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Revisited Box Counting Technique in Bayesian Sense

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Abstract Fractal patterns appear in a wide variety of sources across nature. The unusual characteristic of fractals is that they entail non-integer dimension. The Box Counting method is one of the often used approach to estimate the fractal dimension of a signal. Thanks to the relationship between entropy and the fractal dimension, it is possible to employ entropy in estimating the fractal dimension. In this paper, we propose to utilize Bayesian estimate of Hartley entropy of a finite sample in fractal dimension estimation. This method was tested on artificial fractals generated by recursive expansion of appropriate matrices.

Keywords Unbiased estimation · Hartley entropy · Shannon entropy · Box Counting

1 Introduction

A fractal is an object whose so-called fractal dimension exceeds its topological dimension and its Hausdorff dimension is non-integer at the same time. The Box Counting method [6] can be used for estimating the fractal dimension due to the relationship

$$\ln C(a) = A - D_0 \ln a, \tag{1}$$

where $a > 0$ is a box size and $C(a)$ is a number of covering elements. Capacity dimension [1] D_0 is estimated as a slope of the line computed by the least

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square method. These estimates tend to be biased especially for small values of a. We propose to enhance the Box Counting method by Bayesian estimation of Hartley entropy H_0, which offers better estimate of capacity dimension D_0.

2 Multinomial Distribution and Naive Entropy Estimates

A multinomial distribution [5] model plays the main role in investigating of point set structures. Let $n \in \mathbb{N}$ be a number of distinguished events. Let $p_j > 0$ be a probability of the j^{th} event for $j = 1, ..., n$ satisfying $\sum_{j=1}^{n} p_j = 1$. Then the random variable j has a multinomial distribution $\mathrm{Mul}(p_1, ..., p_n)$. After realization of multinomial distribution sample of size $N \in \mathbb{N}$, we can count the events and obtain $N_j \in \mathbb{N}_0$ as the number of j^{th} event occurrences for $j = 1, ..., n$ satisfying $\sum_{j=1}^{n} N_j = N$. Therefore, we define the number of various events in a sample as $K = \sum_{N_j>0} 1 \leq \min(n, N)$. Revising Hartley [8] and Shannon [8] entropy definitions

$$H_0 = \ln n, \tag{2}$$

$$H_1 = -\sum_{j=1}^{n} p_j \ln p_j, \tag{3}$$

we can perform a direct but naive estimation of them as

$$\hat{H}_{0,\text{naive}} = \ln K, \tag{4}$$

$$\hat{H}_{1,\text{naive}} = -\sum_{N_j>0} \frac{N_j}{N} \ln \frac{N_j}{N}. \tag{5}$$

The main disadvantage of the naive estimates is their biasness. The random variable $K \in \{1, ..., n\}$ is capped by n, which causes $\mathrm{E}\hat{H}_{0,\text{naive}} = \mathrm{E} \ln K < \mathrm{E} \ln n = \ln n = H_0$. Hence, the naive estimate of Hartley entropy $\hat{H}_{0,\text{naive}}$ is negatively biased. On the other hand, the traditional Box Counting Technique is based on this estimate. There we plot the logarithm of the covering element number $C(a) \in \mathbb{N}$ against the logarithm of the covering element size $a > 0$ and then estimate their dependency in the linear form $\ln C(a) = A_0 - \hat{D}_{0,\text{naive}} \ln a$. Recognizing equivalence $C(a) = K$ leads to $\ln C(a) = \ln K = \hat{H}_{0,\text{naive}}$ and then $\hat{H}_{0,\text{naive}} = A_0 - \hat{D}_{0,\text{naive}} \ln a$. Defining $\hat{D}_{0,\text{naive}}$ as an estimate of capacity dimension and recognizing the occurrence of $\hat{H}_{0,\text{naive}}$ in the Box Counting procedure [6], we are not surprised to be victims of the bias of Hartley entropy estimate.

