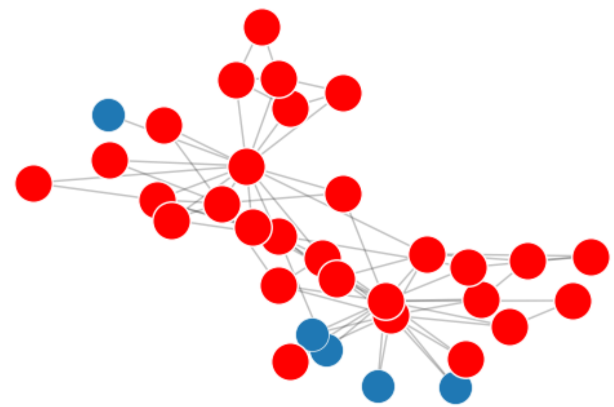
critical regime

$p = 0.218$
$(p_c = 0.218)$



supercritical regime

$p = 0.436$
$(p_c = 0.218)$



Epidemiological models

In a similar fashion, epidemiologists have a long history studying how pathogens spread in a population, and how such population reacts to it over time.

Epidemiological models (SIR)

The idea is classifying the population based on their health state. Typically, nodes can be:

- **Susceptible (S):** Healthy/passive nodes that may be infected
- **Infected (I):** Contagious/active nodes
- **Recovered (R):** Nodes who have recovered develop immunity

Nodes can **transition** between states:

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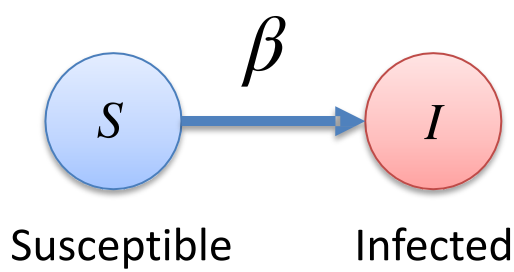
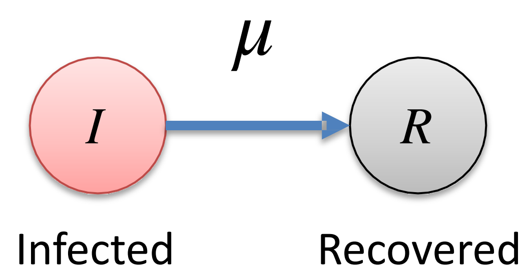
Epidemiological models (SIR)

At each time step, infected nodes infect each of their susceptible neighbors with probability β and recover with μ.

Transition probabilities:

- β: infection probability from infected to susceptible
- μ: recovery probability from an infected node





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Epidemiological models (SIR)

Properties:

- **Reproduction number:** depends on infection AND recovery rates
  - Mixed populations (typical connectivity)

$$R_0 = \frac{\beta}{\mu}\langle k \rangle$$

  - Heterogeneous populations (spread-out degrees)

