Equivariance II: Steerable CNNs

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University of Amsterdam, AMLab

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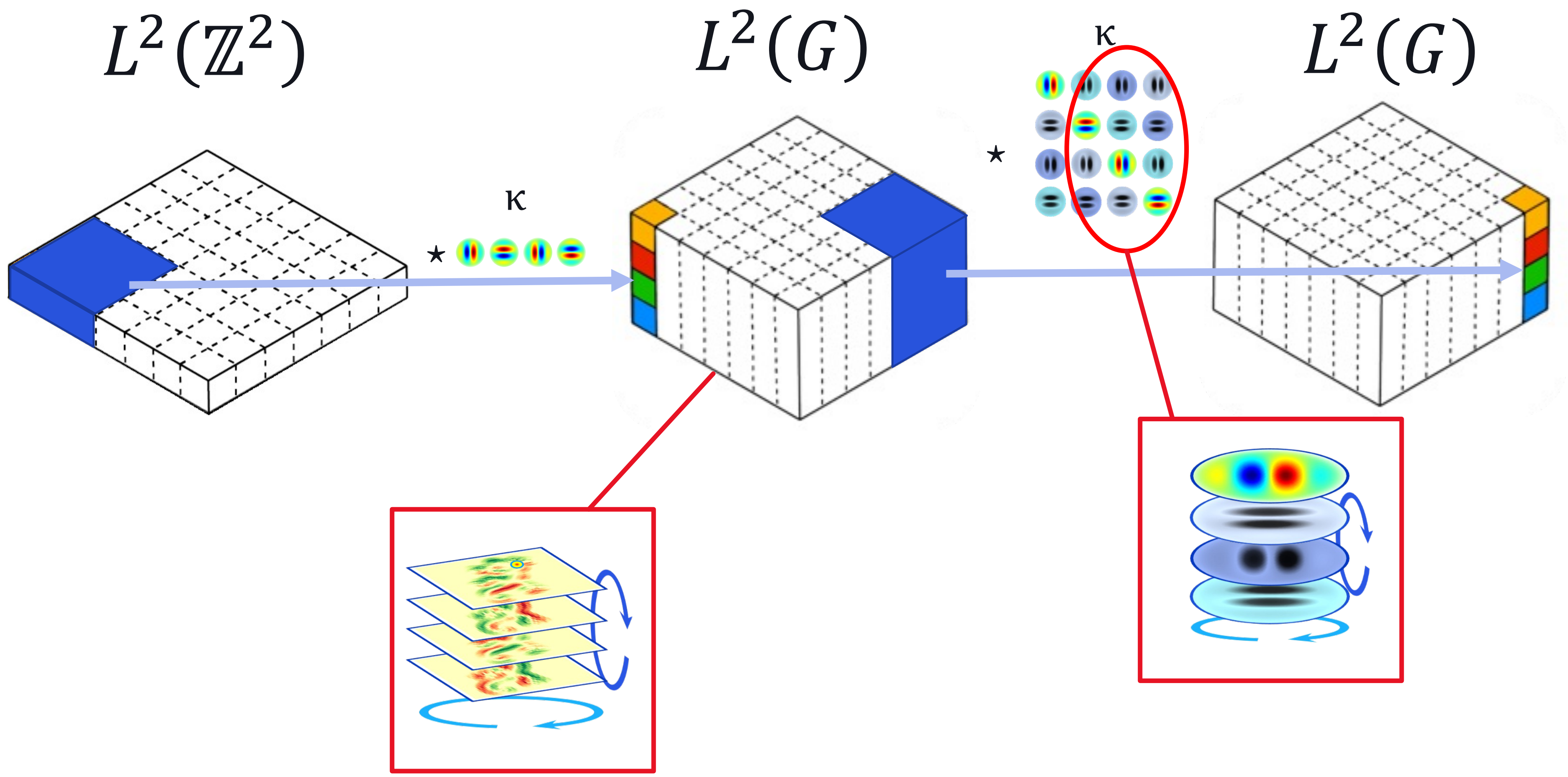
*Qualcomm AI Research is an initiative of Qualcomm Technologies, Inc.

Brief Recap: Lifting Convolution and Group Convolution



Brief Recap: Group Convolution is all you need

We have seen that Group Convolution provides an **equivariant linear map**

$$[\kappa \star \cdot]: L^2(X) \to L^2(G)$$

Group-CNNs are *universal approximators* for equivariant functions

Siamak Ravanbakhsh. Universal Equivariant Multilayer Perceptrons. *International Conference on Machine Learning (ICML), 2020.*

Haggai Maron, Ethan Fetaya, Nimrod Segol, and Yaron Lipman. On the Universality of Invariant Networks. *International Conference on Machine Learning (ICML), 2019*

Wataru Kumagai and Akiyoshi Sannai. Universal Approximation Theorem for Equivariant Maps by Group CNNs, *2020*

Sho Sonoda, Isao Ishikawa, and Masahiro Ikeda. Universality of group convolutional neural networks based on ridgelet analysis on groups. *Advances in Neural Information Processing Systems (NeurIPS), 2022*

Brief Recap: Group Convolution is all you need

We have seen that Group Convolution provides an **equivariant linear map**

$$[\kappa \star \cdot]: L^2(X) \rightarrow L^2(G)$$

Any equivariant linear map of this form is a convolution! (typically, under some minor conditions)

- For any **compact** group G (Kondor and Trivedi, 2018), (Cohen et al., 20120)
- For any **Lie** group (Bekker, 2020)
- Unimodular Lie groups (Aronsson, 2022)

Risi Kondor and Shubhendu Trivedi. On the generalization of equivariance and convolution in neural networks to the action of compact groups. *International Conference on Machine Learning (ICML), 2018.*

Taco S. Cohen, Mario Geiger, and Maurice Weiler. A general theory of equivariant CNNs on homogeneous spaces. *Advances in Neural Information Processing Systems (NeurIPS), 2019*

Erik J Bekkers. B-spline CNNs on Lie groups. *International Conference on Learning Representations (ICLR), 2020.*

Jimmy Aronsson. Homogeneous vector bundles and G-equivariant convolutional neural networks. *Sampling Theory, Signal Processing, and Data Analysis, 2022*

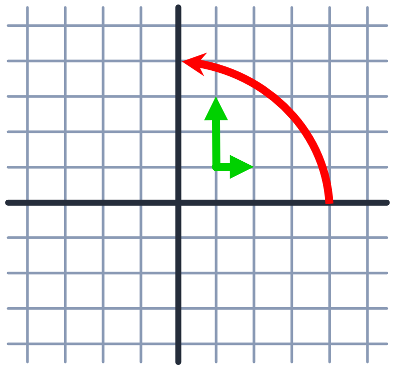
Group Convolution: Challenges

$$[\kappa \star f](g) = \int_{x \in X} \kappa(g^{-1}\boldsymbol{x})\ f(\boldsymbol{x}) d\mu(x)$$

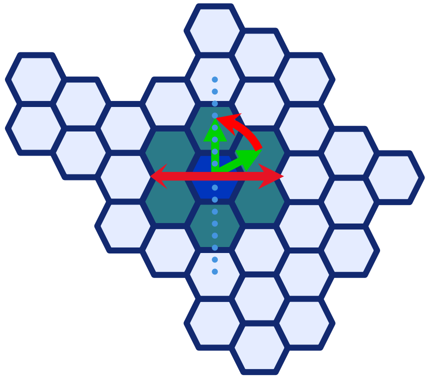
Integrate over X

Transform filter $\kappa \in L^2(X)$
by any group element
$g \in G$

Parameterize
features f
and filters $\kappa \in L^2(X)$



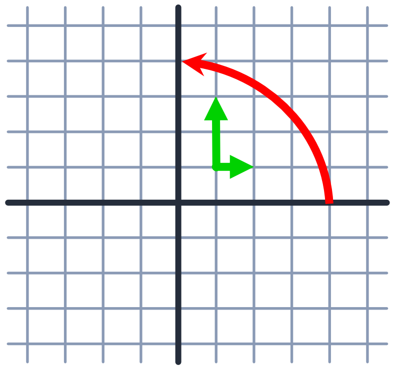
$p4m = (\mathbb{Z}^2, +) \rtimes D_4$



$p6m = (\mathbb{Z}^2, +) \rtimes D_6$

- Often some form of discretization is required, causing artifacts and harming equivariance

Group Convolution: Challenges



$p4m = (\mathbb{Z}^2, +) \rtimes D_4$

$$[\kappa \star f](g) = \int_{x \in X} \kappa(g^{-1}\,\boldsymbol{x})\ f(\boldsymbol{x}) d\mu(x)$$

- Often some form of discretization is required, causing artifacts and harming equivariance
- What if input or output are not in $L^2(\mathbb{R}^n)$ or $L^2(G)$?
  - e.g. vector fields?
- The group G might be infinit or too large to fully enumerate
  - $G = SO(3)$ rotations of a point cloud
  - $n!$ permutations of a set or nodes of a graph

Agenda

Steerable Filters

Steerable Fields and Representation Theory

Group Convolution to Steerable Convolution

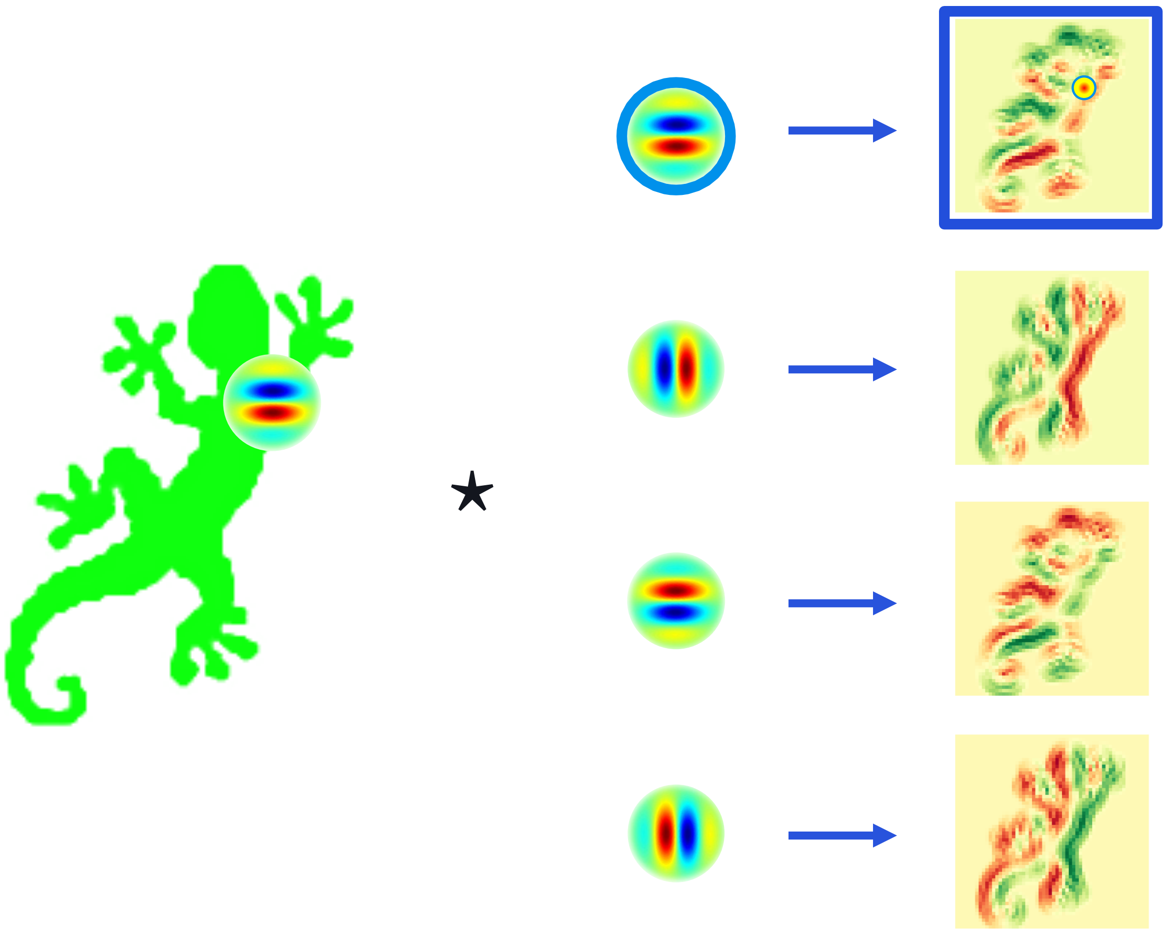
Steerable CNNs

Solving the kernel constraint

Instances and Applications

Group Convolution

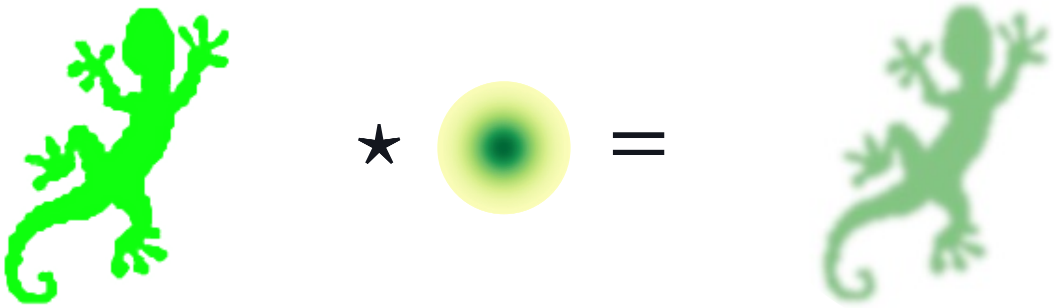
$$[\kappa \star \cdot]: L^2(X) \to L^2(G)$$



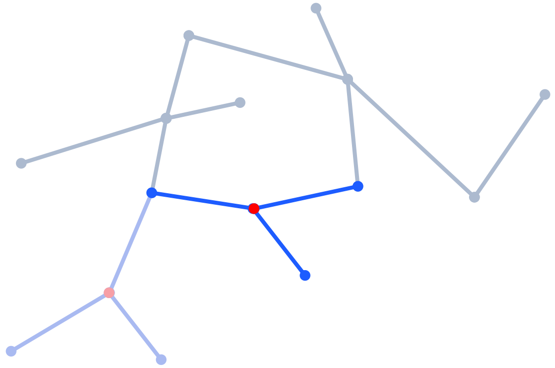
Isotropic Filters

$$[\kappa \star \cdot]: L^2(X) \to L^2(X)$$

- Isotropic filters: the response does not change when rotated



- Not very expressive
- Analogous to Graph Message Passing: no directional dependence!



Steerable Filters

- Filter can be rotated via linear combination

$$\kappa(r, \phi) = w(r)\cos(\phi + \beta)$$

- e.g. filters used for edge-detection in classical Computer Vision

William T. Freeman and Edward H. Adelson. The design and use of steerable filters.
IEEE Transactions on Pattern Analysis & Machine Intelligence, 1991

Steerable Filters

$$
\begin{aligned}
[R_\theta . \kappa](r,\phi) &= \kappa(r, \phi - \theta) \\
&= w(r)\cos(\phi + \beta - \theta) \\
&= w(r)\cos(\phi+\beta)\cos\theta - w(r)\sin(\phi+\beta)\sin\theta \\
&= \cos\theta\, \kappa(r,\phi) - \sin\theta\, \kappa\left(r, \phi - \frac{\pi}{2}\right) \\
&= \cos\theta\, \kappa(r,\phi) - \sin\theta\, R_{\frac{\pi}{2}} . \kappa(r,\phi)
\end{aligned}
$$

- Filter can be rotated via linear combination

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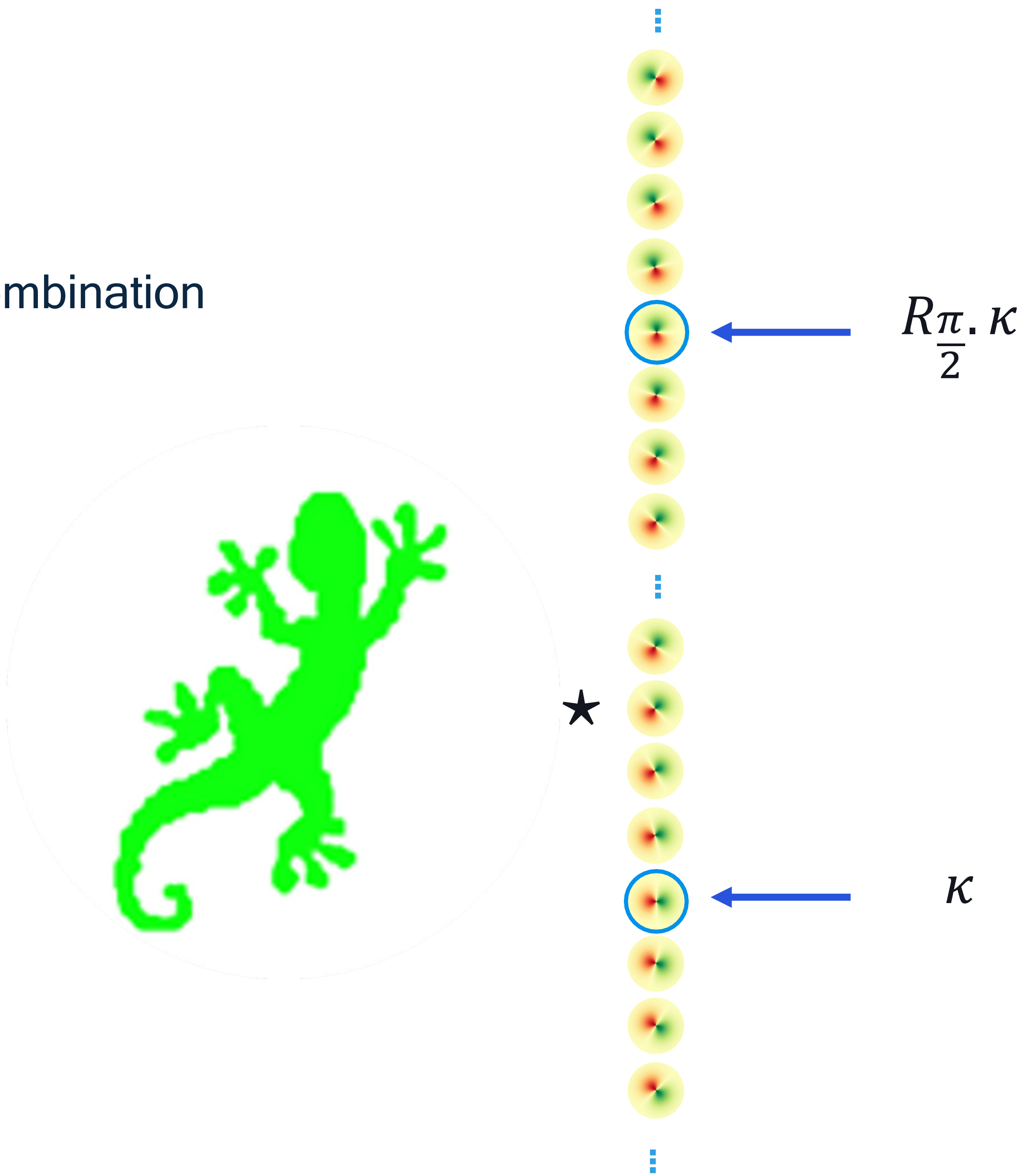