A similar situation is the case of Shannon entropy estimation. There are several approaches how to decrease the bias of $\hat{H}_{1,\text{naive}}$ to be closer to a theoretical value of Shannon entropy H_1. Miller [4] modified the naive estimate $\hat{H}_{1,\text{naive}}$ using a first-order Taylor expansion resulting in

$$\hat{H}_{1,\text{M}} = \hat{H}_{1,\text{naive}} + \frac{K-1}{2N}. \tag{6}$$

Lately, Harris [4] improved the formula to

$$\hat{H}_{1,\mathrm{H}} = \hat{H}_{1,\mathrm{naive}} + \frac{K-1}{2N} - \frac{1}{12N^2}\left(1 - \sum_{p_j>0}\frac{1}{p_j}\right). \tag{7}$$

Finally, we can estimate the capacity and information dimensions according to relation

$$\hat{H}_d = A_d - \hat{D}_d \ln a, \tag{8}$$

where $\hat{H}_d$ is any estimate of H_d. Therefore, we can also estimate Hausdorff dimension D_H using inequalities $D_1 \le D_\mathrm{H} \le D_0$ under the assumption that $\hat{D}_1 \le D_\mathrm{H} \le \hat{D}_0$ for any "good" estimates $\hat{D}_0$, $\hat{D}_1$ of capacity and information dimensions, respectively. The next section is oriented to Bayesian estimation of H_0 and H_1, which are essential for evaluating $\hat{D}_0$ and $\hat{D}_1$.

3 Bayesian Estimation of Hartley Entropy

We suppose Dirichlet distribution [5] of a random vector $\mathbf{p} = (p_1, ..., p_n)$ satisfying $p_j \ge 0$, $\sum_{j=1}^{n} p_j = 1$, with $\alpha_j = \alpha^* > 0$. Using properties of multinomial and its conjugate distribution — the Dirichlet distribution, we can calculate probability estimate $\hat{\mathrm{p}}(K|n,N)$ of the random variable $K \in \mathbb{N}$ for $K \le \min(n,N)$ as

$$\begin{aligned}\hat{\mathrm{p}}(K \mid n, N) &= \mathrm{prob}\left(\sum_{N_j>0} 1 = K \,\middle|\, n, \sum_{j=1}^{n} N_j = N\right)\\ &= \binom{n}{K}\frac{\Gamma(N+1)\Gamma(n\alpha^*)}{\Gamma(N+n\alpha^*)}\sum_{\mathbf{N}\in\mathbb{D}_{K,N}}\prod_{j=1}^{K}\frac{\Gamma(N_j+\alpha^*)}{\Gamma(N_j+1)\Gamma(\alpha^*)}.\end{aligned} \tag{9}$$

Derivation of (9) is included in the Appendix 8.1. When $N \ge K+2$, we can calculate

$$S_{K,N} = \sum_{n=K}^{\infty} \hat{\mathrm{p}}\,(K \mid n, N). \tag{10}$$

When the number of events is constrained as $n \le n_{\max}$, we apply an alternative formula

$$S^*_{K,N} = \sum_{n=K}^{n_{\max}} \hat{\mathrm{p}}\,(K \mid n, N). \tag{11}$$

Convergence of the infinite series (10) is proved in the Appendix 8.2. Having a knowledge of K, N where $N \ge K+2$, we can calculate a Bayesian density

$$\hat{\mathrm{p}}\,(n \mid K, N) = \frac{\hat{\mathrm{p}}\,(K \mid n, N)}{S_{K,N}}, n \ge K \tag{12}$$

Thereafter, Bayesian estimate of Hartley entropy comes out as

$$\begin{aligned}\hat{H}_{0,\mathrm{Bayes}} &= \mathrm{E}H_0 = \sum_{n=K}^{\infty} \hat{\mathrm{p}}\left(n \mid K, N\right) \ln n = \sum_{n=K}^{\infty} \frac{\hat{\mathrm{p}}\left(K \mid n, N\right) \ln n}{S_{K,N}} \\ &= \frac{\sum_{n=K}^{\infty} \hat{\mathrm{p}}\left(K \mid n, N\right) \ln n}{\sum_{n=K}^{\infty} \hat{\mathrm{p}}\left(K \mid n, N\right)} > \ln K,\end{aligned} \tag{13}$$

which is a convergent sum as well. We gain an equivalent formula by substituting $n = K + j$

$$\hat{H}_{0,\mathrm{Bayes}} = \frac{\sum_{j=0}^{\infty} b_j \ln\left(K+j\right)}{\sum_{j=0}^{\infty} b_j}, \tag{14}$$

where

$$b_j = \binom{K+j}{j} \frac{\mathrm{B}\left((K+j)\alpha^*, N\right)}{\mathrm{B}\left(K\alpha^*, N\right)}. \tag{15}$$