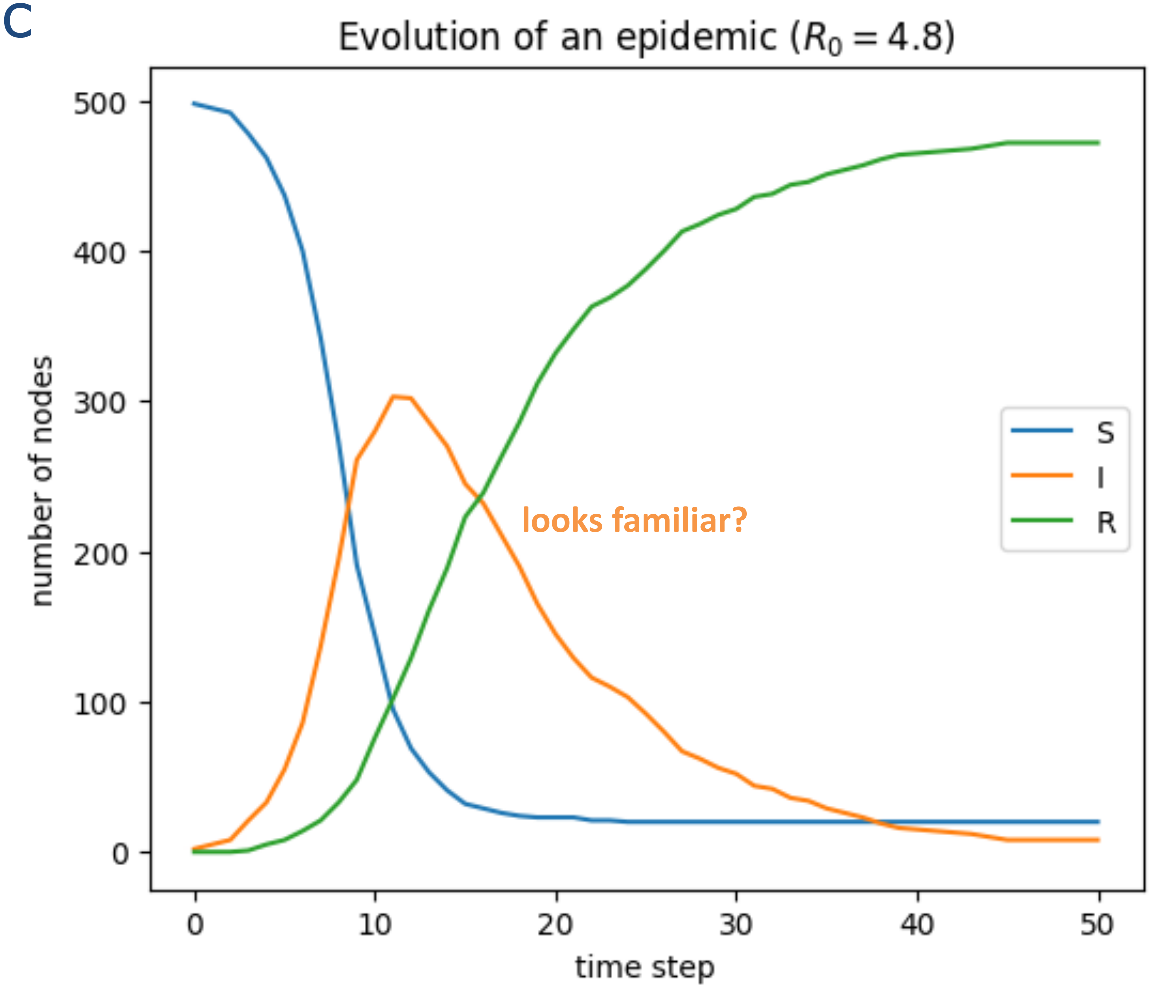
degree variance

$$R_0 = \frac{\beta}{\mu}\frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle}$$

Epidemiological models (SIR)

Reproduction number:

- $R_0 > 1$: exponential growth
- $R_0 < 1$: no epidemic



Critical infectivity:

$$\beta_c = \mu \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$$

Spread-out connectivities

$$\beta_c = \frac{\mu}{\langle k \rangle}$$

well-mixed populations

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Spreading on mixed populations

SIR model:

- R_0 represents the average number of susceptibles infected by an infected during its infectious period in a fully susceptible population
- First thing epidemiologist measure to see how dangerous is the pathogen

Disease	Transmission	R_0
Measles	Airborne	12-18
Pertussis	Airborne droplet	12-17
Diptheria	Saliva	6-7
Smallpox	Social contact	5-7
Polio	Fecal-oral route	5-7
Rubella	Airborne droplet	5-7
Mumps	Airborne droplet	4-7
HIV/AIDS	Sexual contact	2-5
SARS	Airborne droplet	2-5
Influenza (1918 strain)	Airborne droplet	2-3

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4 | Minimizing/Maximizing spreading

Immunization

Who and how much do we have to vaccinate to stop the spreading for a given β, μ, and p_k.

- **Random vaccination**: equivalent to change the effective degree from $k \to (1-g)k$.

$$\frac{\beta}{\mu}(1-g_c) \approx \frac{\langle k \rangle}{\langle k^2 \rangle} \Longrightarrow g_c \approx 1 - \frac{\mu}{\beta}\frac{\langle k \rangle}{\langle k^2 \rangle} \approx 1$$

- Since $\langle k^2 \rangle \gg \langle k \rangle$ in real networks, then we need to vaccinate randomly a lot of nodes to stop the spreading.

How to design strategies?

The idea is identifying what effect nodes with certain characteristics have on network dynamics when removing/promoting them.

We will analyse two types of strategies:

- **Immunization**: to minimize spread
- **Seeding**: for censoring or maximizing spread

We'll take the SIR model as reference.