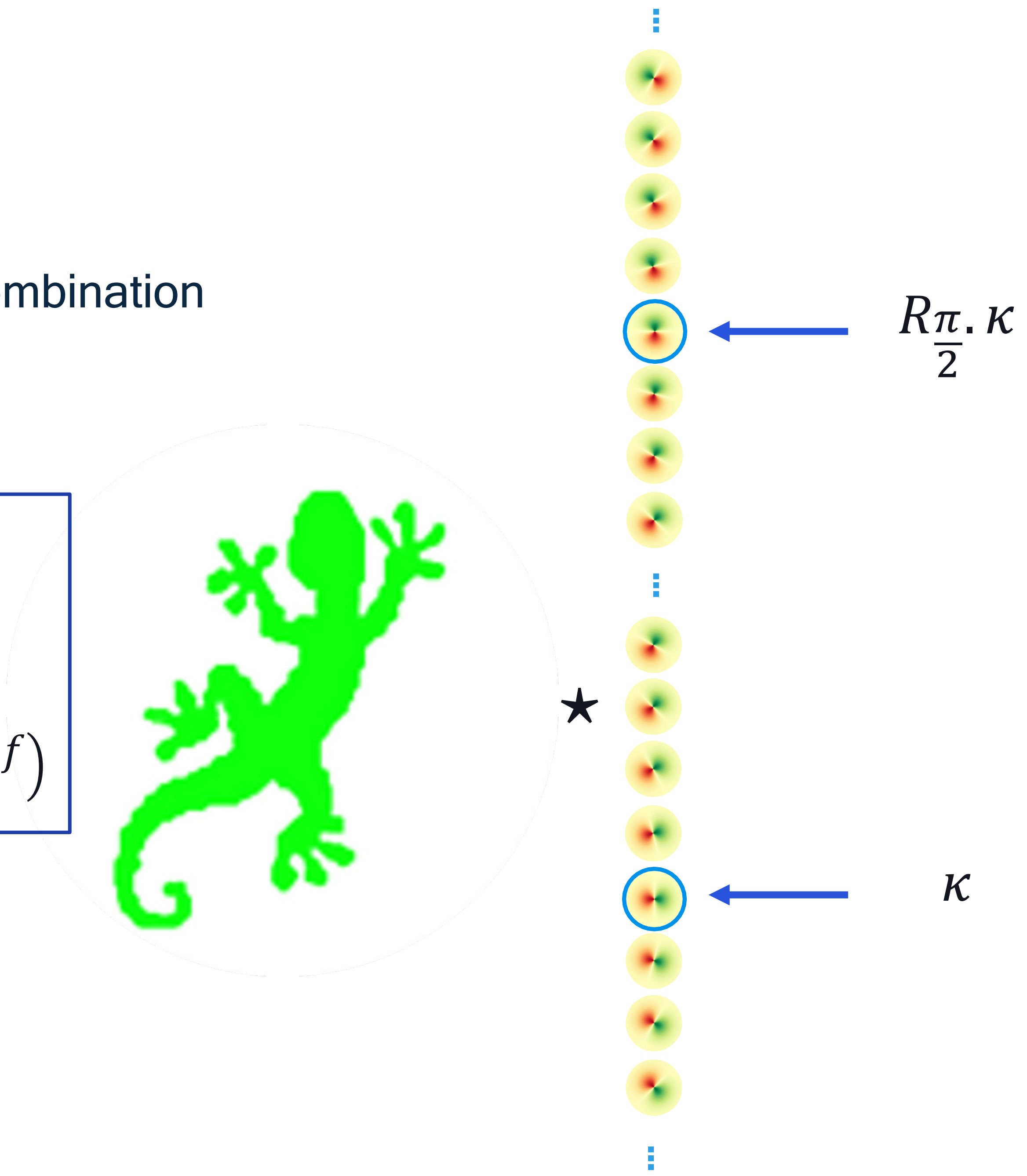
Steerable Filters

- Filter can be rotated via linear combination

Thanks to *linearity* of convolution operator:

$$R_\theta . \kappa \star f = \left(\cos\theta\, \kappa - \sin\theta\, R_{\frac{\pi}{2}} . \kappa\right) \star f$$

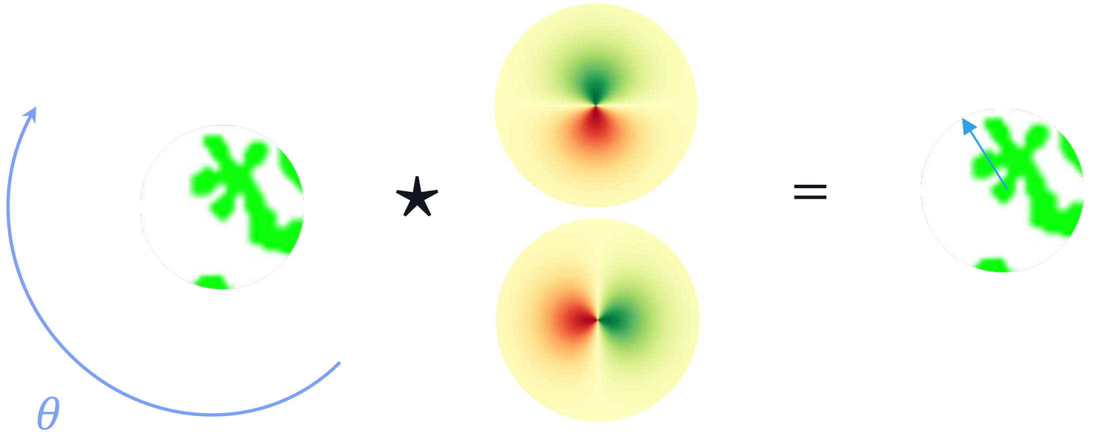
$$= \cos\theta\, (\kappa \star f) - \sin\theta \left(R_{\frac{\pi}{2}} . \kappa \star f\right)$$



Steerable Filters

- Filter can be rotated via linear combination

$$\rho(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\boxed{R_\theta . \kappa \star f = \cos\theta\,(\kappa \star f) - \sin\theta\,\left(R_{\frac{\pi}{2}} . \kappa \star f\right)}$$

Steerable Filters

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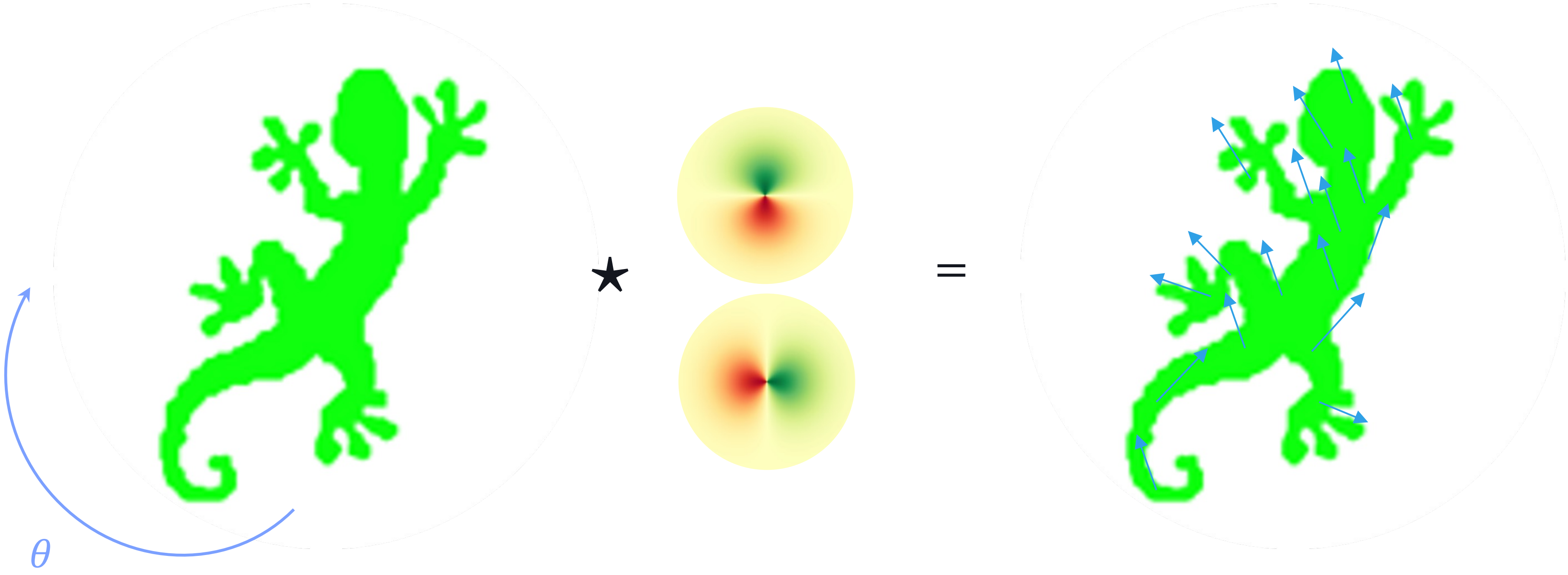
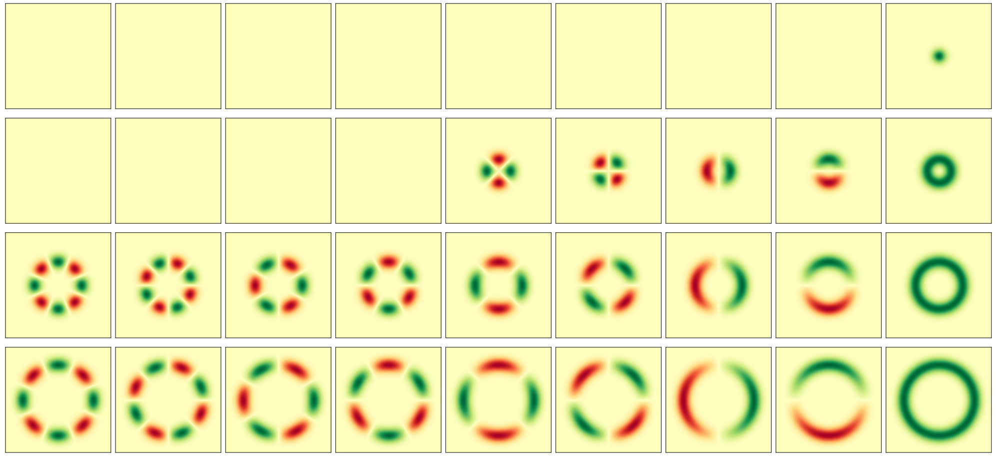


$$\boxed{R_\theta . \kappa \star f = \cos\theta\,(\kappa \star f) - \sin\theta\left(R_{\frac{\pi}{2}} . \kappa \star f\right)}$$

Steerable Filters: Other examples

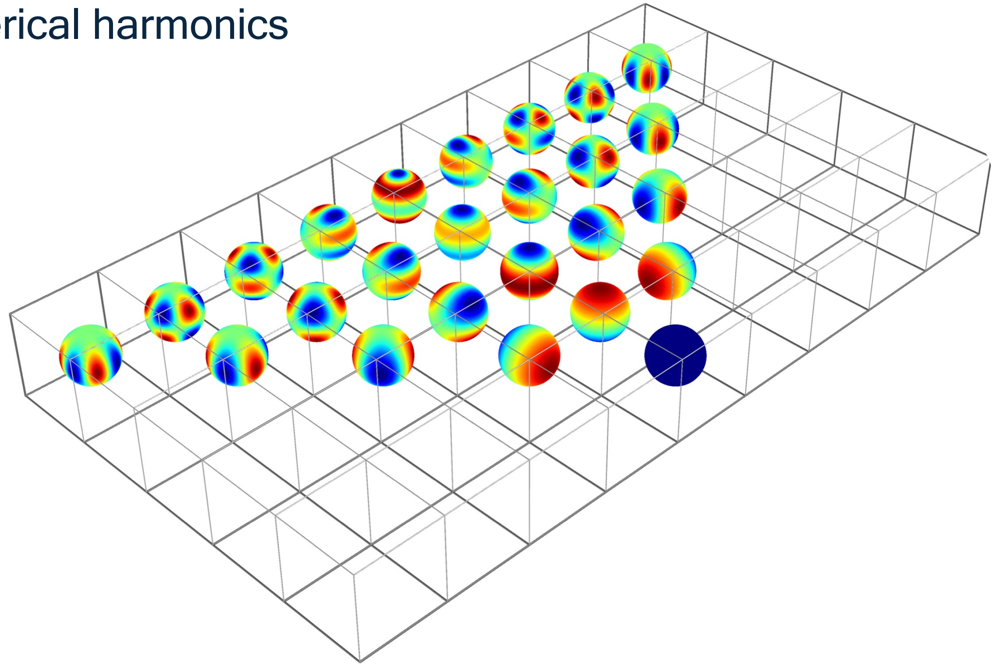
- Filter can be rotated via linear combination

Circular harmonics



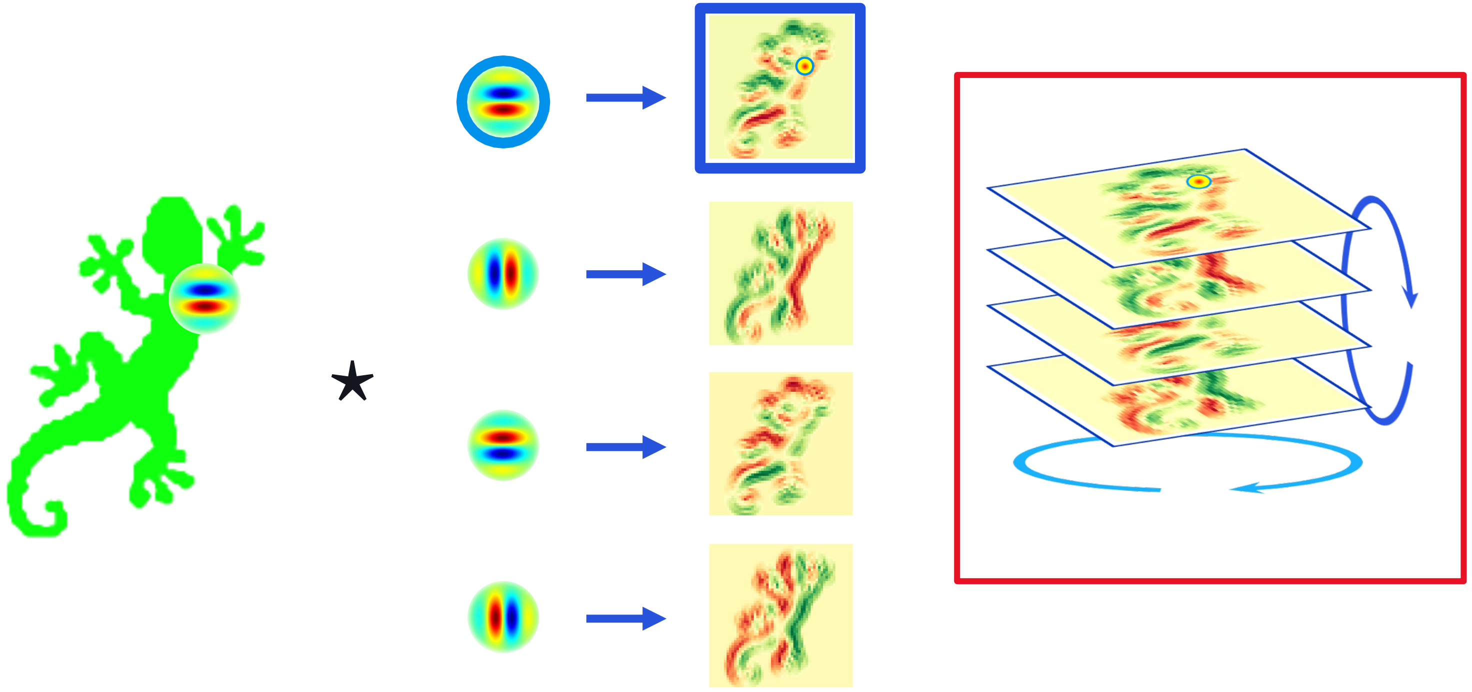
$$\rho_k(\theta) = \begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix}$$

Spherical harmonics



Steerable Filters: Output of Group Convolution

- Filter can be rotated via linear combination



$$\rho(r^0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$$\rho(r^2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

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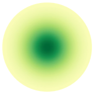
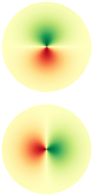
Instances and Applications

Feature Fields

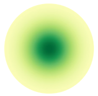
- Interpret features as a multi-channels signal $f: \mathbb{R}^n \to \mathbb{R}^d$

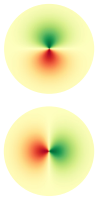
[ $\star$ ]: $\mathbb{R}^2 \to \mathbb{R}^1$

[ $\star$ ]: $\mathbb{R}^2 \to \mathbb{R}^2$

Feature Fields

- Interpret features as a multi-channels signal $f: \mathbb{R}^n \to \mathbb{R}^d$
- Signal transforms under *(point) symmetry group* G according to a *transformation law*
  - symmetry group G : e.g. rotations or reflections
  - N.B.: before we used G to also indicate translations
  - For now, we will implicitly consider equivariance to $(\mathbb{R}^n, +) \rtimes G$

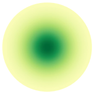
[ $\star$ ]: $\mathbb{R}^2 \to \mathbb{R}^1$

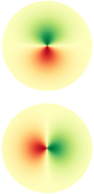
[ $\star$ ]: $\mathbb{R}^2 \to \mathbb{R}^2$

Feature Fields and Steerable CNNs

- Interpret features as a multi-channels signal $f: \mathbb{R}^n \rightarrow \mathbb{R}^d$
- Signal transforms under *(point) symmetry group* G according to a *transformation law*
  - symmetry group G : e.g. rotations or reflections
- Type of transformation identified by a **representation** of G $\quad \rho: G \rightarrow \mathbb{R}^{d \times d}$

$$[g.f](x) = \rho(g) f(g^{-1}x)$$

[ $\star$ ]: $\mathbb{R}^2 \rightarrow \mathbb{R}^1$

[ $\star$ ]: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

Taco S. Cohen and Max Welling, Steerable CNNs,
International Conference on Learning Representations (ICLR), 2017

Feature Fields and Steerable CNNs

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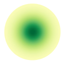
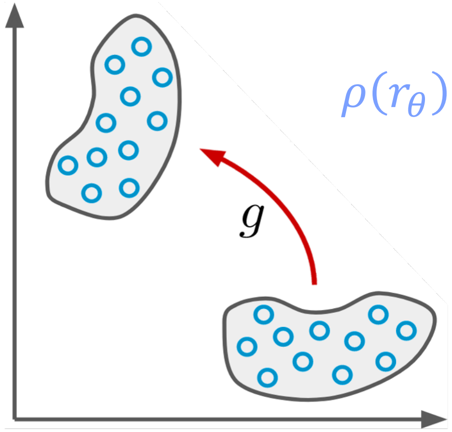
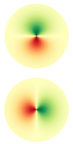
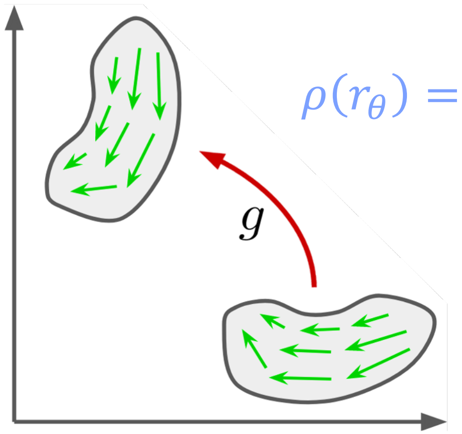
$f = [$  $\star$  $]: \mathbb{R}^2 \rightarrow \mathbb{R}^1$



$\rho(r_\theta) = 1$

$f = [$  $\star$  $]: \mathbb{R}^2 \rightarrow \mathbb{R}^2$



$\rho(r_\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

Source: e2cnn

Definition: **Representation** of a Compact Group

A *(real)* *representation* $\rho: G \to GL(\mathbb{R}^d)$ of a *compact group* G is a map which associates to each element $g \in G$ an *invertible* $d \times d$ matrix s.t.:

- $\forall a, b \quad \rho(a)\rho(b) = \rho(ab)$
- $\forall a \ \ \rho(a)^{-1} = \rho(a^{-1})$
- $\rho(e) = \mathrm{Id}_{d \times d}$

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Can assume w.l.o.g. *orthogonal* representations, i.e. that $\rho(g)^{-1} = \rho(g)^T$

Describes the action of a group G on a *vector space* $V = \mathbb{R}^d$

E.g. representations of the rotation group $SO(2)$

- *Trivial representation* $\rho(r_\theta) = 1$
- *Standard representation* $\rho(r_\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