Convergence of the sums in (13) is proved in Appendix 8.2. Particular coefficients b_j can also be generated recursively

$$\begin{aligned}b_0 &= 1 \\ b_j &= \frac{K+j}{j} \frac{\Gamma((K+j)\alpha^*)}{\Gamma((K+j)\alpha^* - \alpha^*)} \frac{\Gamma(N + (K+j)\alpha^* - \alpha^*)}{\Gamma(N + (K+j)\alpha^*)} b_{j-1} \\ b_j &= b_{j-1} \frac{K+j}{j} \prod_{u=0}^{N-1} \left(1 - \frac{\alpha^*}{(K+j)\alpha^* + u}\right).\end{aligned} \tag{16}$$

4 Bayesian Estimation of Shannon Entropy

In the case when the number of events n is known, we perform Bayesian estimation of Shannon entropy for arbitrary $\alpha_j = \alpha^* > 0$ as

$$-\sum_{i=1}^{M} \frac{\Gamma(N+\alpha)}{\Gamma(n_i+\alpha_i)} \frac{\Gamma(n_i+\alpha_i+1)}{\Gamma(N+\alpha+1)} \left(\psi^{(0)}(n_i+\alpha_i+1) - \psi^{(0)}(N+\alpha+1)\right) \tag{17}$$

$$\begin{aligned}\hat{H}_{1,\mathrm{n}} &= \mathrm{E}H_1(K=n) \\ &= -\sum_{j=1}^{n} \left(\frac{N_j+\alpha^*}{N+n\alpha^*} \left(\psi(N_j+\alpha^*+1) - \psi(N+n\alpha^*+1)\right)\right),\end{aligned} \tag{18}$$

where ψ is digamma function. However, when the number of events n is unknown, we can use K as a lower estimate of n and perform the final Bayesian estimation as

$$\hat{H}_{1,\mathrm{Bayes}} = \sum_{n=K}^{\infty} \mathrm{p}\left(n \mid K, N\right) \hat{H}_{1,\mathrm{n}}, \tag{19}$$

which is also a convergent sum for $N \geq K + 2$.

Substituting $n = K + j$, we obtain an adequate formula

$$\hat{H}_{1,\mathrm{Bayes}} = \frac{\sum_{j=0}^{\infty} b_j \hat{H}_{1,K+j}}{\sum_{j=0}^{\infty} b_j}. \tag{20}$$

Unfortunately, asymptotic expansion of (20) depends on individual frequencies N_j. But $\hat{H}_{1,n} \leq \ln n$, hence $\hat{H}_{1,K+j} \leq \ln K + j$, which implies the convergence of $\sum_{j=0}^{\infty} b_j \hat{H}_{1,K+j}$ based on majority rule and (14).

5 Revisited Box Counting Method

Let $\mathbb{F} \subset \mathbb{R}^m$ be a set of N points placed into m-dimensional rectangular grid of element size $a > 0$. Let $\hat{H}_{0,\mathrm{Bayes}}$ be an unbiased estimate of Hartley entropy H_0. Fitting the linear model

$$\hat{H}_{0,\mathrm{Bayes}} = A - \hat{D}_0 \ln a \tag{21}$$

via the method of least squares is called Revisited Box Counting.

Revisited Box Counting can be modified by using $\hat{H}_{1,\mathrm{Bayes}}$ instead of $\hat{H}_{0,\mathrm{Bayes}}$ which comes to estimation of information dimension [7] according to

$$\hat{H}_{1,\mathrm{Bayes}} = A - \hat{D}_1 \ln a. \tag{22}$$

6 Experimental Part

The Revisited Box Counting technique will be tested on models of deterministic self-similar 2D fractal sets. They are generated by recursive expansion of binary matrix $\mathbb{G}_{u,v} \in \{0,1\}^{v \times v}$, where u a is the number of non-zero elements (units), $v > 1$ is a matrix dimension, and $v < u < v^2$.

Recursive expansion of $\mathbb{G}_{u,v}$ generates a binary matrix which represents fractal set $\mathbb{F}_{u,v}$ of a similarity dimension $D_\mathrm{S} = D_\mathrm{H} = D_0 = D_1 = \frac{\log u}{\log v}$. Depth h of recursion depends on v and should be appropriate to computer memory size. The structures involved in the research are depicted in Fig. 1

At first, adequate point sets of given depth h were generated. Then, they were randomly rotated around the origin, and finally they were randomly shifted. Afterwards, a grid of size a was put on the data points and entropy estimates were calculated. Due to physical interpretation of entropy, the estimates were averaged over 20 realizations and mean values of entropy were calculated.