Immunization

Assume we *vaccinate* a node so it stops spreading.

- **Randomly vaccinating** a proportion g of the population is equivalent to reducing the degree

$$k \to k(1-g)$$

Immunization

Assume we *vaccinate* a node so it stops spreading.

- **Randomly vaccinating** a proportion g of the population is equivalent to reducing the degree

$$k \to k(1-g)$$

- This reduces the basic reproduction number:

$$R_0 \approx \frac{\beta}{\mu}\frac{\langle k^2\rangle(1-g)}{\langle k\rangle} \quad \underbrace{\Longrightarrow}_{R_0=1} \quad \boxed{g_c \approx 1 - \frac{\mu}{\beta}\frac{\langle k\rangle}{\langle k^2\rangle}}$$

Critical proportion to stop epidemic

- In empirical networks $\langle k^2\rangle \gg \langle k\rangle$, so $g_c \approx 1$

Most population would have to be vaccinated!

Immunization

Assume we *vaccinate* a node so it stops spreading.

- **Targeted vaccination:** instead, aim for the highest degree nodes to largely reduce variance $\langle k^2 \rangle$.

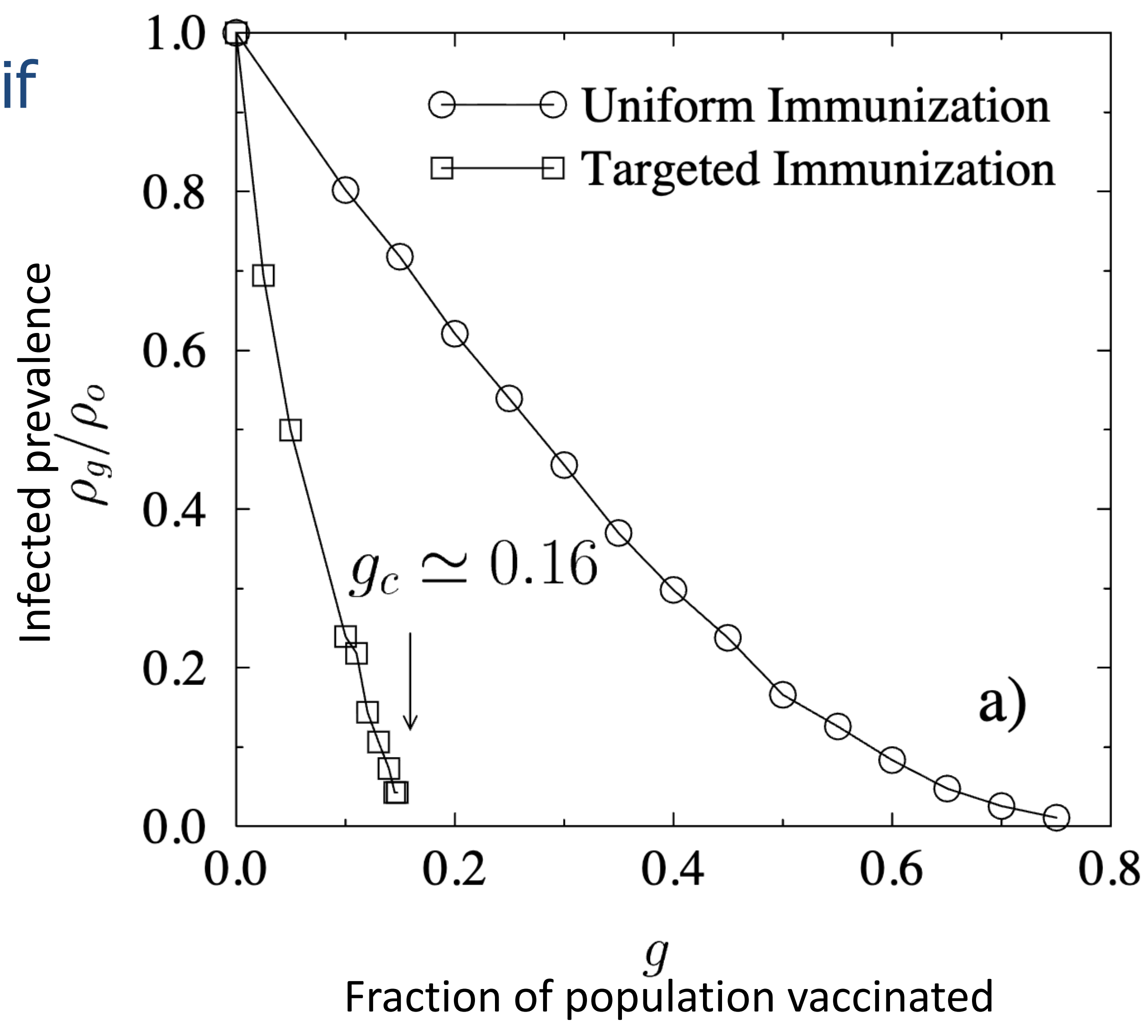
$$\boxed{g_c \approx 1 - \frac{\mu}{\beta} \frac{\langle k \rangle_{k \leq k_M}}{\langle k^2 \rangle_{k \leq k_M}}} \not\approx 1$$