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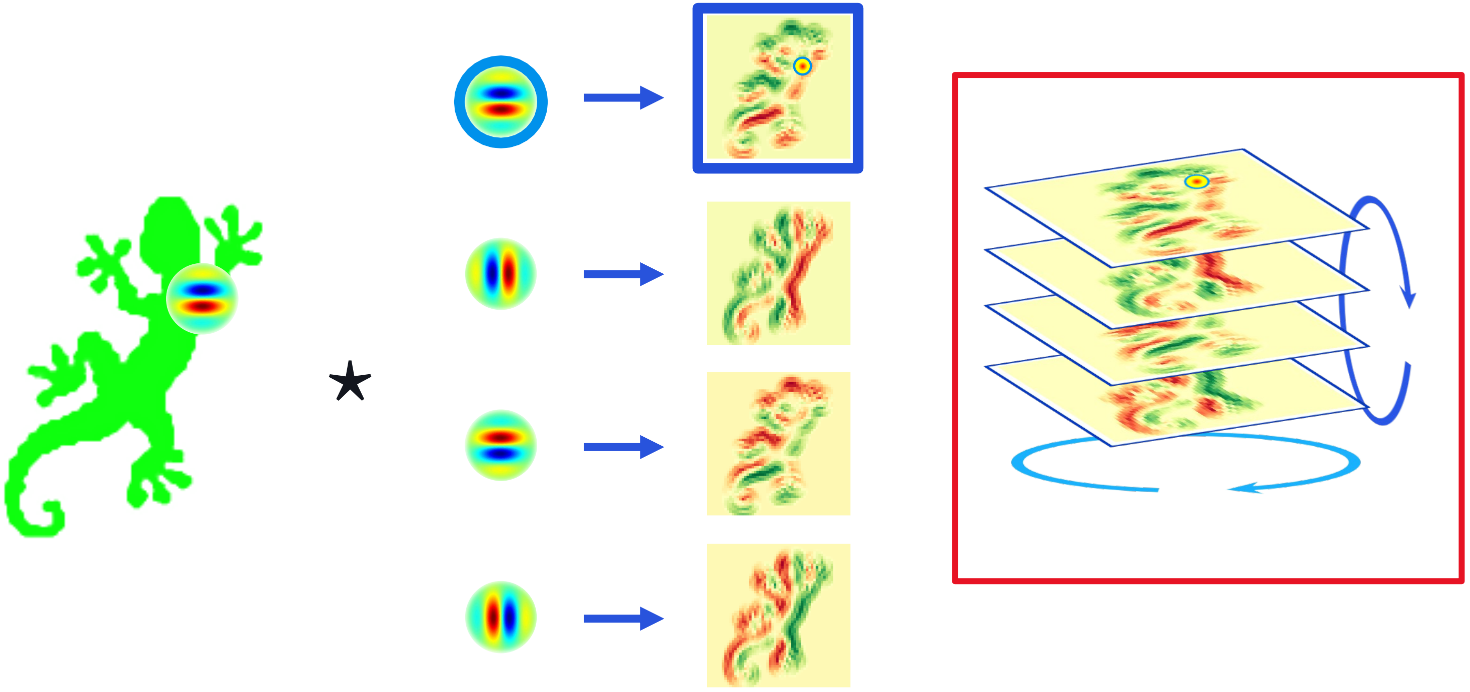
From Group Convolution to Steerable Filters

- Filter can be rotated via linear combination



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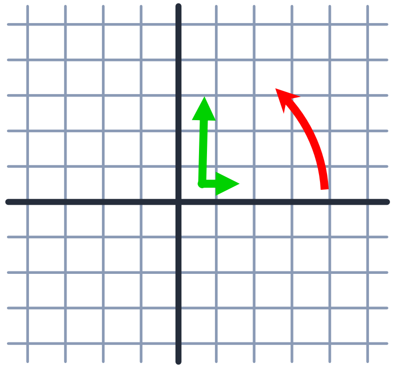
Semi-Direct Product group convolution

Equivariance group $(\mathbb{R}^n, +) \rtimes G$



$SE(2) = (\mathbb{R}^2, +) \rtimes SO(2)$

$$[\kappa \star f](t, g) = \int_{x \in \mathbb{R}^n} \kappa\left(g^{-1}(\boldsymbol{x} - t)\right) f(\boldsymbol{x})\, dx$$

$$[\kappa \star f](t, g) = \int_{x \in \mathbb{R}^n,\, h \in G} \kappa(g^{-1}(\boldsymbol{x} - t), g^{-1}h)\ f(x, h)\, dx$$

Semi-Direct Product group convolution

Equivariance group $(\mathbb{R}^n, +) \rtimes G$

$f: (\mathbb{R}^n, +) \rtimes G \to \mathbb{R}$

$f(t, g) \in \mathbb{R}$

Semi-Direct Product group convolution

Equivariance group $(\mathbb{R}^n, +) \rtimes G$

$f: (\mathbb{R}^n, +) \rtimes G \to \mathbb{R}$

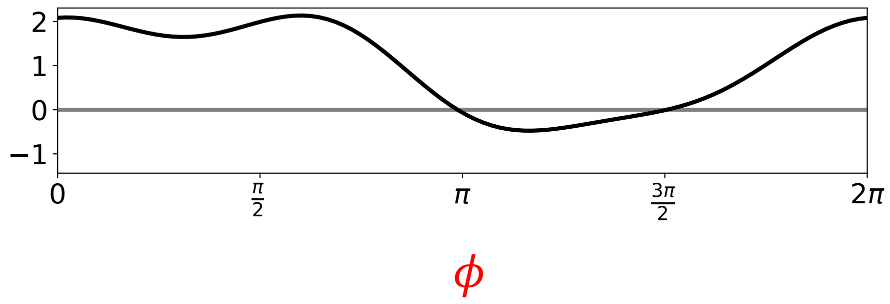
$f(t, \cdot): G \to \mathbb{R}$

Semi-Direct Product group convolution

Equivariance group $(\mathbb{R}^2, +) \rtimes SO(2)$

$f: (\mathbb{R}^2, +) \rtimes SO(2) \rightarrow \mathbb{R}$

$f(t, \cdot): [0, 2\pi) \rightarrow \mathbb{R}$



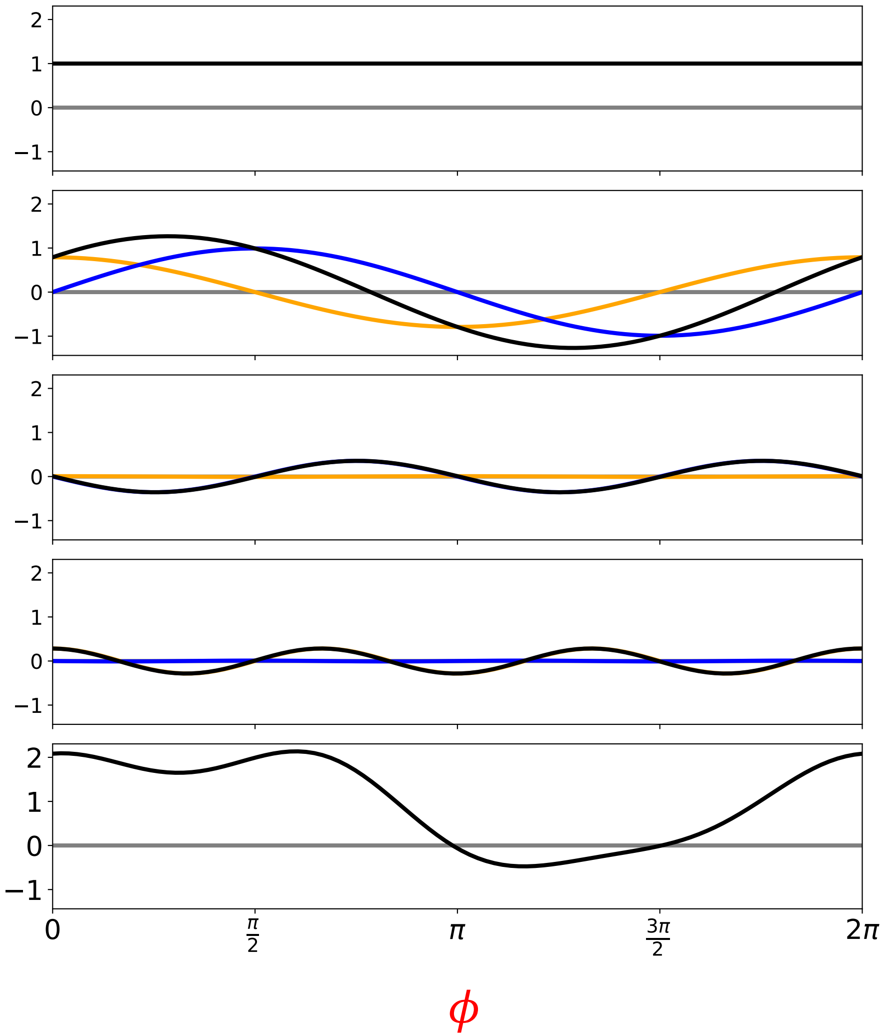
Semi-Direct Product group convolution

Equivariance group $(\mathbb{R}^2, +) \rtimes SO(2)$

$$f: (\mathbb{R}^2, +) \rtimes SO(2) \to \mathbb{R}$$

$$f(t, \cdot): [0, 2\pi) \to \mathbb{R}$$

$$f(t, r_\phi) = w_0(t) + \sum_{k=1} (\cos k\,\phi, \sin k\,\phi) \begin{pmatrix} w_{2k}(t) \\ w_{2k+1}(t) \end{pmatrix}$$



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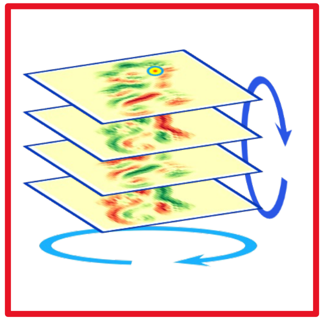
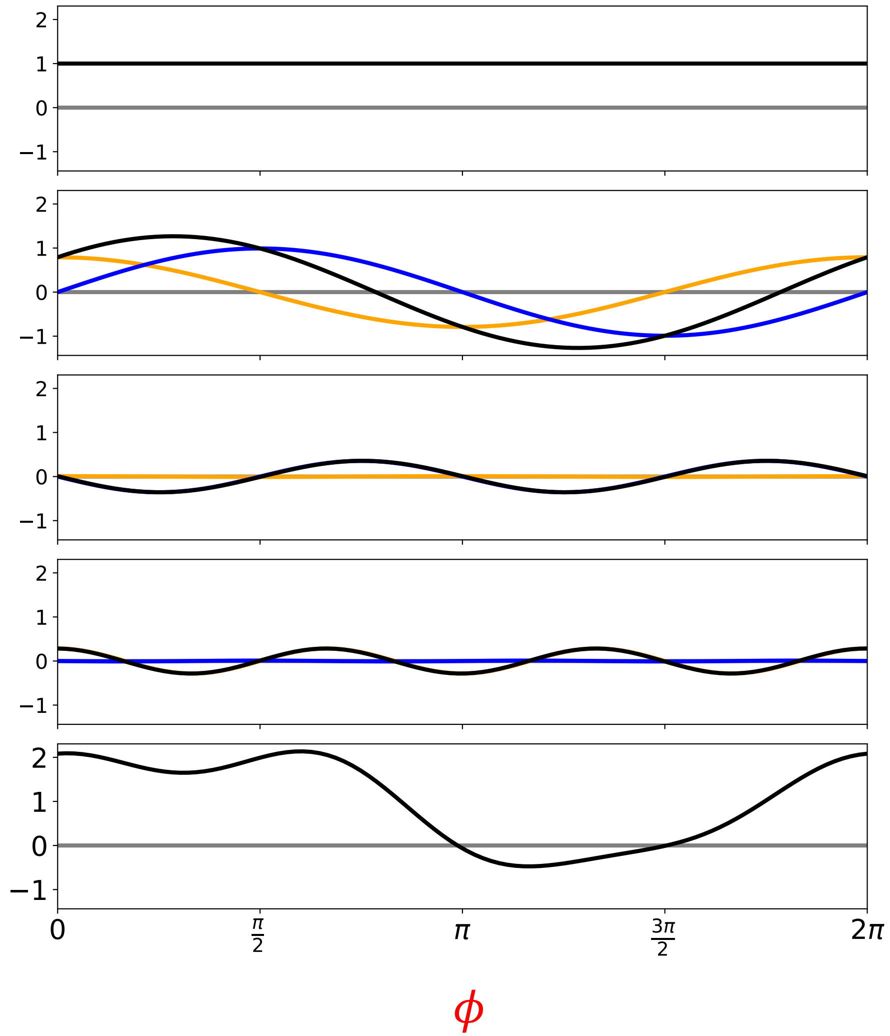


$$[r_\theta . f](t, r_\phi) = f(r_\theta^{-1} t, r_\theta^{-1} r_\phi)$$



Semi-Direct Product group convolution

Equivariance group $(\mathbb{R}^2, +) \rtimes SO(2)$

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$$[r_\theta . f](t, r_\phi) = w_0\left(r_\theta^{-1} t\right) + \sum_{k=1} \left(\cos k\,(\phi - \theta), \sin k\,(\phi - \theta)\right) \begin{pmatrix} w_{2k}\left(r_\theta^{-1} t\right) \\ w_{2k+1}\left(r_\theta^{-1} t\right) \end{pmatrix}$$

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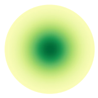
Semi-Direct Product group convolution

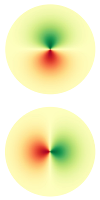
Equivariance group $(\mathbb{R}^2, +) \rtimes SO(2)$

$f : (\mathbb{R}^2, +) \rtimes SO(2) \to \mathbb{R}$

$f(t, \cdot) : [0, 2\pi) \to \mathbb{R}$

$$[r_\theta . f](t, r_\phi) = \underbrace{w_0(r_\theta^{-1} t)}\ + \sum_{k=1} (\cos k\,\phi\,, \sin k\,\phi) \underbrace{\begin{bmatrix} \cos k\theta & -\sin k\theta \\ \sin k\theta & \cos k\theta \end{bmatrix} \begin{pmatrix} w_{2k}(r_\theta^{-1} t) \\ w_{2k+1}(r_\theta^{-1} t) \end{pmatrix}}$$

[ $\star$ ]: $\mathbb{R}^2 \to \mathbb{R}^1$

[ $\star$ ]: $\mathbb{R}^2 \to \mathbb{R}^2$

$[g . f](x) = \rho(g) f(g^{-1} x)$

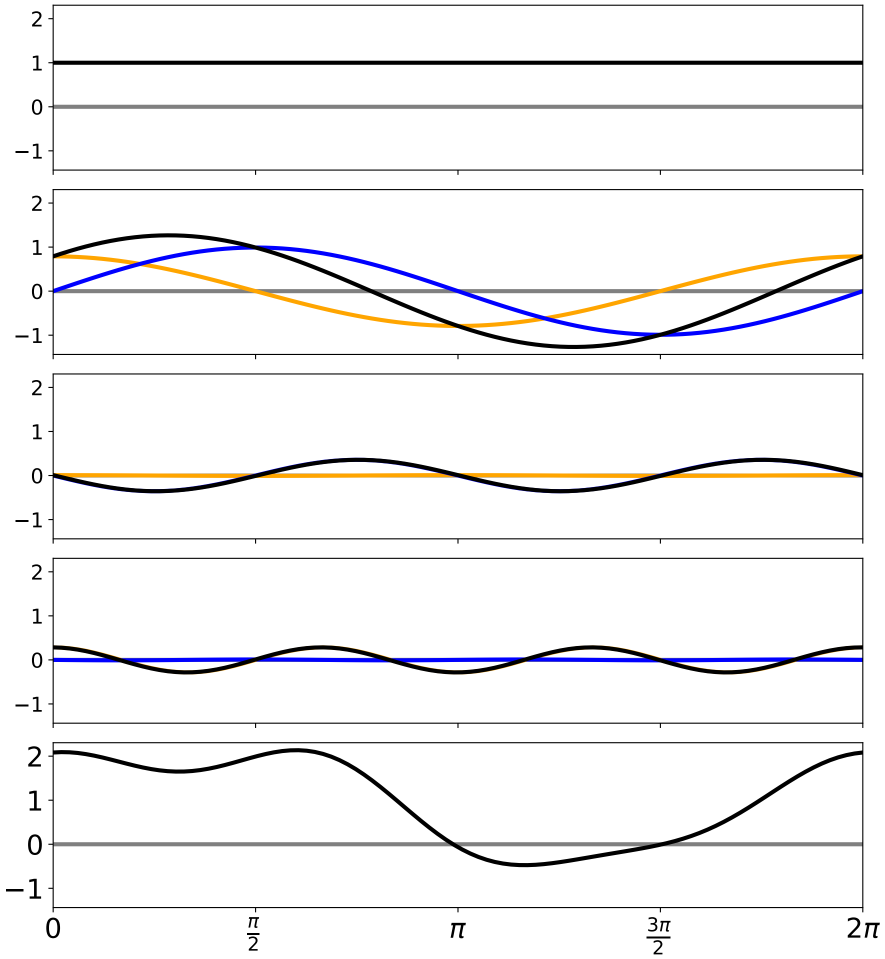
From Group Convolution to Steerable Filters

$$\rho_0(\theta) = 1 \qquad\qquad w_0(t)$$

$$\rho_1(\theta) = \begin{bmatrix} \cos 1\theta & -\sin 1\theta \\ \sin 1\theta & \cos 1\theta \end{bmatrix} \qquad\qquad \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$$

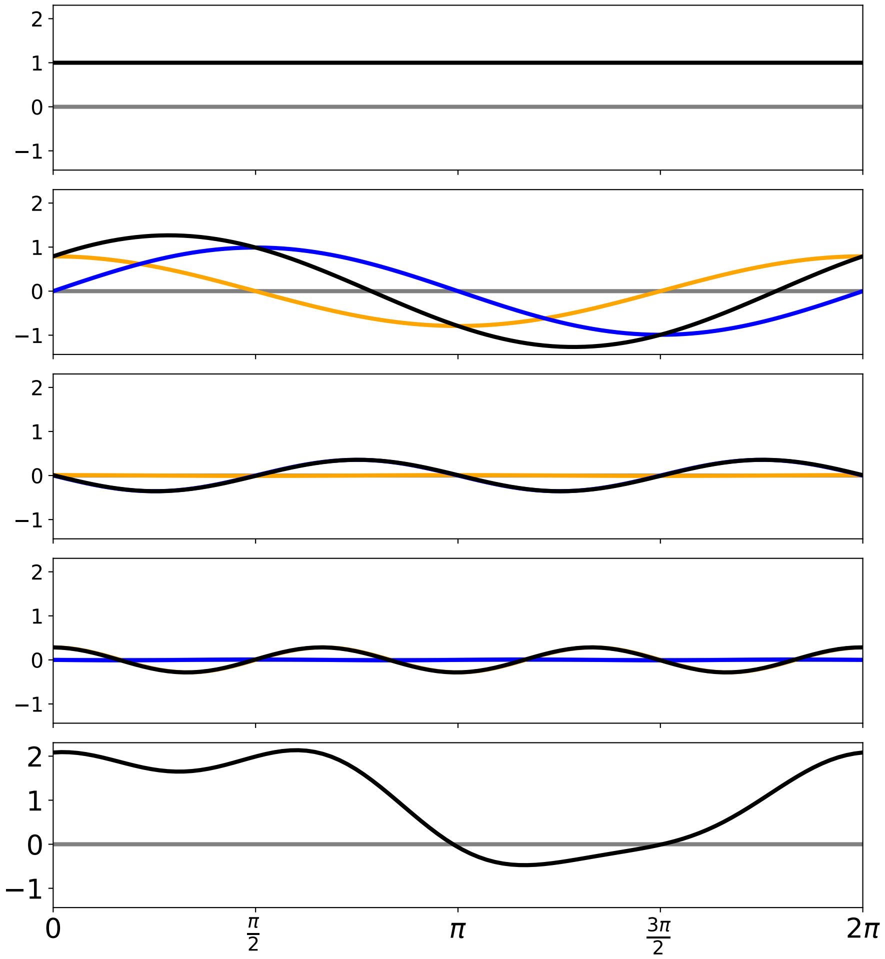
$$\rho_2(\theta) = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \qquad\qquad \begin{pmatrix} w_3(t) \\ w_4(t) \end{pmatrix}$$

$$\rho_3(\theta) = \begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix} \qquad\qquad \begin{pmatrix} w_5(t) \\ w_6(t) \end{pmatrix}$$