The relationship between $\hat{D}_\mathrm{naive}$ and optimum value of α was studied on the

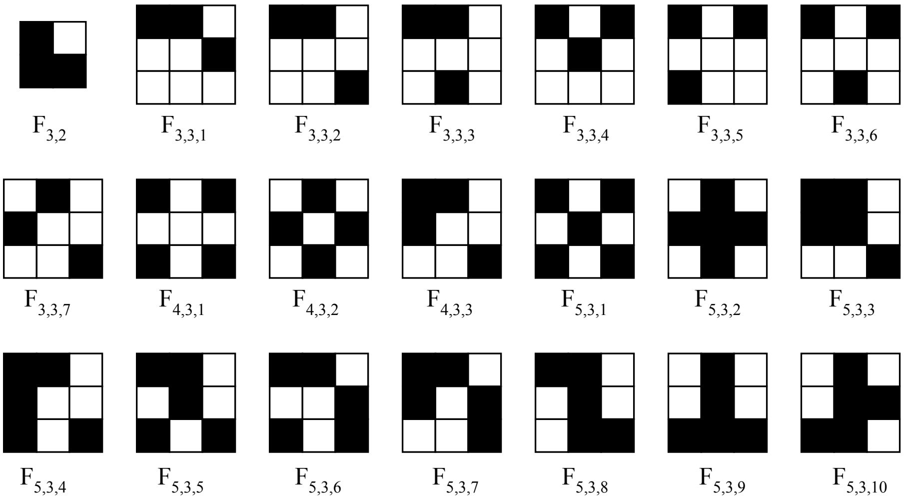

Fig. 1 Table of the fractals involved in the research

aforementioned fractals for the grid of size $a = 12, 16, 20, ..., 480, 500$. The results of optimization are collected in Tab. 1. We suppose, the relationship can be approximated by linear, exponential, or power model respectively as

$$\begin{aligned} \alpha &= A + B\hat{D}_{0,\text{naive}}, \\ \ln\alpha &= A + B\hat{D}_{0,\text{naive}}, \\ \ln\alpha &= A + B\ln\hat{D}_{0,\text{naive}}. \end{aligned} \tag{23}$$

Linear regression was used for the estimation of unknown parameters A and B. The values are included in Tab. 2 together with correlation coefficient r. The exponential model had the best correlation and will be used for corrected estimation. But the differences among models are not statistically significant. Using exponential model, we calculated α_{model} from $\hat{D}_{0,\text{naive}}$, then recalculated $\hat{D}_0$, and tested hypothesis $\text{H}_0 : \text{E}\hat{D}_0 = D_0$ via two-sided t-test. The resulting values are also collected in Tab. 1 as $\hat{D}_0$ and p_{value}. In this case of multiple hypothesis testing we had to apply False Discovery Rate (FDR) [2] methodology on critical level $p_{crit} = 0.05$. We were not able to reject any hypotheses and therefore the improved $\hat{D}_0$ estimate was not biased in our experiments.

7 Conclusion

In this paper we developed the Bayesian estimator $\hat{H}_{0,\text{Bayes}}$ of Hartley entropy for Dirichlet prior. This estimate enables to estimate $\hat{D}_0$ with suppressed bias in comparison with naive box-counting estimate. The novel methodology is based on the box-counting estimate $\hat{D}_{0,\text{naive}}$ which helps to specify the Dirichlet prior and finally reestimate the capacity dimension. This procedure is rec-