Critical proportion to stop epidemic

$\langle k \rangle_{k \leq k_M}$: average degree when ignoring degrees above k_M.

Immunization

Infection decays way faster if vaccination is targeted.



Pastor-Satorras, Romualdo, and Alessandro Vespignani. "Immunization of complex networks." Physical review E 65.3 (2002): 036104.

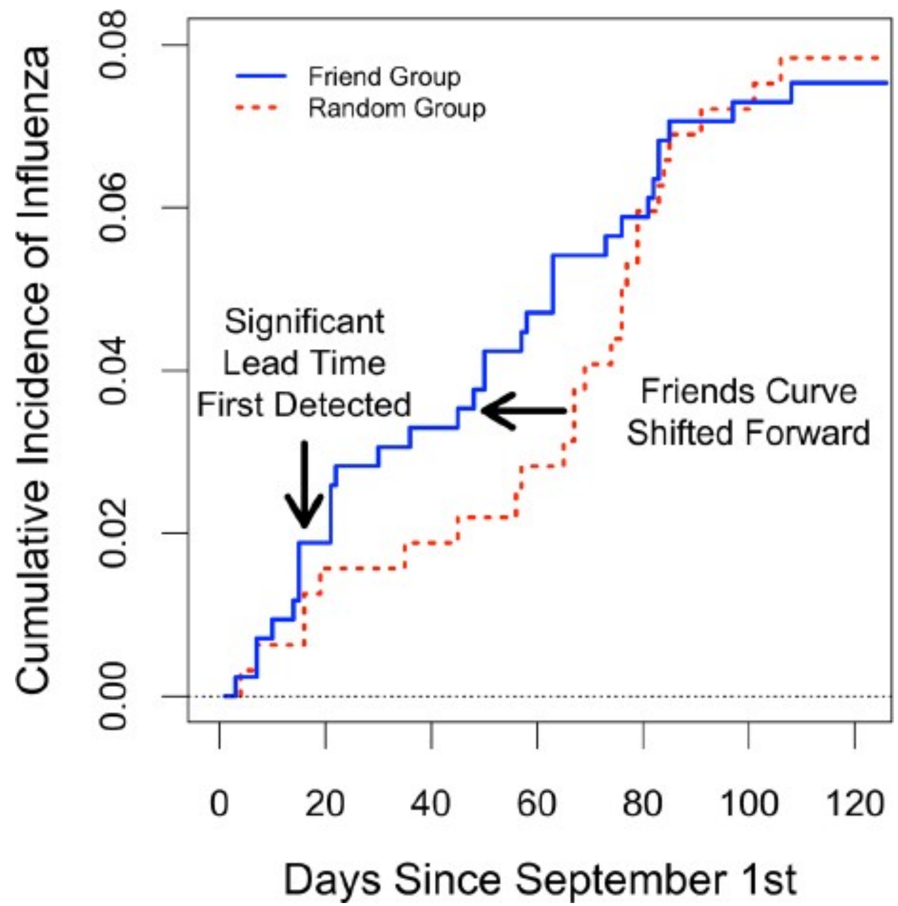
Sensors

On the other hand, nodes with a higher degree get infected faster.