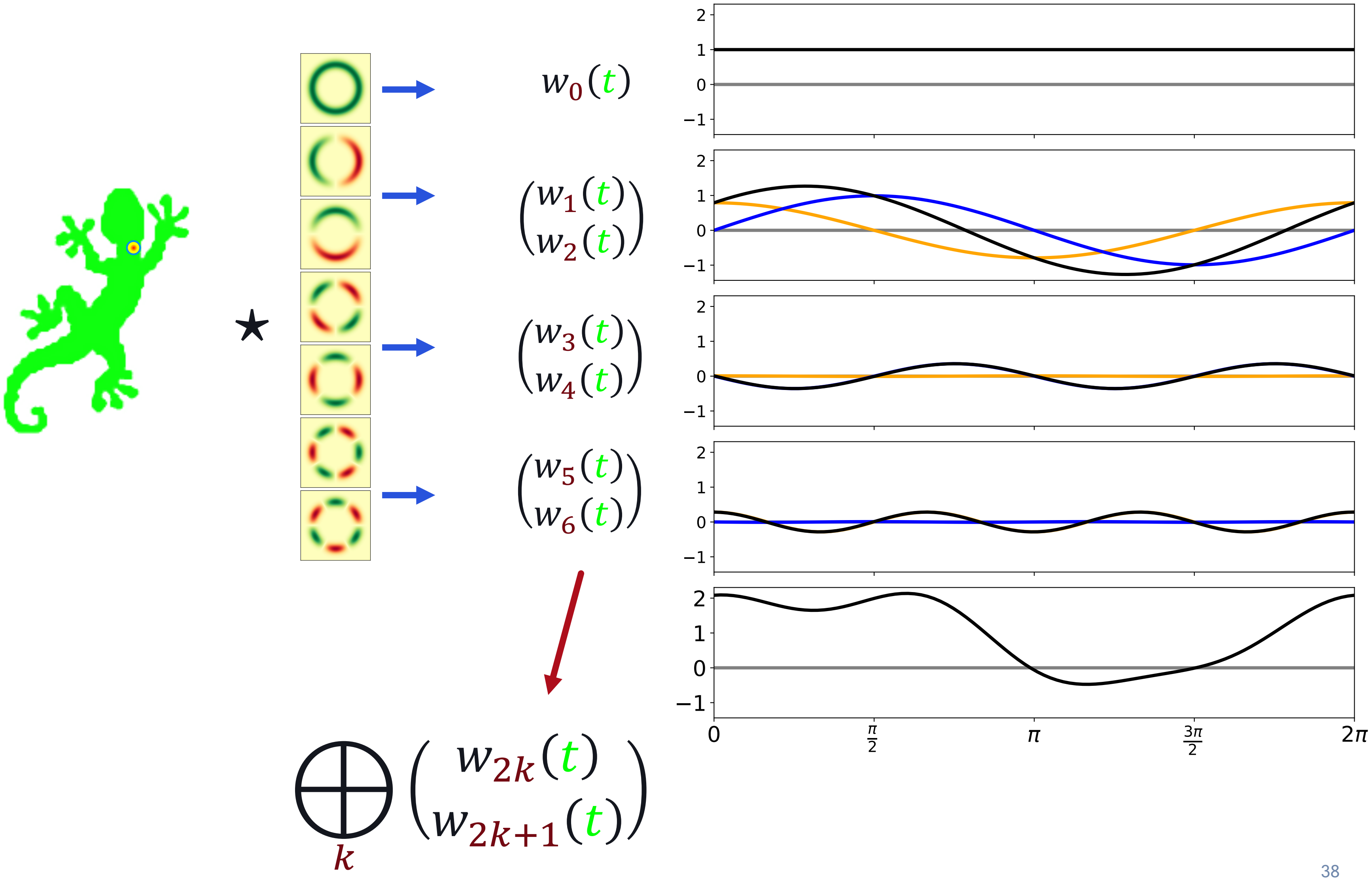
Direct Sum

$$\rho(\theta) = \begin{bmatrix} 1 & & & \\ & \begin{bmatrix} \cos 1\theta & -\sin 1\theta \\ \sin 1\theta & \cos 1\theta \end{bmatrix} & & \\ & & \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} & \\ & & & \begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix} \end{bmatrix} \qquad \begin{matrix} w_0(t) \\ \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix} \\ \begin{pmatrix} w_3(t) \\ w_4(t) \end{pmatrix} \\ \begin{pmatrix} w_5(t) \\ w_6(t) \end{pmatrix} \end{matrix}$$

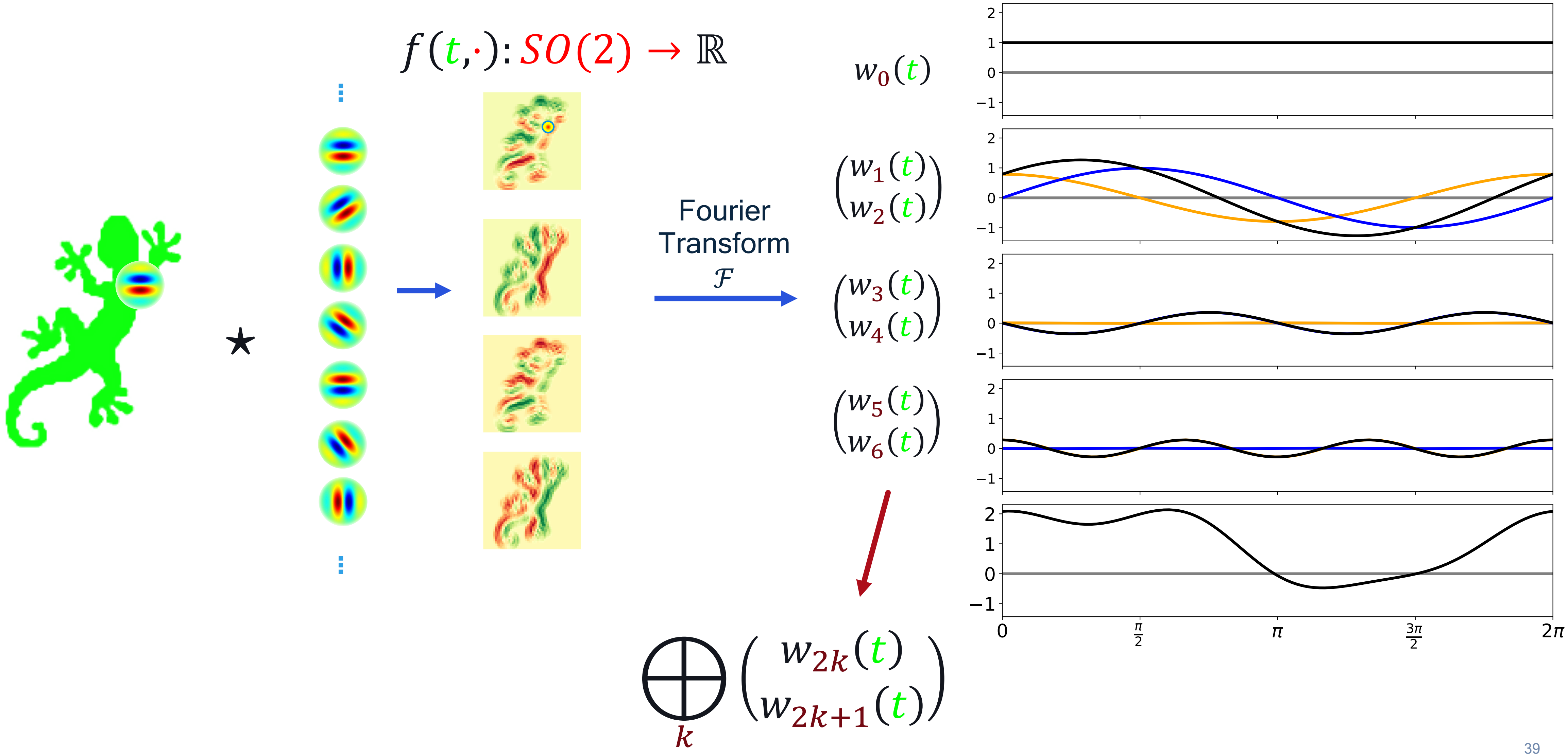


$$\rho(\theta) = \bigoplus_k \rho_k(\theta) \qquad\qquad \bigoplus_k \begin{pmatrix} w_{2k}(t) \\ w_{2k+1}(t) \end{pmatrix}$$

Recap


★
$w_0(t)$
$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$
$\begin{pmatrix} w_3(t) \\ w_4(t) \end{pmatrix}$
$\begin{pmatrix} w_5(t) \\ w_6(t) \end{pmatrix}$
$\bigoplus_k \begin{pmatrix} w_{2k}(t) \\ w_{2k+1}(t) \end{pmatrix}$
2
1
0
−1
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

Recap



$f(t,\cdot): SO(2) \to \mathbb{R}$
$\star$
Fourier Transform
$\mathcal{F}$
$w_0(t)$
$\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}$
$\begin{pmatrix} w_3(t) \\ w_4(t) \end{pmatrix}$
$\begin{pmatrix} w_5(t) \\ w_6(t) \end{pmatrix}$
$\bigoplus_k \begin{pmatrix} w_{2k}(t) \\ w_{2k+1}(t) \end{pmatrix}$
0
$\frac{\pi}{2}$
π
$\frac{3\pi}{2}$
2π

Formally: the Regular Representation

- Recall that $L^2(G)$ is the vector space of square integrable functions on G
- $L^2(G)$ carries an **orthogonal action** of G

$$g: L^2(G) \to L^2(G), \qquad f \mapsto g.f$$
$$[g.f](\boldsymbol{x}) := f(g^{-1}\boldsymbol{x})$$

- This is the *Regular Representation* of G

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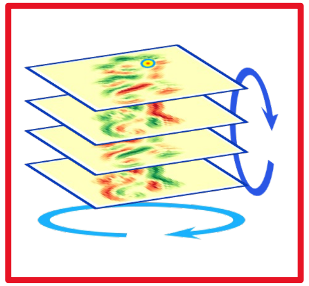
- This is the *Regular Representation* of G
- When G is a **finite group** it looks like permutation matrices (e.g. C_4)



$$\rho(r^0) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \rho(r^1) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \rho(r^2) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \rho(r^3) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Formally: the Fourier Transform (Peter-Weyl theorem)

- Let G be a *compact* group
- There is a set of *irreducible representations* denoted $\hat{G}$
  - analogous to frequencies in classical Fourier Transform
  - e.g. circular harmonics or Wigner-D Matrices
- The matrix coefficients form an *orthogonal basis* for $L^2(G)$

$$f(g) = \sum_{\rho \in \hat{G}} \sqrt{d_\rho} \langle \hat{f}(\rho), \rho(g) \rangle$$

$$\rho_0(\theta) = 1 \qquad \rho_1(\theta) = \begin{bmatrix} \cos 1\theta & -\sin 1\theta \\ \sin 1\theta & \cos 1\theta \end{bmatrix} \qquad \rho_2(\theta) = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \qquad \rho_3(\theta) = \begin{bmatrix} \cos 3\theta & -\sin 3\theta \\ \sin 3\theta & \cos 3\theta \end{bmatrix}$$

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$$f(g) = \sum_{\rho \in \hat{G}} \sqrt{d_\rho} \langle \hat{f}(\rho), \rho(g) \rangle$$

$\hat{f}(\rho) \in \mathbb{R}^{d_\rho \times d_\rho}$
Contains the weights

$$\langle A, B \rangle = \sum_{ij} A_{ij} B_{ij} = Tr(AB^T)$$

inner product between matrices

Formally: the Fourier Transform (Peter-Weyl theorem)

- Let G be a *compact* group
- There is a set of *irreducible representations* denoted $\hat{G}$
  - analogous to frequencies in classical Fourier Transform
  - e.g. circular harmonics or Wigner-D Matrices
- The matrix coefficients form an *orthogonal basis* for $L^2(G)$

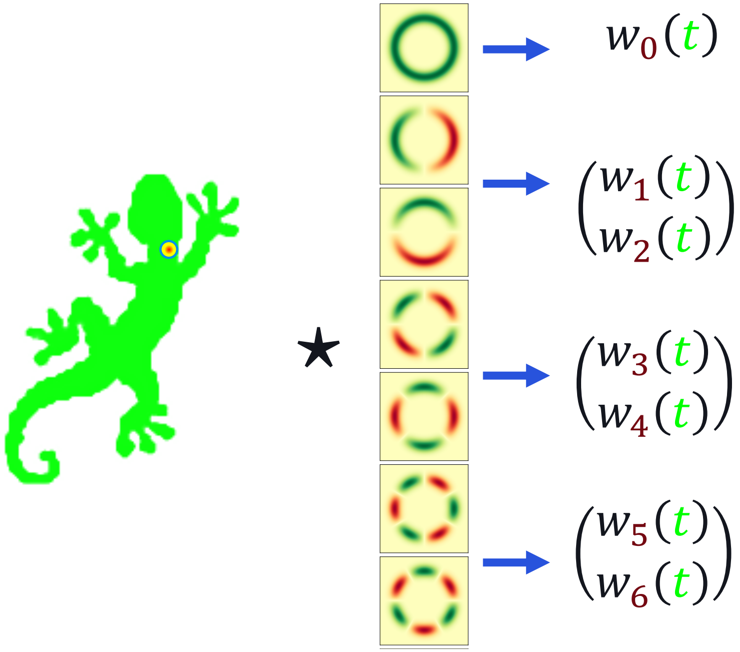
$$f(g) = \sum_{\rho \in \hat{G}} \sqrt{d_\rho} \langle \hat{f}(\rho), \rho(g) \rangle$$

Caveat: this is actually a basis only in $\mathbb{C}$, but sometimes it has redundant entries in $\mathbb{R}$, e.g.

$$\rho_1(\theta) = \begin{bmatrix} \cos 1\theta & -\sin 1\theta \\ \sin 1\theta & \cos 1\theta \end{bmatrix}$$

What did we find?

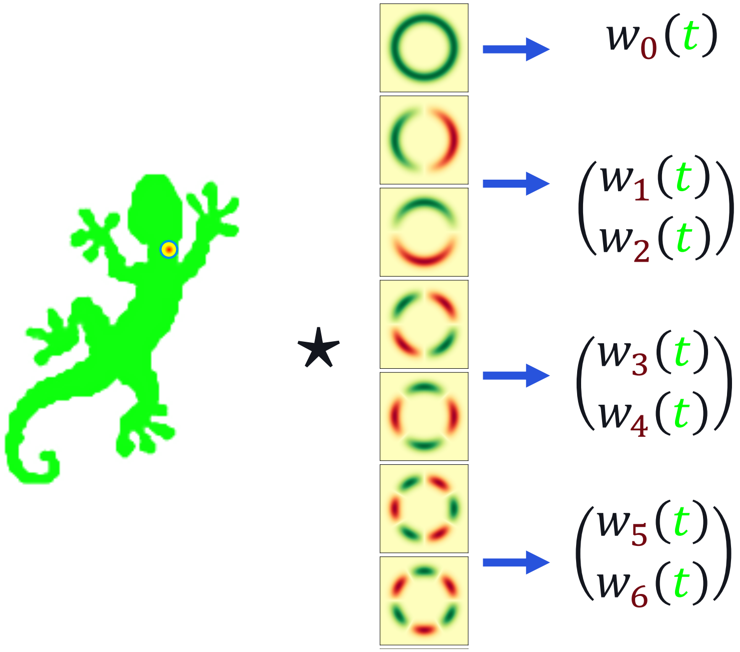
- Group convolution with steerable filters produces smaller steerable features
  - No need to store redundant activations
- So far, only studied lifting convolution $L^2(\mathbb{R}^n) \to L^2\big((\mathbb{R}^n,+) \rtimes G\big)$



$$[g.f](x) = \rho(g) f(g^{-1}x)$$

What did we find?

- Group convolution with steerable filters produces smaller steerable features
  - No need to store redundant activations
- So far, only studied lifting convolution $L^2(\mathbb{R}^n) \to L^2\big((\mathbb{R}^n,+) \rtimes G\big)$
  - Input is a **scalar field**
  - Recover lifting convolution when using output regular representation
- What about other input fields? Intermediate layers?



$$[g.f](x) = \rho(g) f(g^{-1}x)$$

Problem

Given a choice of input and output **steerable feature types**, what convolution do we need to use?

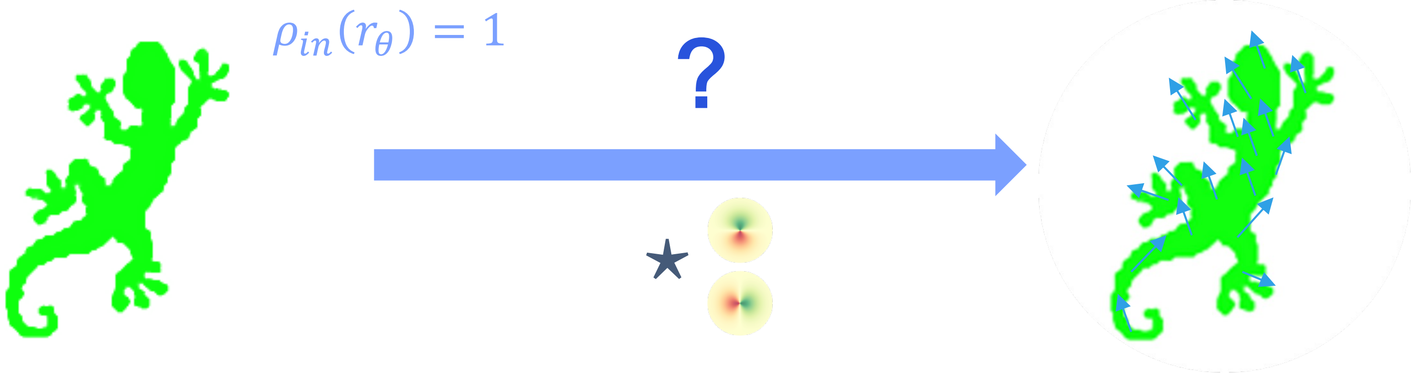
- What kind of filters produce an output feature map with the *desired transformation type*?