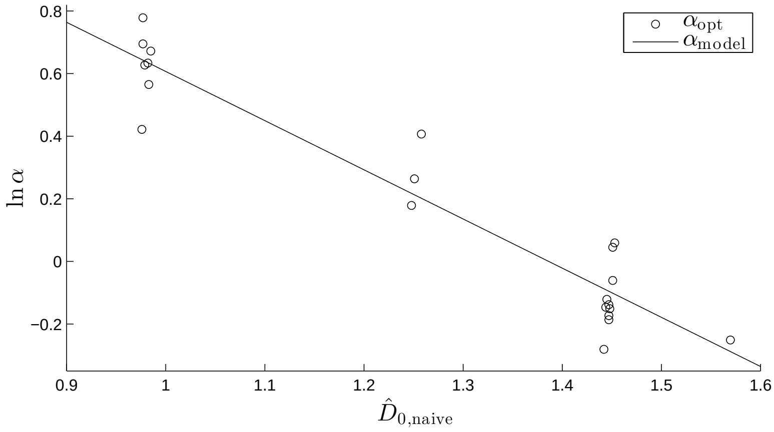

Fig. 2 Optimum α values of exponential model and its linear regression

Table 1 Optimum α values and their exponential model

Fractal	h	D_0	$\hat{D}_{0,\mathrm{naive}}$	α_{opt}	α_{model}	p_{value}
$F_{3,2}$	11	1.585	1.567	0.778	0.749	0.392
$F_{3,3,1}$	7	1	0.982	1.885	1.886	0.995
$F_{3,3,2}$	7	1	0.977	2.178	1.901	0.131
$F_{3,3,3}$	7	1	0.977	2.004	1.901	0.667
$F_{3,3,4}$	7	1	0.985	1.958	1.878	0.743
$F_{3,3,5}$	7	1	0.976	1.525	1.904	0.129
$F_{3,3,6}$	7	1	0.983	1.760	1.883	0.698
$F_{3,3,7}$	7	1	0.979	1.873	1.895	0.929
$F_{4,3,1}$	7	1.262	1.251	1.302	1.236	0.789
$F_{4,3,2}$	7	1.262	1.248	1.196	1.242	0.850
$F_{4,3,3}$	7	1.262	1.258	1.502	1.223	0.283
$F_{5,3,1}$	7	1.465	1.447	0.871	0.908	0.653
$F_{5,3,2}$	7	1.465	1.447	0.830	0.908	0.427
$F_{5,3,3}$	7	1.465	1.451	0.941	0.903	0.544
$F_{5,3,4}$	7	1.465	1.445	0.886	0.911	0.720
$F_{5,3,5}$	7	1.465	1.444	0.864	0.913	0.367
$F_{5,3,6}$	7	1.465	1.451	1.046	0.903	0.185
$F_{5,3,7}$	7	1.465	1.453	1.061	0.900	0.019
$F_{5,3,8}$	7	1.465	1.447	0.841	0.908	0.122
$F_{5,3,9}$	7	1.465	1.442	0.755	0.916	0.009
$F_{5,3,10}$	7	1.465	1.448	0.860	0.907	0.186

Table 2 Comparison of the models fitting the relationship between $\hat{D}_{0,\mathrm{naive}}$ and α

model	A	B	r
linear	3.905	–2.066	–0.9542
exponential	2.178	–1.571	–0.9553
power	0.605	–1.892	–0.9524

ommended for 2D structures with $1 \leq \hat{D}_0 \leq 1.6$ and can be easily extended for information dimension D_1 estimation and higher dimensions.

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References

1. Harris, B.: The statistical estimation of entropy in the non-parametric case. Wisconsin University Madison, Mathematics Research Center, p. 318 (1975).
2. Wendel, J.: Note on the gamma function. Amer. Math. Monthly 55, 563-564 (1948).
3. Renyi, A., *On measures of entropy and information.* In Proc. Fourth Berkeley Symp. Math. Stat. Prob., 1960, volume 1, page 547, Berkeley, 1961. University of California Press.
4. Baker, G. L. and Gollub, J. B. *Chaotic Dynamics: An Introduction, 2nd ed.* Cambridge, England: Cambridge University Press, 1996.
5. Ott, E. *Chaos in Dynamical Systems.* New York: Cambridge University Press, p. 79, 1993.
6. Mandelbrot, B. B. *The Fractal Geometry of Nature.* W.H. Freeman and Company, 1982.
7. S. Kotz, N. Balakrishnan, and N. L. Johnson. *Continuous Multivariate Distributions. Volume 1: Models and Applications.* New York: Wiley, 2000. ISBN 0-471-18387-3.
8. Bromwich, T. J. I'A. and MacRobert, T. M. *An Introduction to the Theory of Infinite Series, 3rd ed.* New York: Chelsea, p. 39, 1991.