We can use high-degree nodes as early sensors of the epidemic.

friendship paradox → select high-degree nodes

e



N. A. Christakis & J. H. Fowler (2010) PLoS ONE 5(9) e12948

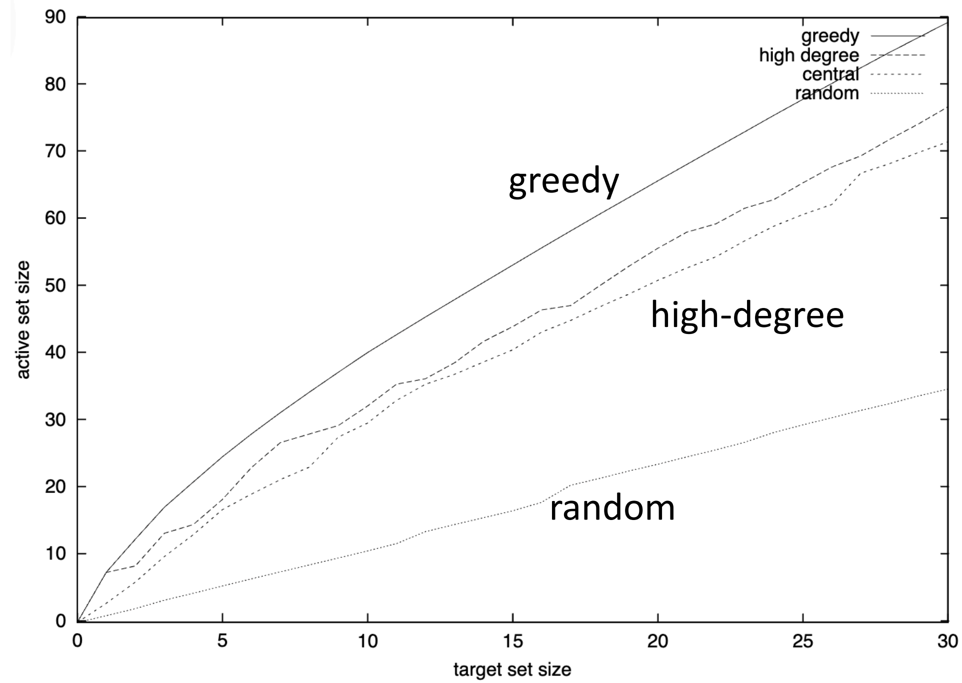
uc3m

Seeding

Similarly, important nodes will likely influence the network faster and wider.

High-degree nodes tend to be good, but

- No high-centrality of other types
- Connected only to other high-degree nodes (assortativity)



Kempe, David, Jon Kleinberg, and Éva Tardos. "Maximizing the spread of influence through a social network." *Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*. 2003.

4 | Complex contagion

Complex contagion

So far, we have explored models of *simple contagion:* nodes move/spread just by getting in contact with their neighbors.