$f_{in}: \mathbb{R}^2 \to \mathbb{R}^1$

$f_{out}: \mathbb{R}^2 \to \mathbb{R}^2$

$\rho_{in}(r_\theta) = 1$

$\rho_{out}(r_\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$



Agenda

Steerable Filters

Steerable Fields and Representation Theory

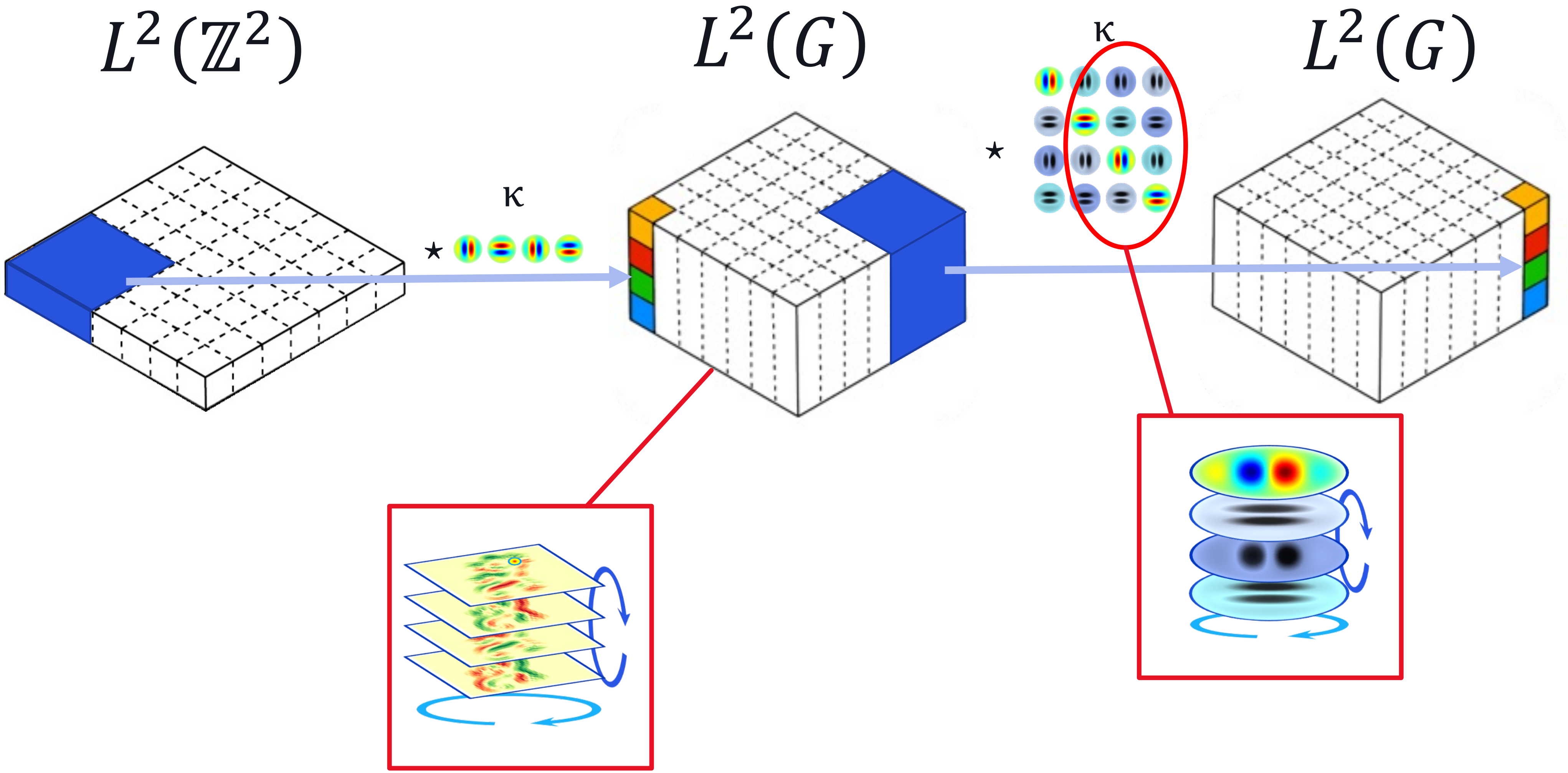
Group Convolution to Steerable Convolution

Steerable CNNs

Solving the kernel constraint

Instances and Applications

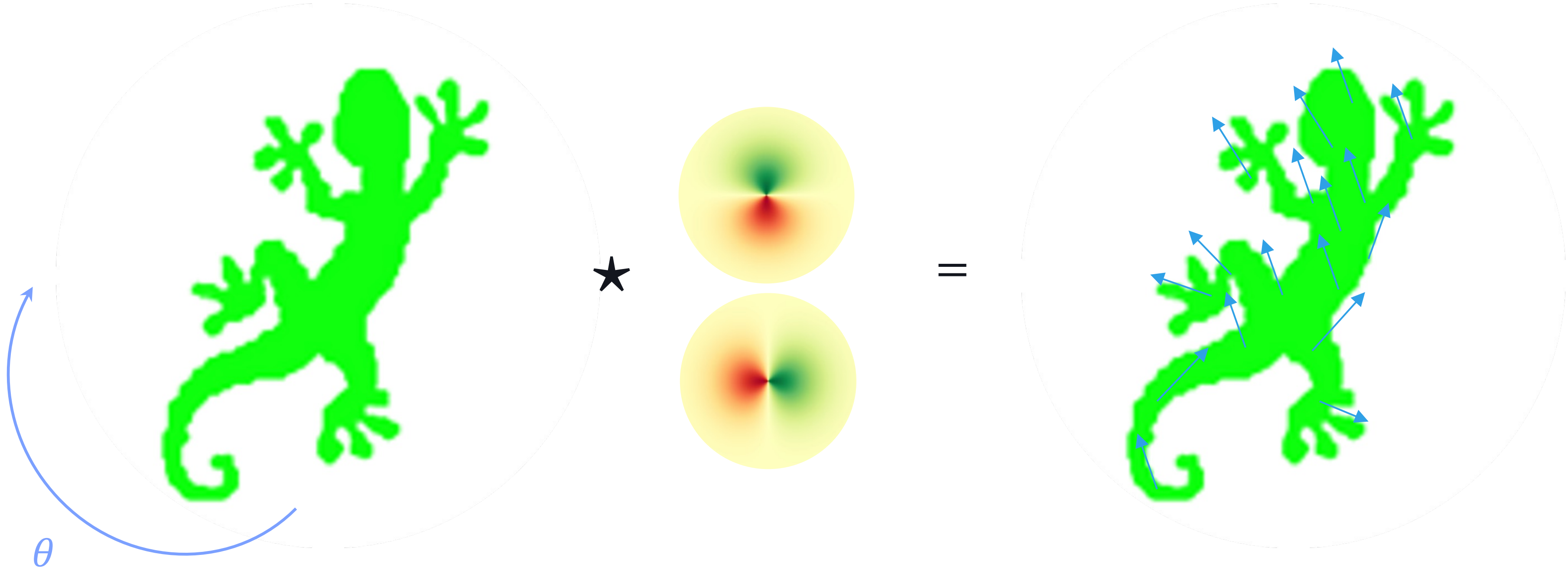
Brief Recap: Lifting Convolution and Group Convolution



Brief recap: Steerable Filters

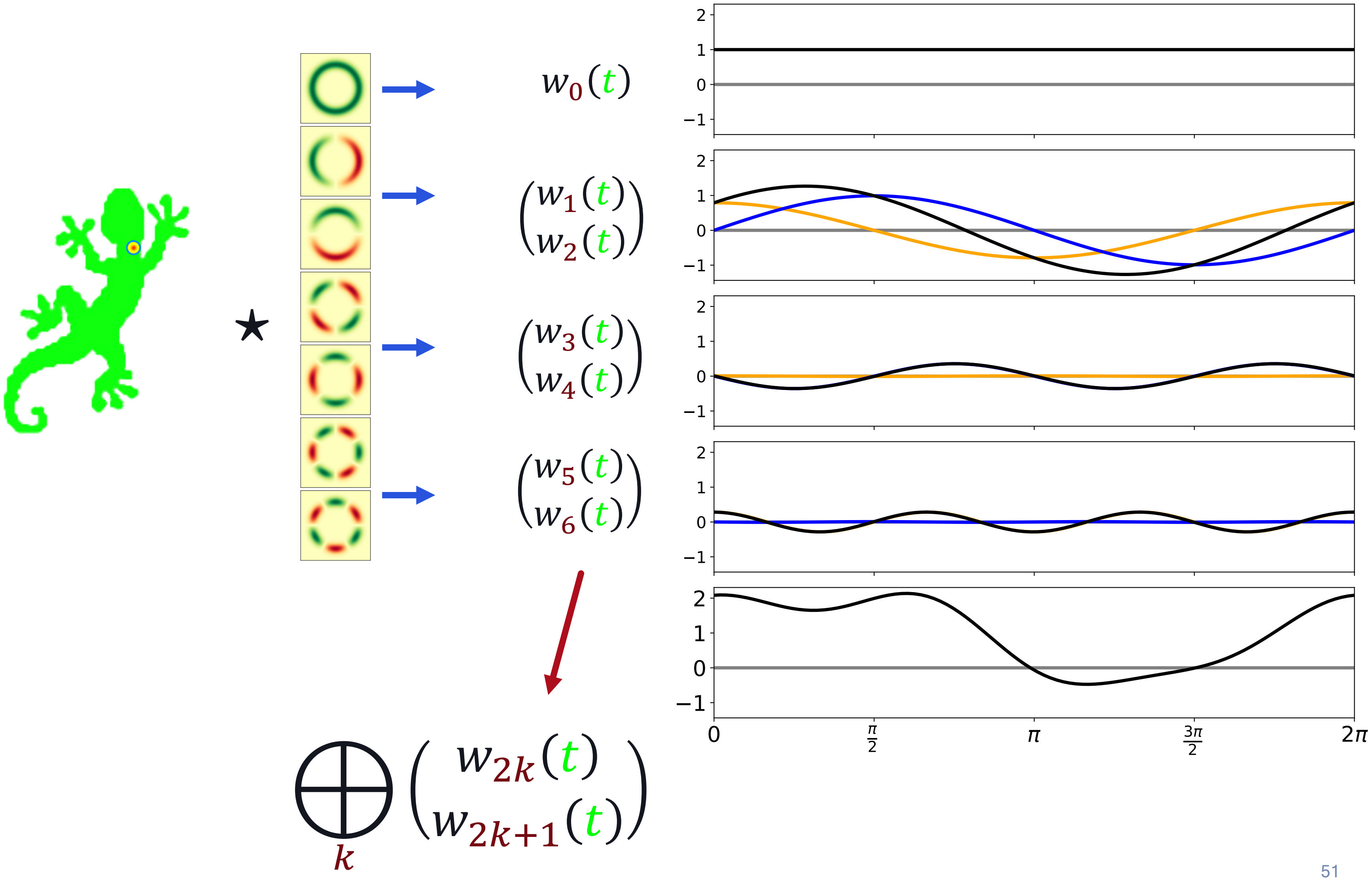
- Filter can be rotated via linear combination

$$\rho(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

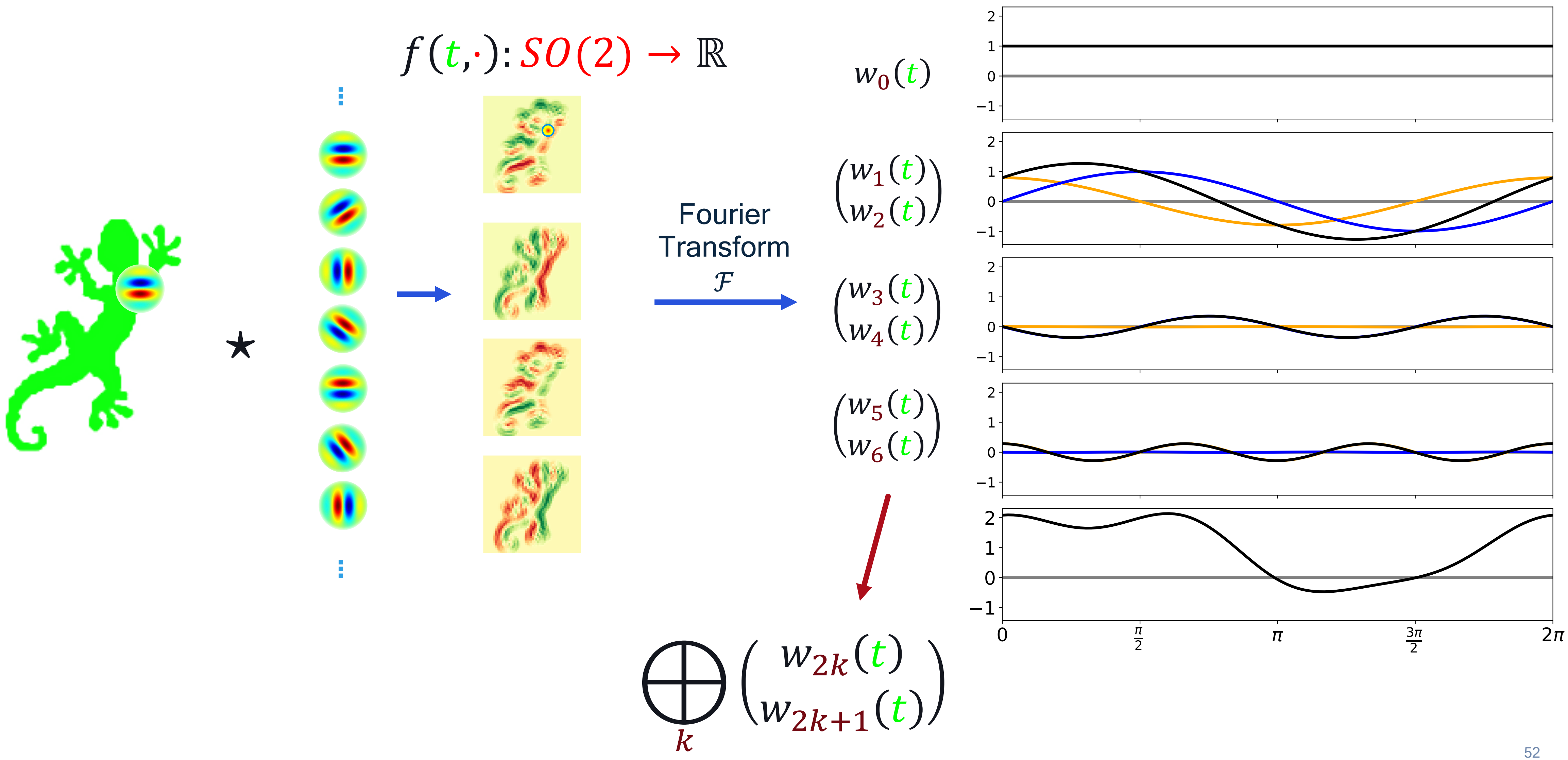


$$\boxed{R_\theta.\kappa \star f = \cos\theta\,(\kappa \star f) - \sin\theta\left(R_{\frac{\pi}{2}}.\kappa \star f\right)}$$

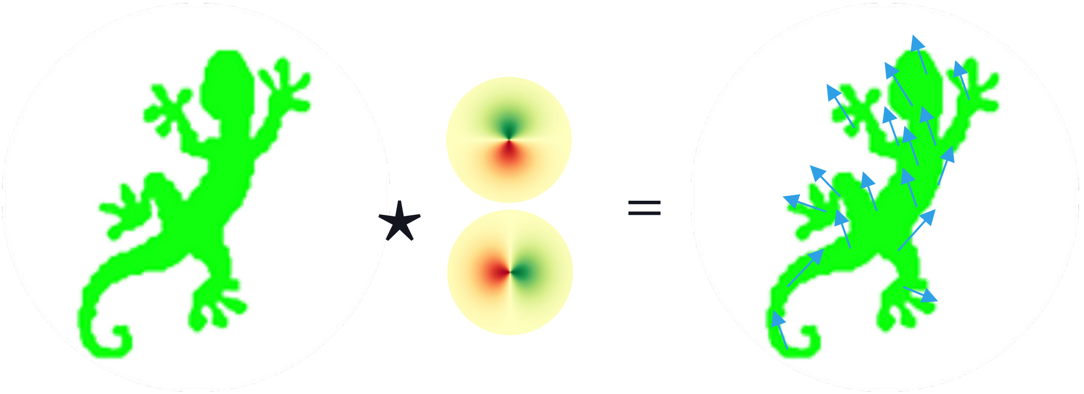
Brief Recap: steerable convolution



Recap: steerable convolution

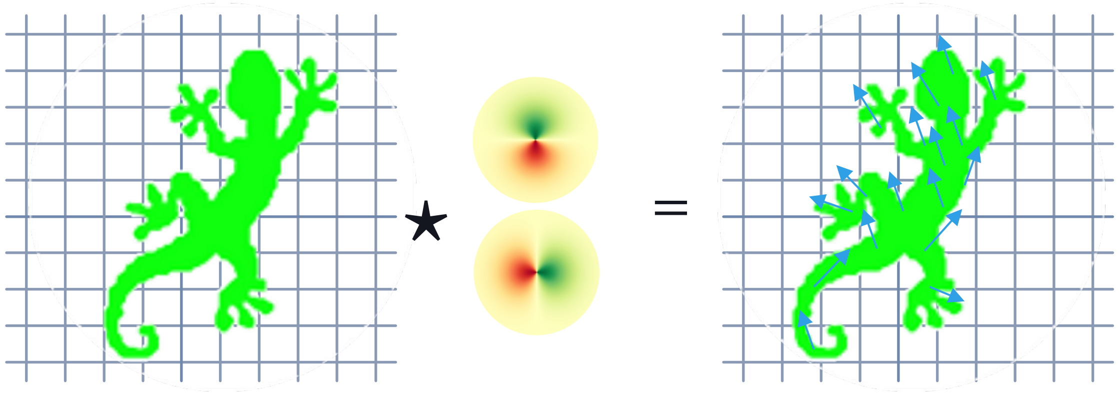


Convolution and Message Passing



$$[\kappa \star f](y) = \sum_{x \in \mathbb{Z}^n} \kappa(x - y)\ f(x)$$

Convolution and Message Passing



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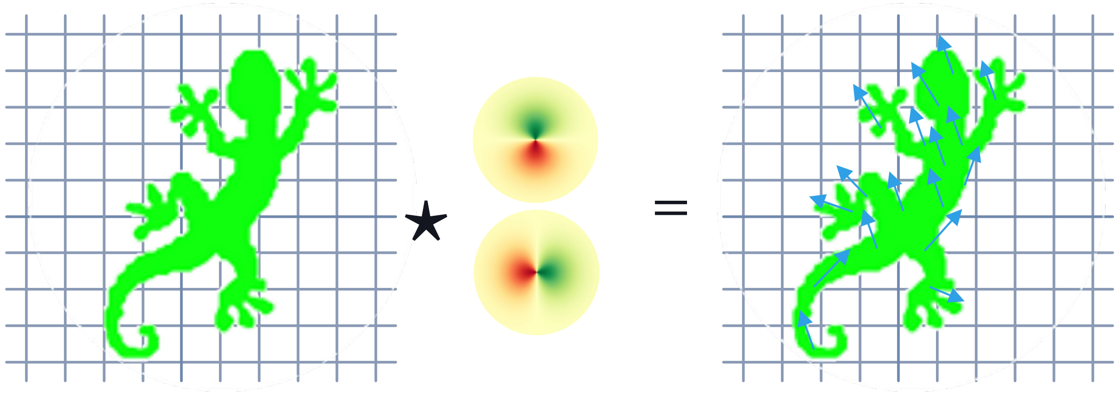
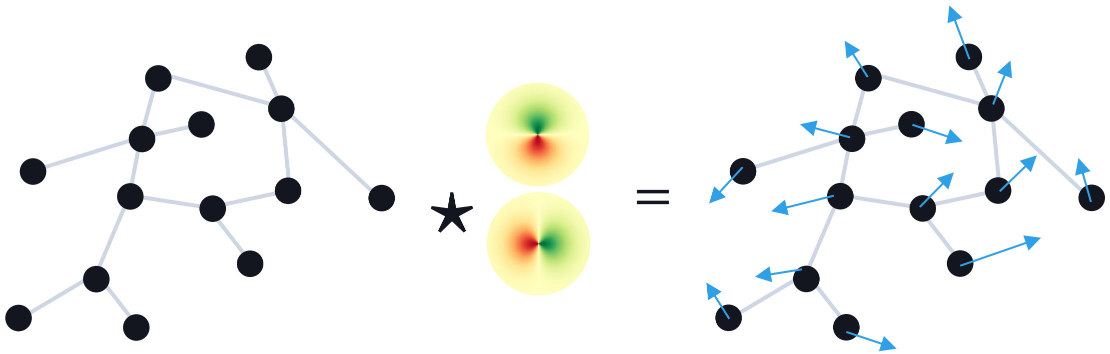
Convolution and Message Passing





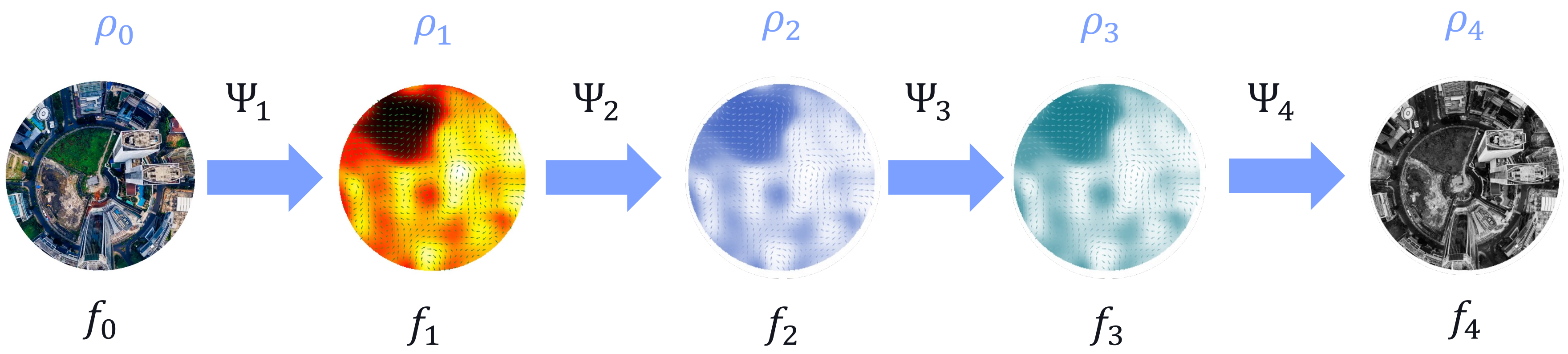
$$[\kappa \star f](y) = \sum_{x \in \mathbb{Z}^n} \kappa(x - y)\ f(x)$$

$$[\kappa \star f](i) = \sum_{j \in N_i} \kappa\left(x_i - x_j\right)\ f_j$$

Nathaniel Thomas, Tess Smidt, Steven Kearnes, Lusann Yang, Li Li, Kai Kohlhoff, and Patrick Riley. Tensor field networks: Rotation-and translation-equivariant neural networks for 3d point clouds. (2018)

Feature Fields and Steerable CNNs

- Symmetry group G
- An intermediate feature is a multi-channels signal $f_l \colon \mathbb{R}^n \to \mathbb{R}^d$
- Associated with its own transformation law ρ_l
- Steerable CNN is equivariant when each layer Ψ_l commutes with its input and output transformations



Steerable CNNs

- Standard ***convolution*** with G-*steerable filter* K guarantees also G **equivariance**

$$K: \mathbb{R}^n \to \mathbb{R}^{d_{out} \times d_{in}}$$

$$K(g.x) = \rho_{out}(g) K(x) \rho_{in}(g)^T$$

Steerability Constraint

Maurice Weiler, Mario Geiger, Max Welling, Wouter Boomsma and Taco Cohen,
3D Steerable CNNs: Learning Rotationally Equivariant Features in Volumetric Data,
Conference on Neural Information Processing Systems (NIPS), 2018

Taco Cohen, Mario Geiger and Maurice Weiler,
A General Theory of Equivariant CNNs on Homogeneous Spaces,
Conference on Neural Information Processing Systems (NeurIPS), 2019

Steerable CNNs

- Standard *convolution* with G *-steerable filter* K guarantees also G **equivariance**

$$K: \mathbb{R}^n \to \mathbb{R}^{d_{out} \times d_{in}}$$

$$K(g.x) = \rho_{out}(g) K(x) \rho_{in}(g)^T$$

Steerability Constraint

Q: How do we parameterize G-steerable filters?