8 Appendix

8.1 Derivation of $\hat{\mathrm{p}}(K|n,N)$ in (9)

Let $\mathbb{Q}_n = \{\mathbf{q} \in (\mathbb{R}_0^+)^n | \sum_{j=1}^n q_j = 1\}$ be a support set of a Dirichlet-distributed random variable $\mathbf{p} \in \mathbb{Q}_n$ with parameters α_j, for $j = 1, ..., n$. The conditional probability of an integer K satisfying $1 \leq K \leq \min(n, N)$ is

$$\mathrm{p}(K \mid n, N) = \mathrm{prob}\left(\sum_{N_j > 0} 1 = K \,\middle|\, n, \sum_{j=1}^{n} N_j = N \right). \tag{24}$$

The vector of N_j can be reorganized to begin with positive values:

$$\mathrm{p}(K \mid n, N) = \binom{n}{K} \mathrm{prob}\left(\forall j = 1, ..., n : N_j > 0 \Leftrightarrow j \leq K \,\middle|\, n, \sum_{j=1}^{K} N_j = N \right). \tag{25}$$

Let $\mathbb{D}_{K,N} = \{\mathbf{x} \in \mathbb{N}^K | \sum_{j=1}^K x_j = N\}$ be the domain of $\mathbf{N} = (N_1, ..., N_K) \in \mathbb{D}_{K,N}$. Using the mean value of a multinomial distribution over $\mathbb{Q}_n$, we obtain an unbiased estimate of $\mathrm{p}(K \mid n, N)$ as

$$\begin{aligned}\hat{\mathrm{p}}(K \mid n, N) &= \binom{n}{K}\mathrm{E}\left(\sum_{\mathbf{N}\in\mathbb{D}_{K,N}} \binom{N}{N_1, ..., N_K} \prod_{j=1}^{K} p_j^{N_j} \prod_{j=k+1}^{n} p_j^0\right) \\ &= \binom{n}{K} \sum_{\mathbf{N}\in\mathbb{D}_{K,N}} \binom{N}{N_1, ..., N_K}\mathrm{E}\left(\prod_{j=1}^{K} p_j^{N_j}\right).\end{aligned} \tag{26}$$

Using the generalized Beta function

$$B(\mathbf{x}) = \int_{\mathbf{p}\in\mathbb{Q}_m} \prod_{j=1}^{m} p_j^{x_j-1}\mathrm{d}\mathbf{p} = \frac{\prod_{j=1}^{m} \Gamma(x_j)}{\Gamma(\sum_{j=1}^{m} x_j)}, \tag{27}$$

we can calculate

$$\begin{aligned}\mathrm{E}\left(\prod_{j=1}^{K} p_j^{N_j}\right) &= \frac{\int_{\mathbf{p}\in\mathbb{Q}_n} B(\boldsymbol{\alpha})^{-1} \prod_{j=1}^{K} p_j^{N_j+\alpha_j-1} \prod_{j=K+1}^{n} p_j^{\alpha_j-1}\mathrm{d}\mathbf{p}}{\int_{\mathbf{p}\in\mathbb{Q}_n} B(\boldsymbol{\alpha})^{-1} \prod_{j=1}^{n} p_j^{\alpha_j-1}\mathrm{d}\mathbf{p}} \\ &= \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)} \prod_{j=1}^{K} \frac{\Gamma(N_j+\alpha_j)}{\Gamma(\alpha_j)},\end{aligned} \tag{28}$$

where α is the sum of all α_j. Therefore,

$$\begin{aligned}\hat{\mathrm{p}}(K \mid n, N) &= \binom{n}{K} \sum_{\mathbf{N}\in\mathbb{D}_{K,N}} \frac{N!}{\prod_{j=1}^{K} N_j!} \frac{\Gamma(\alpha)}{\Gamma(N+\alpha)} \frac{\prod_{j=1}^{K} \Gamma(N_j+\alpha_j)}{\prod_{j=1}^{K} \Gamma(\alpha_j)} \\ &= \binom{n}{K} \frac{\Gamma(N+1)\Gamma(\alpha)}{\Gamma(N+\alpha)} \sum_{\mathbf{N}\in\mathbb{D}_{K,N}} \prod_{j=1}^{K} \frac{\Gamma(N_j+\alpha_j)}{\Gamma(N_j+1)\Gamma(\alpha_j)}\end{aligned} \tag{29}$$

In this particular paper, we assume $\alpha_j = \alpha^*, \forall j = 1, ..., n$ which results in a simpler form of Equation 29

$$\hat{\mathrm{p}}(K \mid n, N) = \binom{n}{K} \frac{\Gamma(N+1)\Gamma(n\alpha^*)}{\Gamma(N+n\alpha^*)} \sum_{\mathbf{N}\in\mathbb{D}_{K,N}} \prod_{j=1}^{K} \frac{\Gamma(N_j+\alpha^*)}{\Gamma(N_j+1)\Gamma(\alpha^*).} \tag{30}$$