Complex contagion models consider more intricate kind of interactions, such as

1. Bounded rationality
2. Heterogeneous node/agent actions
3. Thresholds for behavior

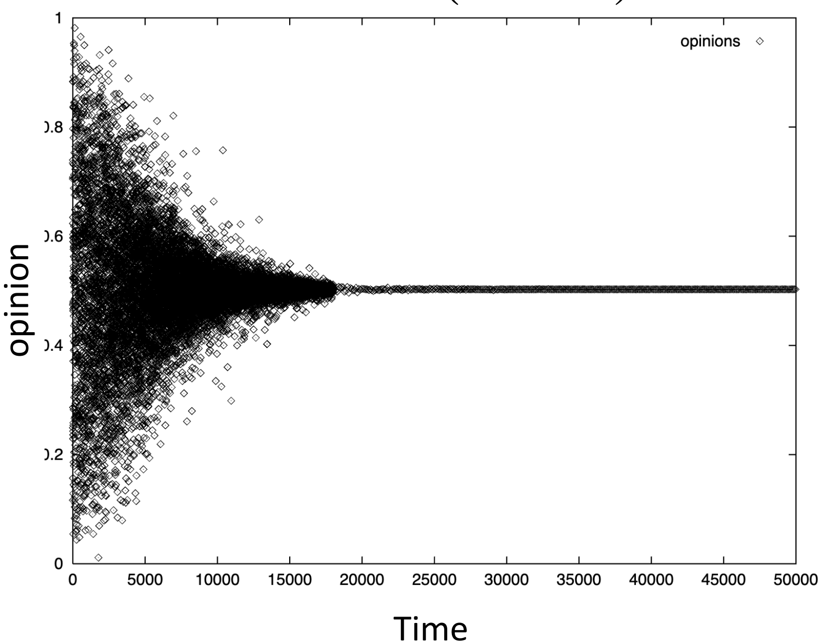
Complex contagion

Example 1: Bounded confidence opinion dynamics

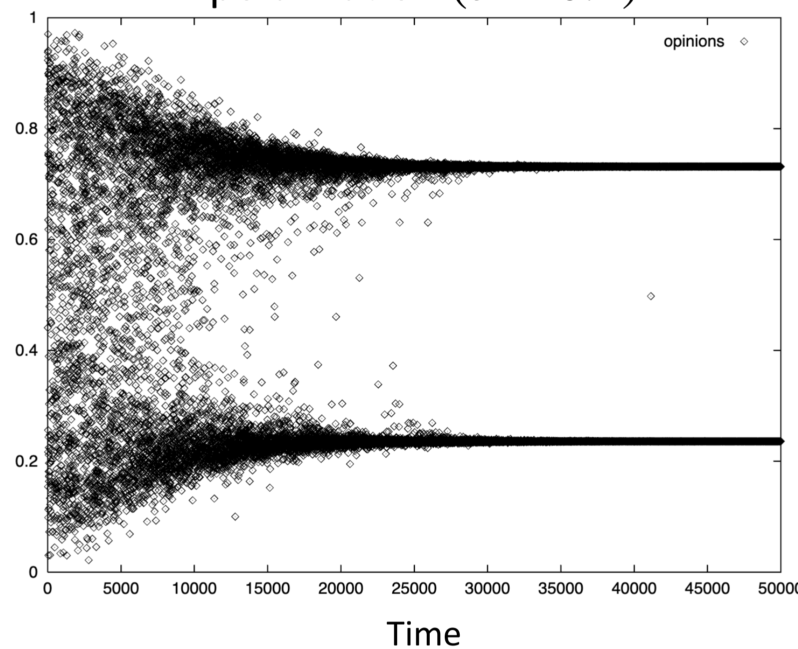
Nodes interact only if their *opinions are sufficiently close*.

- ϵ_i: confidence level of node i

consensus ($\epsilon = 0.5$)



polarization ($\epsilon = 0.2$)



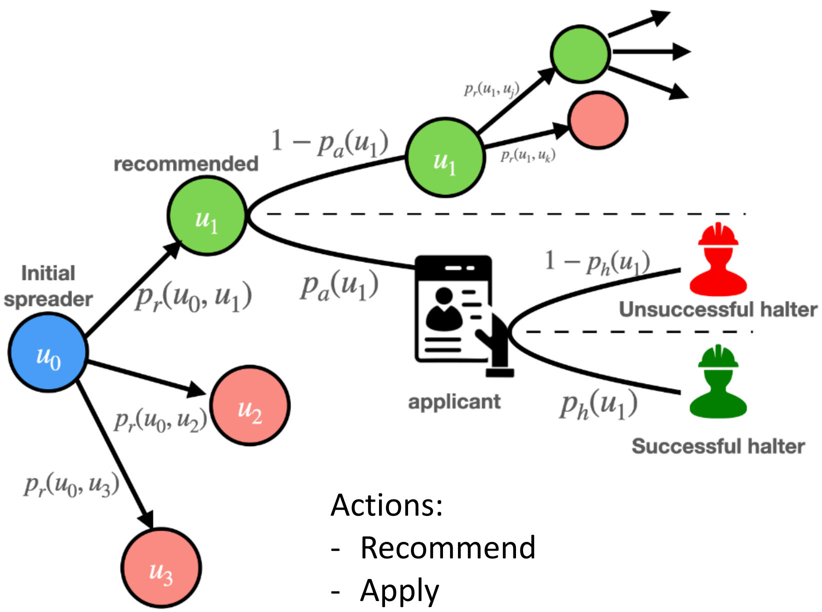
uc3m

Deffuant, Guillaume, et al. "Mixing beliefs among interacting agents." Advances in Complex Systems 3.01n04 (2000): 87-98.