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Solving the steerability constraint

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$

$$K: \mathbb{R}^n \to \mathbb{R}^{d_{out} \times d_{in}}$$

$$K(g.x) = \rho_{out}(g) K(x)\, \rho_{in}(g)^{\mathrm{T}}$$

Solving the steerability constraint

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
  - Column-wise vectorization of the matrix $K(x)$

$$K: \mathbb{R}^n \to \mathbb{R}^{d_{out} \times d_{in}}$$

$$K(g.x) = \rho_{out}(g) K(x)\, \rho_{in}(g)^{\mathrm{T}}$$

$$vec(B\ K\ A^T) = (A \otimes B) vec(K)$$

Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11} \cdot B & \cdots & a_{1n} \cdot B \\ \vdots & \ddots & \vdots \\ a_{n1} \cdot B & \cdots & a_{nn} \cdot B \end{bmatrix}$$

Solving the steerability constraint

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$

$$\kappa: \mathbb{R}^n \rightarrow \mathbb{R}^{d_{out} \cdot d_{in}}$$

$$\kappa(g.x) = [(\rho_{in} \otimes \rho_{out})(g)]\kappa(x)$$

Vectorized Steerability Constraint

Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11} \cdot B & \cdots & a_{1n} \cdot B \\ \vdots & \ddots & \vdots \\ a_{n1} \cdot B & \cdots & a_{nn} \cdot B \end{bmatrix}$$

$$\rho_{in} \otimes \rho_{out} : G \rightarrow \mathbb{R}^{(d_{out} \cdot d_{in}) \times (d_{out} \cdot d_{in})}$$

Steerable Filters

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
- Interpret filters as **linear map** $L^2(\mathbb{R}^n) \to \mathbb{R}^{d_{out} \cdot d_{in}}$

$$f \mapsto [\kappa \star f](y) = \int_{\mathbb{R}^n} \kappa(x) f(x+y)\, dx$$

Steerable Filters

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
- Interpret filters as **linear map** $L^2(\mathbb{R}^n) \to \mathbb{R}^{d_{out} \cdot d_{in}}$

$$f \mapsto [\kappa \star f](y) = \int_{\mathbb{R}^n} \kappa(x) f(x+y)\, dx$$

Inner product of functions in $L^2(\mathbb{R}^n)$

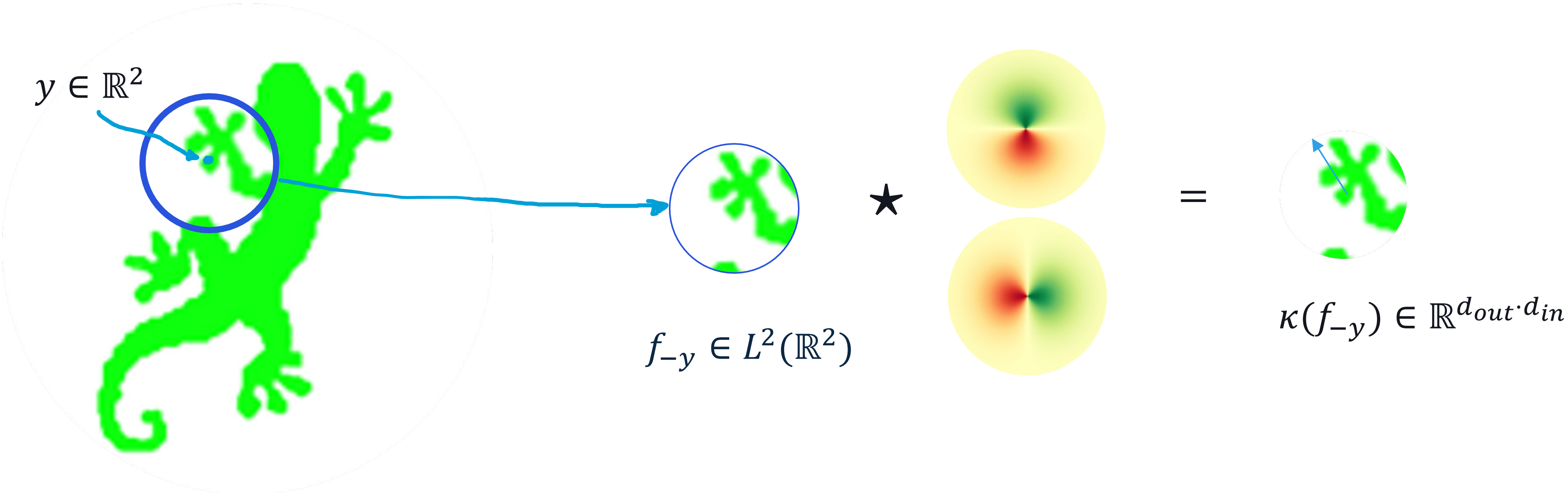
$$= \int_{\mathbb{R}^n} \kappa(x) f_{-y}(x)\, dx$$

Translated function by $-y$
$f_{-y}(x) := f(x+y)$

Steerable Filters

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
- Interpret filters as **linear map** $\kappa: L^2(\mathbb{R}^n) \to \mathbb{R}^{d_{out} \cdot d_{in}}$

$$\kappa: \; f_{-y} \mapsto \kappa\big(f_{-y}\big) = \; [\kappa \star f](y)$$



Solving the steerability constraint

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
- Interpret filters as **linear map** $\kappa: L^2(\mathbb{R}^n) \to \mathbb{R}^{d_{out} \cdot d_{in}}$

$$\kappa(g.x) = [(\rho_{in} \otimes \rho_{out})(g)]\kappa(x)$$

Action of G on $L^2(\mathbb{R}^n)$

Action of G on $\mathbb{R}^{d_{out} \cdot d_{in}}$

Steerable Filters

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
- Interpret filters as linear map $\kappa: L^2(\mathbb{R}^n) \to \mathbb{R}^{d_{out} \cdot d_{in}}$

$$\int_{\mathbb{R}^n} \kappa(g.x) f(x+y)\, dx = \int_{\mathbb{R}^n} \kappa(x) \underbrace{[g.f_{-y}]}(x)\, dx = [\kappa \star g.f](g.y)$$

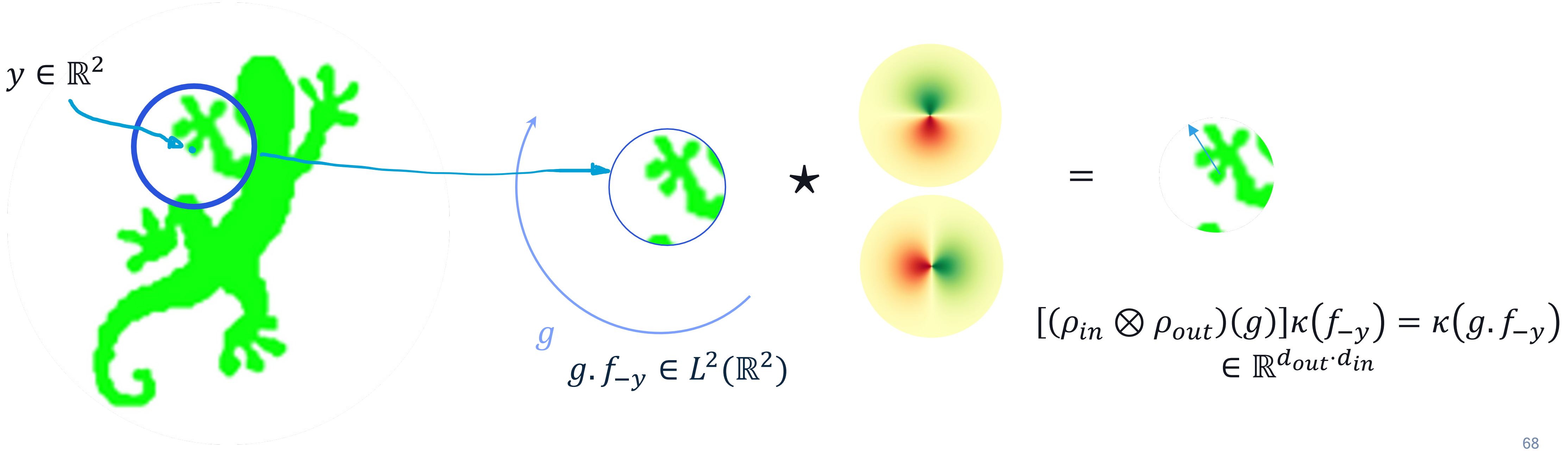
Function translated by $-y$ and rotated by g

$g.f_{-y}(x) \coloneqq f(g^{-1}.x + y)$

Solving the steerability constraint

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
- Interpret filters as **linear map** $\kappa: L^2(\mathbb{R}^n) \to \mathbb{R}^{d_{out} \cdot d_{in}}$

$$\kappa(g.x) = [(\rho_{in} \otimes \rho_{out})(g)]\kappa(x)$$



Solving the steerability constraint

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
- Interpret filters as **linear map** $\kappa\colon L^2(\mathbb{R}^n) \to \mathbb{R}^{d_{out} \cdot d_{in}}$
- IDEA: simply *find intertwiners* between two *(orthogonal)* representations!

$$\kappa(g.x) = [(\rho_{in} \otimes \rho_{out})(g)]\kappa(x)$$

Action of G on $L^2(\mathbb{R}^n)$

Action of G on $\mathbb{R}^{d_{out} \cdot d_{in}}$

Formally: Peter-Weyl Theorem (again)

Any *(orthogonal)* representation ρ can be decomposed in a *block-diagonal* form

- change-of-basis Q
- each block contains a copy of an *irreducible representation* ρ_j

$$\rho(g) = Q^T \underbrace{\begin{pmatrix} \rho_{j_1}(g) & & \\ & \rho_{j_2}(g) & \\ & & \ddots \end{pmatrix}}_{\bigoplus_{j \in I} \rho_j(g)} Q$$

Irreducible Representations (irreps): special set of representations $\hat{G} = \{\rho_j\}_j$ of G

Formally: Schur's Lemma

The space of matrices (*intertwiners*) commuting with two *irreps* $\rho_i, \rho_j \in \hat{G}$

1. is a vector space

2. is empty unless ρ_i and ρ_j are *isomorphic*, i.e. they only differ by a change of basis:

$$\rho_i(g) = Q\,\rho_j(g) Q^T$$

3. If not empty, the space is 1, 2 or 4 dimensional

Solving the steerability constraint

- Consider the vectorized kernel $\kappa(x) = vec(K(x))$
- Interpret filters as **linear map** $\kappa: L^2(\mathbb{R}^n) \rightarrow \mathbb{R}^{d_{out} \cdot d_{in}}$
- IDEA: simply *find intertwiners* between two *(orthogonal)* representations!

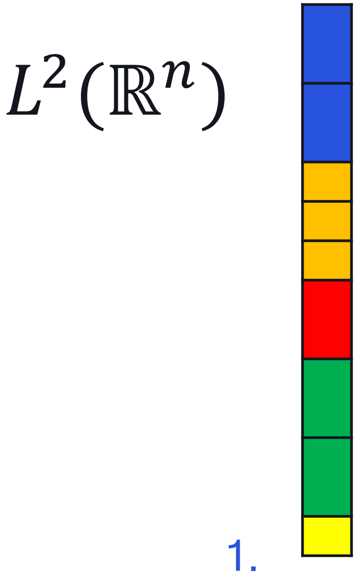
$$\kappa(g.x) = [(\rho_{in} \otimes \rho_{out})(g)]\kappa(x)$$

Action of G on $L^2(\mathbb{R}^n)$

Action of G on $\mathbb{R}^{d_{out} \cdot d_{in}}$

Solving the steerability constraint

1. decompose G's action on $L^2(\mathbb{R}^n)$

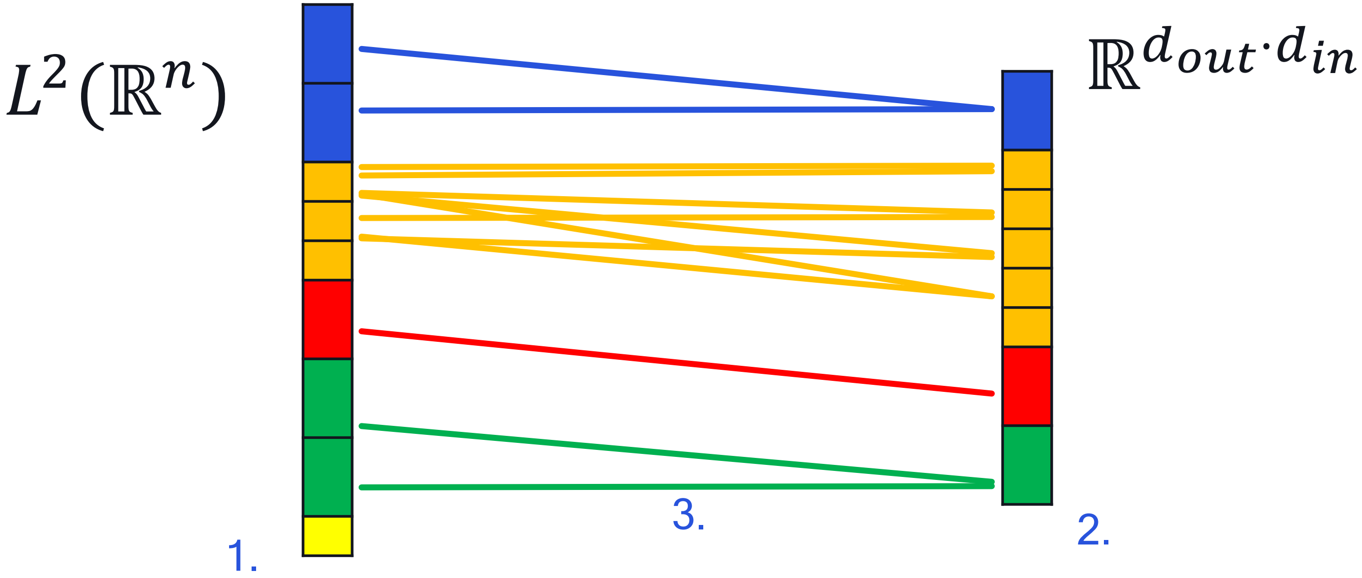


Solving the steerability constraint

1. decompose G's action on $L^2(\mathbb{R}^n)$
2. decompose G's action on $\mathbb{R}^{d_{out} \cdot d_{in}}$

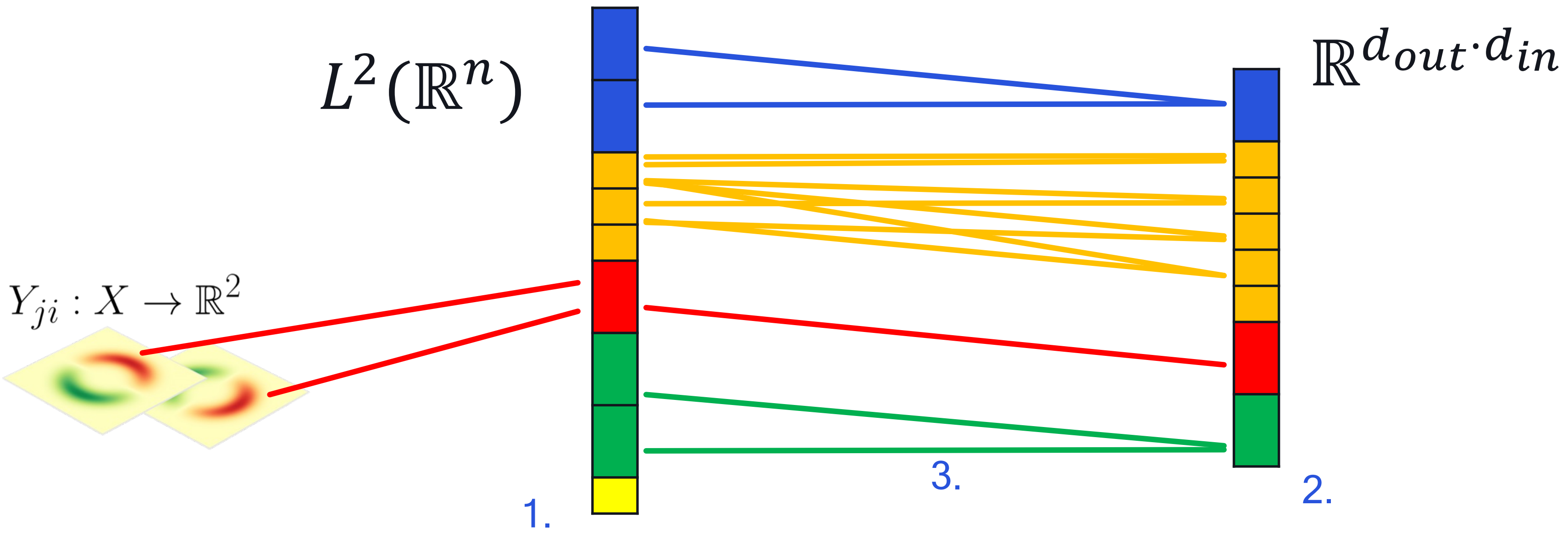
Solving the steerability constraint

1. decompose G's action on $L^2(\mathbb{R}^n)$
2. decompose G's action on $\mathbb{R}^{d_{out} \cdot d_{in}}$
3. Schur's Lemma: linear equivariant maps only between irreps of the same kind