8.2 Convergence of $\sum_{j=0}^{\infty} b_j \ln(K+j)$ in (14) and $\sum_{j=0}^{\infty} b_j$ in (10)

The ratio of coefficients b_j could be expressed as:

$$\begin{aligned}q_j &= \frac{b_j}{b_{j-1}} \frac{\ln(K+j)}{\ln(K+j-1)} \\ &= \frac{(K+j)}{j} \frac{\ln(K+j)}{\ln(K+j-1)} \frac{\Gamma((K+j)\alpha^*)}{\Gamma((K+j-1)\alpha^*)} \frac{\Gamma(N+(K+j-1)\alpha^*)}{\Gamma(N+(K+j)\alpha^*)}.\end{aligned} \tag{31}$$

Starting with inequality proved by Wendel [9]:

$$\forall d \in [0;1], \forall x > 0 : \frac{\Gamma(x+d)}{\Gamma(x)} \leq x^d; \tag{32}$$

that can be generalized for $\delta = D + d$ where $D \in \mathbb{N}_0, d \in [0;1)$ as

$$\frac{\Gamma(x+\delta)}{\Gamma(x)} \leq x^d \prod_{i=0}^{D-1} (x+i+d). \tag{33}$$

We should see the similarity between $\alpha^* = A + a$, where $A \in \mathbb{N}_0, a \in [0;1)$, and δ leading to

$$\begin{aligned} q_j &= \frac{b_j}{b_{j-1}} \frac{\ln(K+j)}{\ln(K+j-1)} \\ &\leq \frac{K+j}{j} \frac{\ln(K+j)}{\ln(K+j-1)} \left(\frac{(K+j-1)\alpha^*}{(K+j-1)\alpha^*+N} \right)^a \\ &\cdot \prod_{i=0}^{A-1} \frac{(K+j-1)\alpha^*+i+a}{(K+j-1)\alpha^*+i+a+N} \end{aligned} \tag{34}$$

$$\begin{aligned} q_j &= \frac{b_j}{b_{j-1}} \frac{\ln(K+j)}{\ln(K+j-1)} \\ &\leq \frac{K+j}{j} \frac{\ln(K+j)}{\ln(K+j-1)} \left(\frac{(K+j-1)\alpha^*}{(K+j-1)\alpha^*+N} \right)^a \end{aligned} \tag{35}$$

The Raabe criterion [3] will state series of positive members $\sum_{n=0}^{\infty} a_n$ as convergent if exists $L = \lim_{n\to\infty} n \left(\frac{a_n}{a_{n+1}} - 1 \right)$ satisfying $L > 1$. Then we can calculate

$$\begin{aligned} L &= \lim_{j\to\infty} j \left(\frac{b_{j-1}}{b_j} \frac{\ln(K+j-1)}{\ln(K+j)} - 1 \right) \\ &\geq \lim_{j\to\infty} j \left(\frac{j}{K+j} \frac{\ln(K+j-1)}{\ln(K+j)} \left(\frac{(K+j-1)\alpha^*+N}{(K+j-1)\alpha^*} \right)^a - 1 \right). \end{aligned} \tag{36}$$

Substitution $x = K + j$ leads to

$$L = \lim_{x\to\infty} (x-K) \left(\frac{x-K}{x} \frac{\ln(x-1)}{\ln(x)} \left(\frac{(x-1)\alpha^*+N}{(x-1)\alpha^*} \right)^a - 1 \right), \tag{37}$$

and finally

$$L = -K + \lim_{x\to\infty} \left(x \left(1 + \frac{N-\alpha^*}{x\alpha^*} \right)^a - x \left(1 - \frac{1}{x} \right)^a \right). \tag{38}$$

Substituting $h = x^{-1} \to 0^+$ and applying l'Hospital rule, we obtain

$$L = -K + \lim_{h\to 0^+} \frac{\left(1 + \frac{N-\alpha^*}{\alpha^*} h \right)^a - (1-h)^a}{h} = N - K. \tag{39}$$

Thus the series $\sum_{j=0}^{\infty} b_j \ln(K+j)$ converges absolutely for $K \leq N-2$ because $L = N - K > 1$. According to majority rule, the series $\sum_{j=0}^{\infty} b_j = \sum_{n=K}^{\infty} \hat{\mathrm{p}}\,(K \mid n, N)$ converges as well.