Complex contagion

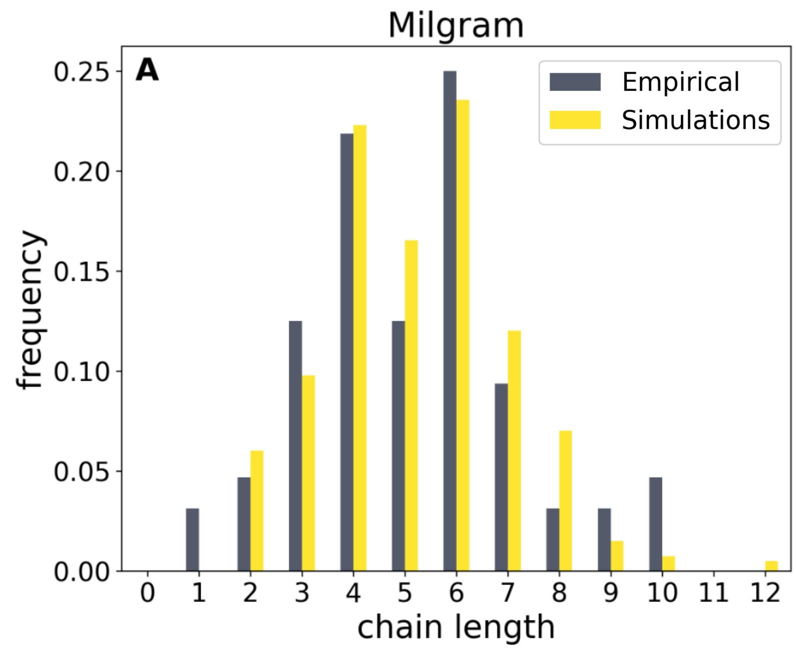
Example 2: Independent Halting Cascades

Nodes may perform *several actions* to complete a task.



Actions:

- Recommend
- Apply
- Halt



Travers and Milgram US letters experiment

uc3m

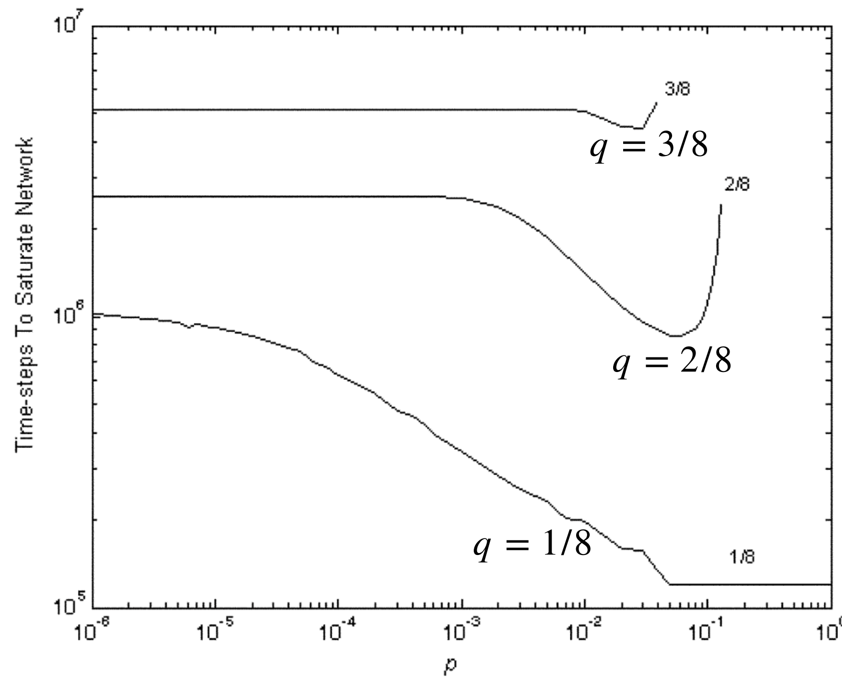
Kolic, Blas, et al. "Incentivized network dynamics in digital job recruitment." Journal of Complex Networks 13.6 (2025): cnaf047.

Complex contagion

Example 3: Thresholded diffusion

Contagion/infections happens only when enough neighbors are infected.

- q : fraction of infected neighbors to get infected
- The larger the q, the slower the propagation



Centola, D., & Macy, M. (2007). Complex contagions and the weakness of long ties. American journal of Sociology, 113(3), 702-734.

5 Summary

Summary

1. Dynamics occur *on top* of network structures.
2. Simple models capture key spreading mechanisms
   - **i.** Random walks → exploration and visitation patterns
   - **ii.** Independent cascade → information spreading
   - **iii.** SIR model → epidemic spreading
3. Dynamics can be summarised through key indicators
   - **i.** Stationary distributions and return times
   - **ii.** Cascade length and size
   - **iii.** Reproduction numbers and critical thresholds
4. Network structure can be exploited to control spreading
5. Complex contagion models exist and are useful.