Solving the steerability constraint

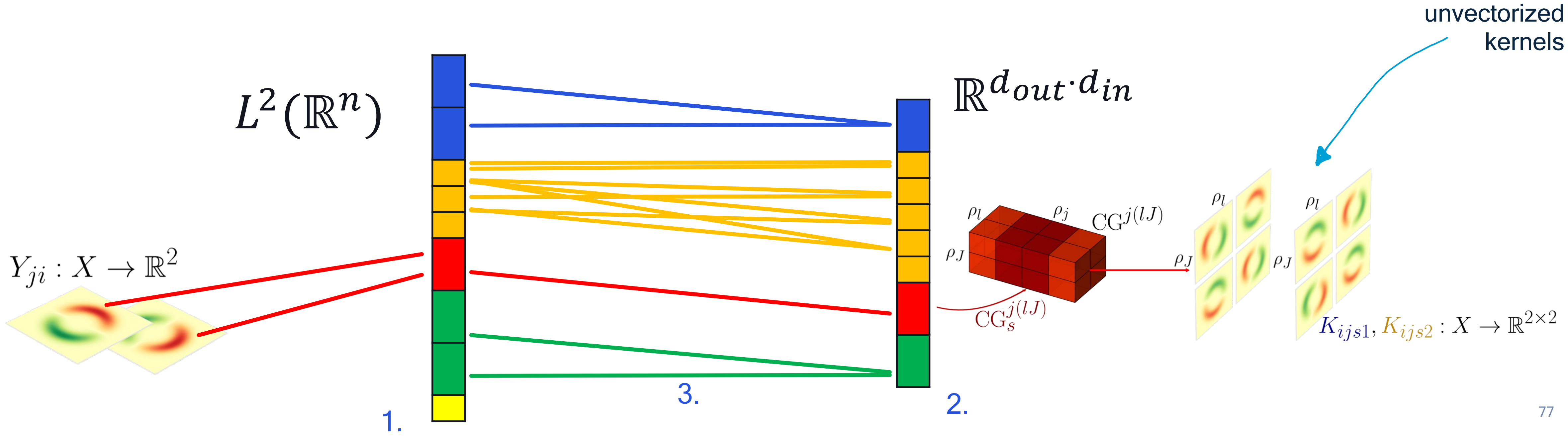
1. decompose G's action on $L^2(\mathbb{R}^n)$
   - Given by a G-steerable basis $\mathcal{B} = \{Y_{ji}: \mathbb{R}^n \to \mathbb{R}^{d_j}\}_{ji}$ for $L^2(\mathbb{R}^n)$ as in (Freeman & Adelson, 1991)
2. decompose G's action on $\mathbb{R}^{d_{out} \cdot d_{in}}$
3. Schur's Lemma: linear equivariant maps only between irreps of the same kind



William T. Freeman and Edward H. Adelson. The design and use of steerable filters. *IEEE Transactions on Pattern Analysis & Machine Intelligence*, 1991

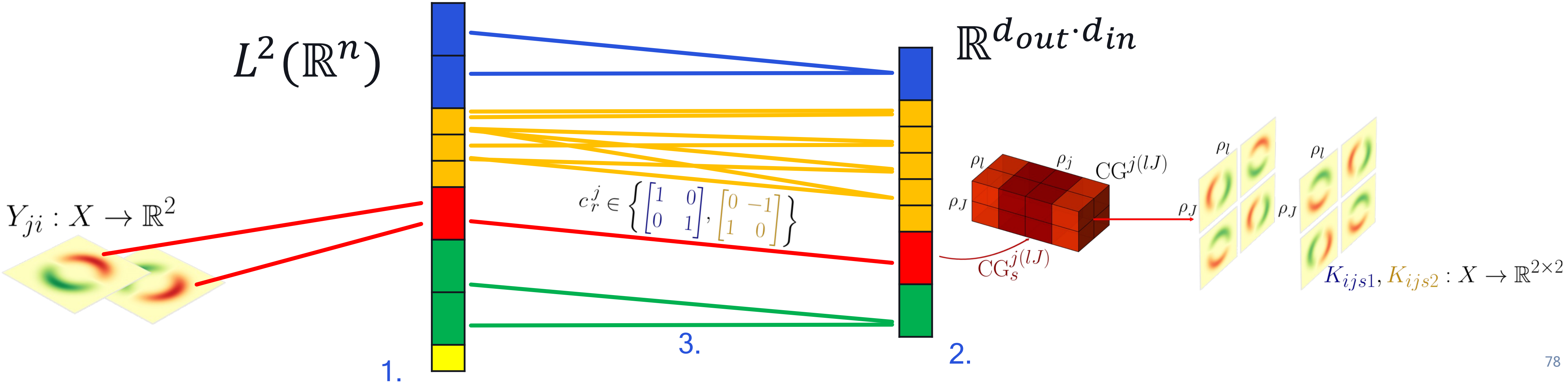
Solving the steerability constraint

1. decompose G's action on $L^2(\mathbb{R}^n)$
2. decompose G's action on $\mathbb{R}^{d_{out} \cdot d_{in}}$
   - Clebsh-Gordan coefficients CG^{lJ} to decompose $[\rho_l \otimes \rho_J](g) = [\mathrm{CG}^{lJ}]^T \left(\oplus_j \oplus_s^{[j(lJ)]} \rho_j(g) \right) \mathrm{CG}^{lJ}$
3. Schur's Lemma: linear equivariant maps only between irreps of the same kind



Solving the steerability constraint

1. decompose G's action on $L^2(\mathbb{R}^n)$
2. decompose G's action on $\mathbb{R}^{d_{out} \cdot d_{in}}$
3. Schur's Lemma: linear equivariant maps only between irreps of the same kind
   - The space of matrices M s.t. $\rho_j(g)M = M\rho_j(g)$ is spanned by a set of matrices $\left\{c_r^j\right\}_r$



Basis for G-steerable kernels

Theorem 2.1 (Basis for G-Steerable Kernels). *Let G be a compact group acting on a space X. Let $\mathcal{B} = \{Y_{ji}\}_{ji}$ be a G-steerable basis for $L^2(X)$. Assume $\rho_{\text{in}} = \rho_l$ and $\rho_{\text{out}} = \rho_J$ in $\widehat{G}$. Under minor conditions, a basis for (vectorized) G-steerable kernels over X is given by $\mathcal{K} = \{\kappa_{jisr}\}_{jisr}$, with:*

$$\kappa_{jisr}(x) = [\mathrm{CG}_s^{j(lJ)}]^T \cdot c_r^j \cdot Y_{ji}(x)$$

Decomposes $\rho_l \otimes \rho_J$
(Clebsh-Gordan coefficients)

Linear maps
equivariant to ρ_j
(Schur's Lemma)

Decomposes $L^2(\mathbb{R}^n)$
(Steerable filters)

Gabriele Cesa, Leon Lang, and Maurice Weiler.
A program to build E (N)-equivariant steerable CNNs.
International conference on learning representations. 2021.

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Generalize Wigner-Eckart theorem of (Lang & Weiler, 2020)
from *homogeneous spaces* to *general G-spaces*

Leon Lang and Maurice Weiler.
A Wigner-Eckart theorem for group equivariant convolution kernels.
International Conference on Learning Representations, 2020

Building G-steerable basis for $L^2(\mathbb{R}^n)$

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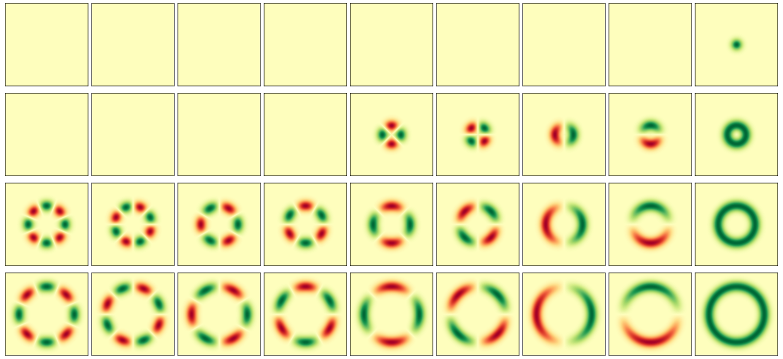
Building G-steerable basis for $L^2(\mathbb{R}^n)$

- Building a G-steerable basis $\mathcal{B}$ for $L^2(\mathbb{R}^n)$ requires an ad-hoc solution for each G
- If $G < G'$, a G'-steerable basis $\mathcal{B}' = \{Y_{j'i'}\}_{j'i'}$ is also G-steerable
- Can always choose $G' = O(n)$
- e.g. $\mathcal{B}'$ contains circular/spherical harmonics combined with a radial profile

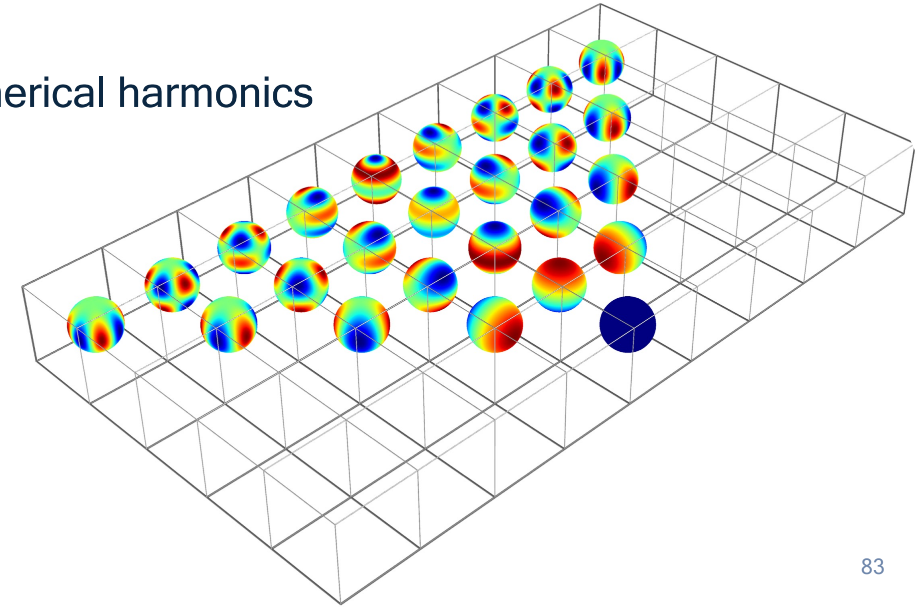
Circular harmonics

$\mathcal{B}' =$



Spherical harmonics

$\mathcal{B}' =$



Agenda

Steerable Filters

Steerable Fields and Representation Theory

Group Convolution to Steerable Convolution

Steerable CNNs

Solving the kernel constraint

Instances and Applications

Question: Equivariant Non-Linearities

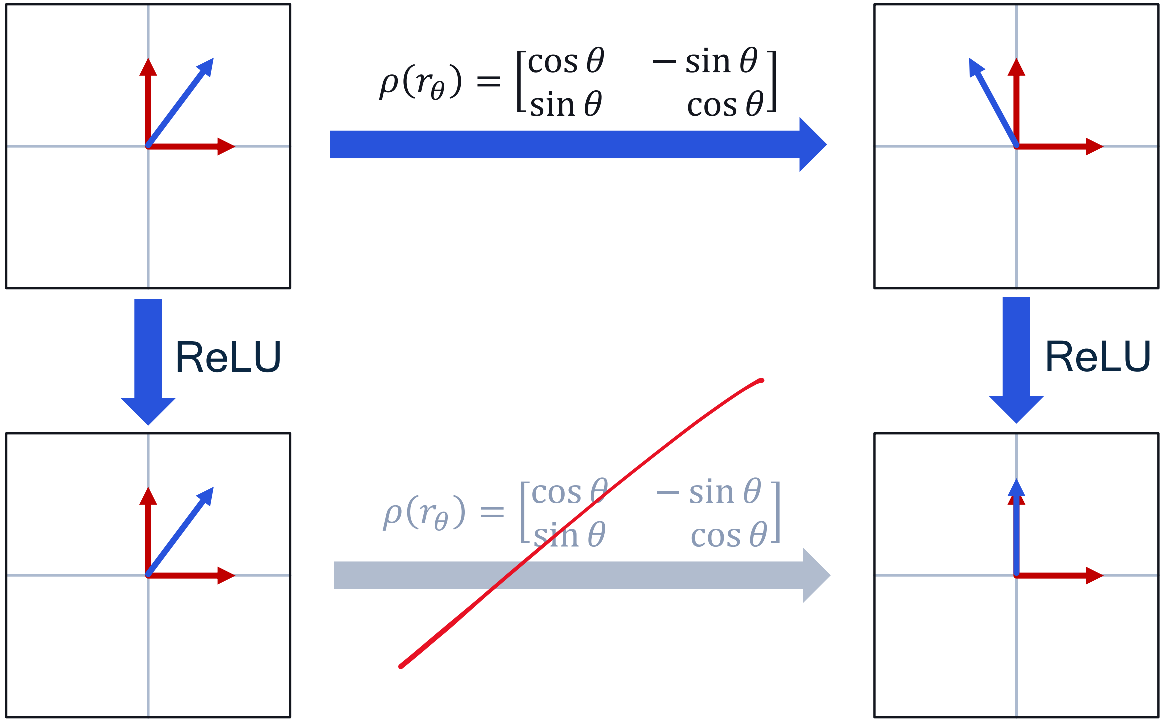
- Intermediate feature $f: \mathbb{R}^n \to \mathbb{R}^d$
- Transforms under representation of G $\quad \rho: G \to \mathbb{R}^{d \times d}$

$$[g.f](x) = \rho(g) f(g^{-1}x)$$

Equivariant Non-Linearities

- Intermediate feature $f: \mathbb{R}^n \rightarrow \mathbb{R}^d$
- Transforms under representation of G $\quad \rho: G \rightarrow \mathbb{R}^{d \times d}$
- We can NOT always use point-wise non-linearities (e.g ReLU)

$$[g.f](x) = \rho(g) f(g^{-1} x)$$



Equivariant Non-Linearities: Other Choices

- Let the intermediate feature $f: \mathbb{R}^n \to \mathbb{R}^d$ transform under $\rho: G \to O(d)$
- The quantity $|f(x)|_2^2 \in \mathbb{R}^+$ is invariant

Daniel E. Worrall, Stephan J. Garbin, Daniyar Turmukhambetov, and Gabriel J. Brostow. Harmonic networks: Deep translation and rotation equivariance. *Conference on Computer Vision and Pattern Recognition (CVPR), 2017.*

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Equivariant Non-Linearities: Other Choices

- Let the intermediate feature $f: \mathbb{R}^n \to \mathbb{R}^d$ transform under $\rho: G \to O(d)$
- The quantity $|f(x)|_2^2 \in \mathbb{R}^+$ is invariant
- **Norm Non-Linearity:** $f(x) \mapsto \sigma(|f(x)|_2^2)\, f(x)$ (Worrall et al., 2017)
- **Gated Non-Linearity:** $f(x), f_g(x) \mapsto \sigma\left(f_g(x)\right) f(x)$ (Weiler et al., 2018)
  - where $f_g(x)$ is another, invariant, feature field transforming under $\rho(g) = 1$

Potentially less expressive

- Not always universal (Finzi et al., 2021)

Daniel E. Worrall, Stephan J. Garbin, Daniyar Turmukhambetov, and Gabriel J. Brostow. Harmonic networks: Deep translation and rotation equivariance. *Conference on Computer Vision and Pattern Recognition (CVPR), 2017.*

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Marc Finzi, Max Welling, and Andrew Gordon Wilson. A practical method for constructing equivariant multilayer perceptrons for arbitrary matrix groups. International conference on machine learning (ICML), 2021.

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- Let the intermediate feature $f: \mathbb{R}^n \to \mathbb{R}^d$ transform under $\rho: G \to O(d)$
- The quantity $|f(x)|_2^2 \in \mathbb{R}^+$ is **invariant**
- Can also use other quadratic invariants: (Kondor et al., 2018)
- **Tensor-Product**: $f(x) \mapsto f(x) \otimes f(x) \in \mathbb{R}^{d^2}$
  - output transforms under $\rho_{out} = \rho \otimes \rho$

$$(x, y, z) \otimes (x, y, z) = vec\begin{pmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{pmatrix}$$

Kronecker product

$$\boldsymbol{a} \otimes \boldsymbol{b} = vec(\boldsymbol{a}\,\boldsymbol{b}^T)$$

$$A \otimes B = \begin{bmatrix} a_{11} \cdot B & \cdots & a_{1n} \cdot B \\ \vdots & \ddots & \vdots \\ a_{n1} \cdot B & \cdots & a_{nn} \cdot B \end{bmatrix}$$

Risi Kondor, Zhen Lin, and Shubhendu Trivedi. Clebsch-gordan nets: a fully Fourier space spherical convolutional neural network. Advances in Neural Information Processing Systems (NeurIPS), 2018.
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  - output transforms under $\rho_{out} = \rho \otimes \rho$
  - universality (Dym and Maron, 2020)
  - More expensive

Kronecker product

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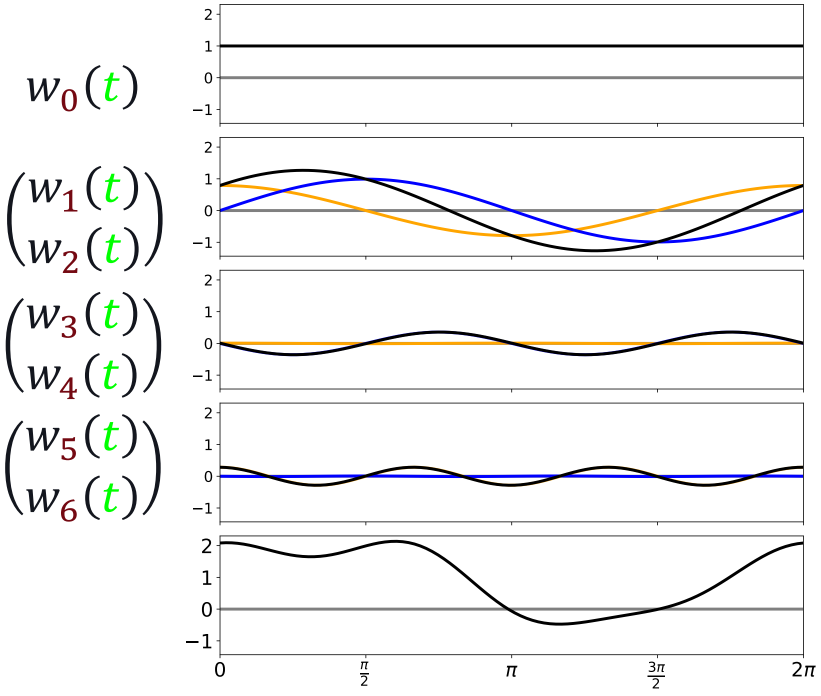
Equivariant Non-Linearities: Fourier Transform based

- Imitate a GCNN
- Choose a **band-limited** subset of irreps $\{\rho_i\}_{i\in I} \subset \hat{G}$
- A feature vector $f(x) \in \mathbb{R}^d$ represents a bandlimited signal in $L^2(G)$
- Apply point-wise non-linearity σ (e.g. ReLU) by:
  - Sampling the signal $f(x)$ on a finite subset $\mathcal{G} \subset G$
  - Applying σ on each sample
  - Reconstruct a band-limited signal from the samples



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  - Reconstruct a band-limited signal from the samples (discrete Fourier Transform)

- Band-limit + sufficient samples to control *reconstruction error*

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- Can also consider functions on homogeneous space X rather than G for reduced complexity.

  Recall Spherical CNNs

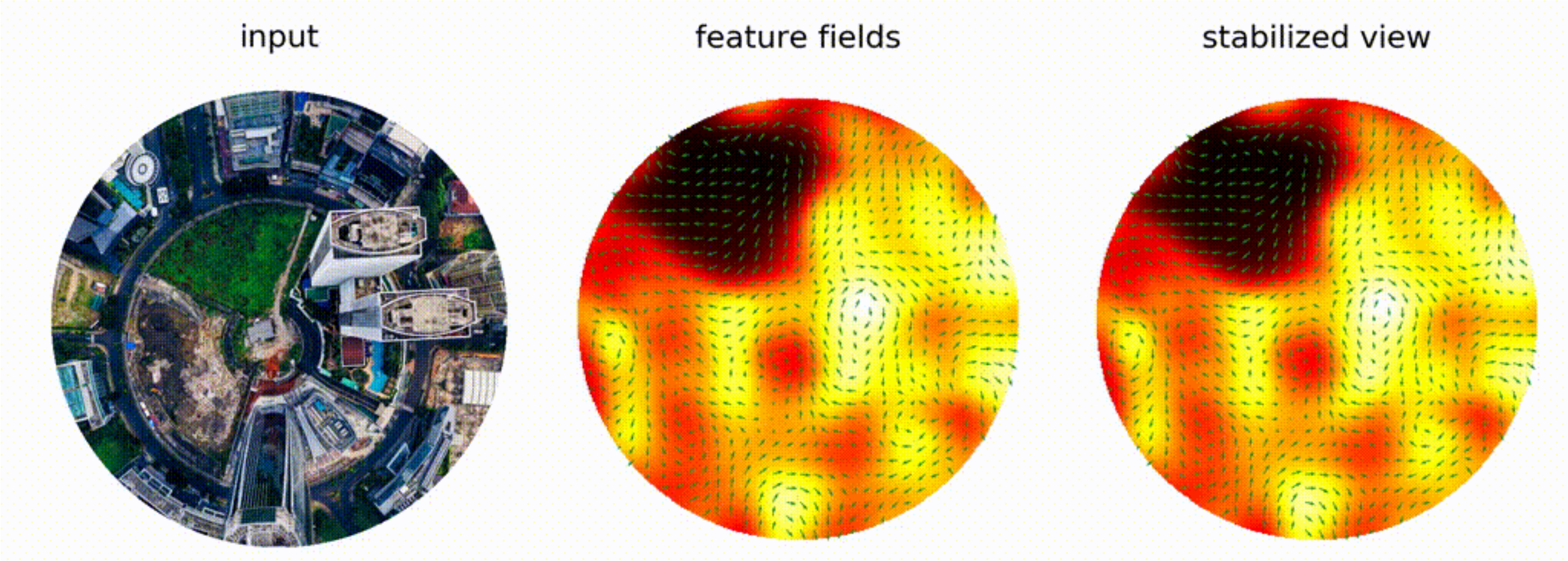
Equivariant Non-Linearities: Take Away

A useful framework to look at the literature

Most models out there differ by *choice of*

- intermediate feature types
- activation functions
- parameterization of convolution filters / messages
- discretization

Outcome: equivariance to continuous rotations



Source: e2cnn

General Program to implement G-equivariance: 2D images

2D rotational symmetries

$(\mathbb{R}^2, +) \rtimes G < E(2)$

#	group	representation		nonlinearity	invariant map	citation	MNIST O(2)	MNIST rot	MNIST 12k
1	$\{e\}$	(conventional CNN)		ELU	-	-	5.53 ± 0.20	2.87 ± 0.09	0.91 ± 0.06
2	C_1					[7, 9]	5.19 ± 0.08	2.48 ± 0.13	0.82 ± 0.01
3	C_2					[7, 9]	3.29 ± 0.07	1.32 ± 0.02	0.87 ± 0.04
4	C_3					-	2.87 ± 0.04	1.19 ± 0.06	0.80 ± 0.03
5	C_4					[6, 1, 7, 9, 10]	2.40 ± 0.05	1.02 ± 0.03	0.99 ± 0.03
6	C_6	regular	ρ_{reg}	ELU	G-pooling	[8]	2.08 ± 0.03	0.89 ± 0.03	0.84 ± 0.02
7	C_8					[7, 9]	1.96 ± 0.04	0.84 ± 0.02	0.89 ± 0.03
8	C_{12}					[7]	1.95 ± 0.07	0.80 ± 0.03	0.89 ± 0.03
9	C_{16}					[7, 9]	1.93 ± 0.04	0.82 ± 0.02	0.95 ± 0.04
10	C_{20}					[7]	1.95 ± 0.05	0.83 ± 0.05	0.94 ± 0.06
11	C_4		$5\rho_{reg} \oplus 2\rho_{quot}^{C_4/C_2} \oplus 3\psi_0$			[1]	2.43 ± 0.05	1.03 ± 0.05	1.01 ± 0.03
12	C_8		$5\rho_{reg} \oplus 2\rho_{quot}^{C_6/C_2} \oplus 2\rho_{quot}^{C_6/C_3} \oplus 2\psi_0$			-	2.03 ± 0.05	0.84 ± 0.05	0.91 ± 0.02
13	C_{12}	quotient	$5\rho_{reg} \oplus 2\rho_{quot}^{C_8/C_2} \oplus 2\rho_{quot}^{C_8/C_4} \oplus 2\psi_0$			-	2.04 ± 0.04	0.81 ± 0.02	0.95 ± 0.02
14	C_{16}		$5\rho_{reg} \oplus 2\rho_{quot}^{C_{12}/C_2} \oplus 2\rho_{quot}^{C_{12}/C_4} \oplus 3\psi_0$			-	2.00 ± 0.01	0.86 ± 0.04	0.98 ± 0.04
15	C_{20}		$5\rho_{reg} \oplus 2\rho_{quot}^{C_{16}/C_2} \oplus 2\rho_{quot}^{C_{16}/C_4} \oplus 4\psi_0$			-	2.01 ± 0.05	0.83 ± 0.03	0.96 ± 0.04
16		regular/scalar	$\psi_0 \xrightarrow{\text{conv}} \rho_{reg} \xrightarrow{G\text{-pool}} \psi_0$	ELU, G-pooling		[6, 25]	2.02 ± 0.02	0.90 ± 0.03	0.93 ± 0.04
17	C_{16}	regular/vector	$\psi_1 \xrightarrow{\text{conv}} \rho_{reg} \xrightarrow{\text{vector pool}} \psi_1$	vector field		[13, 26]	2.12 ± 0.02	1.07 ± 0.03	0.78 ± 0.03
18		mixed vector	$\rho_{reg} \oplus \psi_1 \xrightarrow{\text{conv}} 2\rho_{reg} \xrightarrow[\text{pool}]{\text{vector}} \rho_{reg} \oplus \psi_1$	ELU, vector field		-	1.87 ± 0.03	0.83 ± 0.02	0.63 ± 0.02
19	D_1					-	3.40 ± 0.07	3.44 ± 0.10	0.98 ± 0.03
20	D_2					-	2.42 ± 0.07	2.39 ± 0.04	1.05 ± 0.03
21	D_3					-	2.17 ± 0.06	2.15 ± 0.05	0.94 ± 0.02
22	D_4					[6, 1, 27]	1.88 ± 0.04	1.87 ± 0.04	1.69 ± 0.03
23	D_6	regular	ρ_{reg}	ELU	G-pooling	[8]	1.77 ± 0.06	1.77 ± 0.04	1.00 ± 0.03
24	D_8					-	1.68 ± 0.06	1.73 ± 0.03	1.64 ± 0.02
25	D_{12}					-	1.66 ± 0.05	1.65 ± 0.05	1.67 ± 0.01
26	D_{16}					-	1.62 ± 0.04	1.65 ± 0.02	1.68 ± 0.04
27	D_{20}					-	1.64 ± 0.06	1.62 ± 0.05	1.69 ± 0.03
28	D_{16}	regular/scalar	$\psi_{0,0} \xrightarrow{\text{conv}} \rho_{reg} \xrightarrow{G\text{-pool}} \psi_{0,0}$	ELU, G-pooling		-	1.92 ± 0.03	1.88 ± 0.07	1.74 ± 0.04
29		irreps ≤ 1	$\bigoplus_{i=0}^{1} \psi_i$			-	2.98 ± 0.04	1.38 ± 0.09	1.29 ± 0.05
30		irreps ≤ 3	$\bigoplus_{i=0}^{3} \psi_i$			-	3.02 ± 0.18	1.38 ± 0.09	1.27 ± 0.03
31		irreps ≤ 5	$\bigoplus_{i=0}^{5} \psi_i$			-	3.24 ± 0.05	1.44 ± 0.10	1.36 ± 0.04
32		irreps ≤ 7	$\bigoplus_{i=0}^{7} \psi_i$	ELU, norm-ReLU	conv2triv	-	3.30 ± 0.11	1.51 ± 0.10	1.40 ± 0.07
33		$\mathbb{C}$-irreps ≤ 1	$\bigoplus_{i=0}^{1} \psi_i^{\mathbb{C}}$			[12]	3.39 ± 0.10	1.47 ± 0.06	1.42 ± 0.04
34		$\mathbb{C}$-irreps ≤ 3	$\bigoplus_{i=0}^{3} \psi_i^{\mathbb{C}}$			[12]	3.48 ± 0.16	1.51 ± 0.05	1.53 ± 0.07
35		$\mathbb{C}$-irreps ≤ 5	$\bigoplus_{i=0}^{5} \psi_i^{\mathbb{C}}$			-	3.59 ± 0.08	1.59 ± 0.05	1.55 ± 0.06
36		$\mathbb{C}$-irreps ≤ 7	$\bigoplus_{i=0}^{7} \psi_i^{\mathbb{C}}$			-	3.64 ± 0.12	1.61 ± 0.06	1.62 ± 0.03
37	SO(2)			ELU, squash		-	3.10 ± 0.09	1.41 ± 0.04	1.46 ± 0.05
38				ELU, norm-ReLU		-	3.23 ± 0.08	1.38 ± 0.08	1.33 ± 0.03
39				ELU, shared norm-ReLU	norm	-	2.88 ± 0.11	1.15 ± 0.06	1.18 ± 0.03
40		irreps ≤ 3	$\bigoplus_{i=0}^{3} \psi_i$	shared norm-ReLU		-	3.61 ± 0.09	1.57 ± 0.05	1.88 ± 0.05
41				ELU, gate	conv2triv	-	2.37 ± 0.06	1.09 ± 0.03	1.10 ± 0.02
42				ELU, shared gate		-	2.33 ± 0.06	1.11 ± 0.03	1.12 ± 0.04
43				ELU, gate	norm	-	2.23 ± 0.09	1.04 ± 0.04	1.05 ± 0.06
44				ELU, shared gate		-	2.20 ± 0.06	1.01 ± 0.03	1.03 ± 0.03
45		irreps $= 0$	$\psi_{0,0}$	ELU	-	-	5.46 ± 0.46	5.21 ± 0.29	3.98 ± 0.04
46		irreps ≤ 1	$\psi_{0,0} \oplus \psi_{1,0} \oplus 2\psi_{1,1}$			-	3.31 ± 0.17	3.37 ± 0.18	3.05 ± 0.09
47		irreps ≤ 3	$\psi_{0,0} \oplus \psi_{1,0} \bigoplus_{i=1}^{3} 2\psi_{1,i}$	ELU, norm-ReLU	O(2)-conv2triv	-	3.42 ± 0.03	3.41 ± 0.10	3.86 ± 0.09
48		irreps ≤ 5	$\psi_{0,0} \oplus \psi_{1,0} \bigoplus_{i=1}^{5} 2\psi_{1,i}$			-	3.59 ± 0.13	3.78 ± 0.31	4.17 ± 0.15
49		irreps ≤ 7	$\psi_{0,0} \oplus \psi_{1,0} \bigoplus_{i=1}^{7} 2\psi_{1,i}$			-	3.84 ± 0.25	3.90 ± 0.18	4.57 ± 0.27
50		Ind-irreps ≤ 1	$\text{Ind}\, \psi_0^{SO(2)} \oplus \text{Ind}\, \psi_1^{SO(2)}$			-	2.72 ± 0.05	2.70 ± 0.11	2.39 ± 0.07
51	O(2)	Ind-irreps ≤ 3	$\text{Ind}\, \psi_0^{SO(2)} \bigoplus_{i=1}^{3} \text{Ind}\, \psi_i^{SO(2)}$	ELU, Ind norm-ReLU	Ind-conv2triv	-	2.66 ± 0.07	2.65 ± 0.12	2.25 ± 0.06
52		Ind-irreps ≤ 5	$\text{Ind}\, \psi_0^{SO(2)} \bigoplus_{i=1}^{5} \text{Ind}\, \psi_i^{SO(2)}$			-	2.71 ± 0.11	2.84 ± 0.10	2.39 ± 0.09
53		Ind-irreps ≤ 7	$\text{Ind}\, \psi_0^{SO(2)} \bigoplus_{i=1}^{7} \text{Ind}\, \psi_i^{SO(2)}$			-	2.80 ± 0.12	2.85 ± 0.06	2.25 ± 0.08
54		irreps ≤ 3	$\psi_{0,0} \oplus \psi_{1,0} \bigoplus_{i=1}^{3} 2\psi_{1,i}$	ELU, gate	O(2)-conv2triv	-	2.39 ± 0.05	2.38 ± 0.07	2.28 ± 0.07
55					norm	-	2.21 ± 0.09	2.24 ± 0.06	2.15 ± 0.03
56		Ind-irreps ≤ 3	$\text{Ind}\, \psi_0^{SO(2)} \bigoplus_{i=1}^{3} \text{Ind}\, \psi_i^{SO(2)}$	ELU, Ind gate	Ind-conv2triv	-	2.13 ± 0.04	2.09 ± 0.05	2.05 ± 0.05
57					Ind-norm	-	1.96 ± 0.06	1.95 ± 0.05	1.85 ± 0.07

MNIST Variations

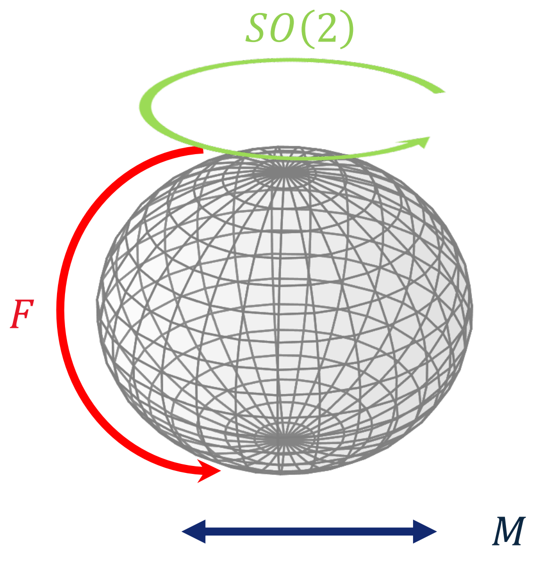
Maurice Weiler* and Gabriele Cesa*.
General E(2)-Equivariant Steerable CNNs,
Neural Information Processing Systems (NeurIPS), 2019

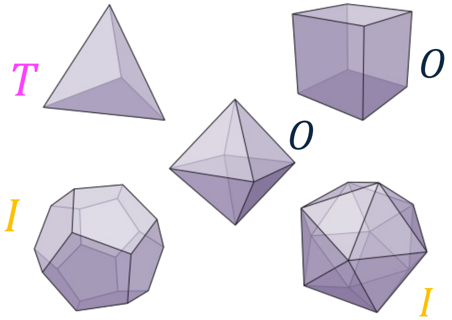
General Program to implement G-equivariance: 3D voxel data

Table 1: Rotated ModelNet10 (O(3) symmetry). $*$ indicates wider models to fix the computational cost.

G	Description	Accuracy
$\{e\}$	Conventional CNN	82.5 ± 1.4
SO(2)	Axial Symmetry	86.9 ± 1.9
$\mathrm{SO}(2) \rtimes \mathrm{F} \cong \mathrm{O}(2)$	Dihedral Symmetry	87.5 ± 0.7
$\mathrm{SO}(2) \rtimes \mathrm{M} \cong \mathrm{O}(2)$	Conical Symmetry	88.5 ± 0.8
$\mathrm{Inv} \times \mathrm{SO}(2)$	Cylindrical Symmetry	86.8 ± 0.7
$\mathrm{Inv} \times \mathrm{SO}(2) \rtimes \mathrm{F}$	Full Cylindrical Symmetry	87.0 ± 1.0
O	Octahedral Symmetry (Winkels & Cohen, 2018)	89.7 ± 0.6
I	Icosahedral Symmetry	90.0 ± 0.6
I	Icosahedral Symmetry (finite orbits basis)	88.2 ± 1.0
SO(3)	Chiral (Tensor product) (Anderson et al., 2019)	86.3 ± 1.0
SO(3)	Chiral (Gated non-linearity)(Weiler et al., 2018b)	88.8 ± 1.2
SO(3)	Chiral (Regular, $\lvert\mathcal{G}\rvert = 96$)	89.1 ± 1.2
SO(3)	Chiral (Regular, $\lvert\mathcal{G}\rvert = 192$)*	89.4 ± 1.4
SO(3)	Chiral (Quotient $\mathcal{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$, $\lvert\mathcal{X}\rvert = 30$)	89.5 ± 1.0
O(3)	Achiral (Regular, $\lvert\mathcal{G}\rvert = 120$)	89.2 ± 0.6
O(3)	Achiral (Regular, $\lvert\mathcal{G}\rvert = 144$)*	89.4 ± 0.7
O(3)	Achiral (Quotient $\mathrm{Inv} \times \mathcal{S}^2 = \mathrm{O}(3)/\mathrm{SO}(2)$, $\lvert\mathcal{X}\rvert = 60$)	88.6 ± 0.9

3D rotational symmetries





Axial rotational symmetries in 3D

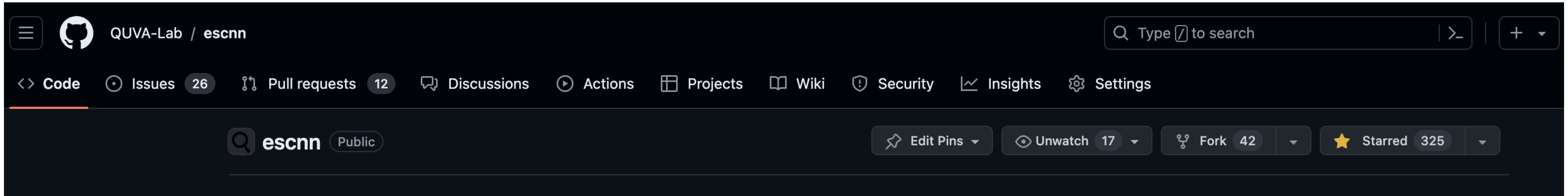
Table 2: ModelNet10 (O(2) symmetry)

G	Description	Accuracy
$\{e\}$	Conventional CNN	91.2 ± 0.5
SO(2)	Azimuthal Symmetry	91.9 ± 0.8
SO(3)	Chiral (Regular, $\lvert\mathcal{G}\rvert = 72$)	89.8 ± 0.6
O(2)	Full Azimuthal Symmetry	92.3 ± 0.4
O(3)	Achiral (Regular, $\lvert\mathcal{G}\rvert = 120$)	89.9 ± 1.0
$\mathrm{C}_2 \rtimes \mathrm{F}$	Klein Group (dihedral symmetry)	91.0 ± 0.6
	VOXNet (Maturana & Scherer, 2015)	92.0
$\mathrm{C}_2 \rtimes \mathrm{F}$	Klein Group (Worrall & Brostow, 2018)	94.2

Gabriele Cesa, Leon Lang, Maurice Weiler,
A Program to build E(n)-Equivariant Steerable CNNs,
International Conference on Representation Learning, 2022

Conclusion

- Complete theoretical description of the space of G-steerable filters
  - For any compact G and any transformation laws ρ_{in}, ρ_{out}
- Algorithm to explicitly construct the steerable convolution layers
- Generic implementation in the form of a library: github.com/QUVA-Lab/escnn



Thank you

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Semi-Direct Product group convolution

Equivariance group $(\mathbb{R}^n, +) \rtimes SO(2)$

$f: (\mathbb{R}^n, +) \rtimes SO(2) \to \mathbb{R}$

$f(t, \cdot): [0, 2\pi) \to \mathbb{R}$

$$f(t, r_\phi) = (0, \cos 1\,\phi, \sin 1\,\phi, \cos 2\,\phi, \dots) \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \\ \dots \end{pmatrix}$$

$$\bigoplus_k \begin{pmatrix} \cos k\,\phi \\ \sin k\,\phi \end{pmatrix} \qquad \bigoplus_k \begin{pmatrix} w_{2k}(t) \\ w_{2k+1}(t) \end{pmatrix}$$

Semi-Direct Product group convolution

Equivariance group $(\mathbb{R}^n, +) \rtimes SO(2)$

$f : (\mathbb{R}^n, +) \rtimes SO(2) \to \mathbb{R}$

$f(t, \cdot) : [0, 2\pi) \to \mathbb{R}$

$$[r_\theta . f](t, r_\phi) = (0, \cos 1\,\phi, \sin 1\,\phi, \cos 2\,\phi, \ldots)\,\rho(\theta) \begin{pmatrix} w_0(t) \\ w_1(t) \\ w_2(t) \\ \ldots \end{pmatrix}$$

$$\bigoplus_k \begin{pmatrix} \cos k\,\phi \\ \sin k\,\phi \end{pmatrix} \qquad \bigoplus_k \rho_k(\theta) \qquad \bigoplus_k \begin{pmatrix} w_{2k}(t) \\ w_{2k+1}(t) \end{pmatrix}$$