REVISITED BOX COUNTING TECHNIQUE IN BAYESIAN SENSE

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Abstract:

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1 Introduction

Fractals are objects, which exceed its topological dimensions and its Hausdorff dimension is not integer. Box Counting method is used for estimating fractal dimension in the form

$$\ln C(a) = A - D_0 \ln a, \tag{1}$$

where $a > 0$ is box size and $C(a)$ is number of covering elements. Capacity dimension D_0 is estimated as slope of the line which is generated by least square method. These estimate is biased especialy for small values of a. Box Counting method can be improved by Bayesian estimation of Hartley entropy H_0, which offers better estimate of capacity dimension D_0.

2 Multinomial Distribution and Naive Entropy Estimates

Multinomial distrubution model plays main role in investigation of point set structures. Let $n \in \mathbb{N}$ be number of distinguish events. Let $p_j > 0$ be probability of j^{th} event for $j = 1, ..., n$ satisfying $\sum_{j=1}^{n} p_j = 1$. Then random variable j has multinomial distribution $\text{Mul}(p_1, ..., p_n)$. After realization of multinomial distribution sample of size $N \in \mathbb{N}$, we can count the events and obtain $N_j \in \mathbb{N}_0$ as number of j^{th} event occurences for $j = 1, ..., n$ satisfying $\sum_{j=1}^{n} N_j = N$. Therefore, we define number of various events in sample as $K = \sum_{N_j>0} 1 \leq \min(n, N)$. Remembering Hartley and Shannon entropy definition as

$$H_0 = \ln n, \tag{2}$$

$$H_1 = -\sum_{j=1}^{n} p_j \ln p_j, \tag{3}$$

we can perform direct but naive estimation of them as

$$\hat{H}_{0,\text{naive}} = \ln K, \tag{4}$$

$$\hat{H}_{1,\text{naive}} = -\sum_{N_j>0} \frac{N_j}{N} \ln \frac{N_j}{N}. \tag{5}$$

The main disadvantage of naive estimates is their biasness. Random variable $K = \{1, ..., n\}$ is upper constrained by n, then $\mathrm{E}\hat{H}_{0,\text{naive}} = \mathrm{E} \ln K < \mathrm{E} \ln n = \ln n = H_0$. Therefore, naive estimate of Hartley entropy $\hat{H}_{0,\text{naive}}$ is negative biased. On the other hand, traditional Box Counting Technique is based on this estimate because we plot logarithm of covering element number $C(a) \in \mathbb{N}$ against logarithm of covering element size $a > 0$ and then estimate their dependency in linear form $\ln C(a) = A_0 - \hat{D}_{0,\text{naive}} \ln a$. Recognizing equivalence $C(a) = K$, we obtain $\ln C(a) = \ln K = \hat{H}_{0,\text{naive}}$ and then $\hat{H}_{0,\text{naive}} = A_0 - \hat{D}_{0,\text{naive}} \ln a$. Defining $\hat{D}_{0,\text{naive}}$ as estimate of capacity dimension and recognizing the occurence of $\hat{H}_{0,\text{naive}}$ in Box Counting procedure, we are not suprised to be victims of the bias of Hartley entropy estimate.

Similar situation is the case of Shannon entropy estimation. There are several approaches how to decrease the bias of $\hat{H}_{1,\text{naive}}$ to be closer to Shannon entropy H_1. Miller [2] modified naive estimate $\hat{H}_{1,\text{naive}}$ using first order Taylor expansion, which produces

$$\hat{H}_{1,\text{M}} = \hat{H}_{1,\text{naive}} + \frac{K-1}{2N}. \tag{6}$$

Lately, Harris [2] improved the formula to

$$\hat{H}_{1,\text{H}} = \hat{H}_{1,\text{naive}} + \frac{K-1}{2N} - \frac{1}{12N^2}\left(1 - \sum_{p_j>0}\frac{1}{p_j}\right) \tag{7}$$

Finally, we can estimate capacity and information dimension according to relation

$$\hat{H}_d = A_d - \hat{D}_d \ln a \tag{8}$$

where $\hat{H}_d$ is any estimate of H_d. Therefore, we can also estimate Hausdorff dimension D_H using inequalities $D_1 \leq D_\text{H} \leq D_0$ and then also supposing $\hat{D}_1 \leq D_\text{H} \leq \hat{D}_0$ for any "good" estimates $\hat{D}_0$, $\hat{D}_1$ of capacity and information dimensions. Next section is oriented to Bayesian estimation of H_0, H_1 for $\hat{D}_0$ and $\hat{D}_1$ evaluations.

3 Bayesian Estimation of Hartley Entropy

We a suppose Dirichlet distribution of random vector $\vec{p} = (p_1, ..., p_n)$ satisfying $p_j \geq 0$, $\sum_{j=1}^n p_j = 1$, with $\alpha_j =]alpha^* \geq 0$. Using properties of multinomial and its conjugate distribution — the Dirichlet distributions, we can calculate probability $\text{p}(K|n,N)$ of the random variable $K \in \mathbb{N}$ for $K \leq \min(n,N)$ as

$$\hat{\text{p}}(K \mid n,N) = \text{prob}\left(\sum_{N_j>0} 1 = K \,\middle|\, n, \sum_{j=1}^n N_j = N\right) = \binom{n}{K}\frac{\Gamma(N+1)\Gamma(n\alpha^*)}{\Gamma(N+n\alpha^*)}\sum_{\vec{N}\in\mathbb{D}_{K,N}}\prod_{j=1}^K \frac{\Gamma(N_j+\alpha^*)}{\Gamma(N_j+1)\Gamma(\alpha^*)}. \tag{9}$$

Derivation of (9) is included in the Appendix A.1. When $N \geq K+2$, we can calculate

$$S_{K,N} = \sum_{n=K}^{\infty} \hat{\text{p}}\,(K \mid n,N). \tag{10}$$

When the number of events is constrained as $n \leq n_{\max}$, we apply an alternative formula

$$S^*_{K,N} = \sum_{n=K}^{n_{\max}} \hat{\text{p}}\,(K \mid n,N). \tag{11}$$

Convergence of the infinite series (10) is proved in the Appendix **??**. Having a knowledge of K, N where $N \geq K+2$, we can calculate a Bayesian density

$$\hat{\text{p}}\,(n \mid K,N) = \frac{\hat{\text{p}}\,(K \mid n,N)}{S_{K,N}}, \tag{12}$$

for $n \geq K$. Therefore, Bayesian estimate of Hartley entropy is

$$\hat{H}_{0,\text{Bayes}} = \text{E}H_0 = \sum_{n=K}^{\infty}\hat{\text{p}}\,(n \mid K,N)\ln n = \sum_{n=K}^{\infty}\frac{\hat{\text{p}}\,(K \mid n,N)\ln n}{S_{K,N}} = \frac{\sum_{n=K}^{\infty}\hat{\text{p}}\,(K \mid n,N)\ln n}{\sum_{n=K}^{\infty}\hat{\text{p}}\,(K \mid n,N)} > \ln K \tag{13}$$

which is also a convergent sum. Substituting $n = K+j$ we gain equivalent formula

$$\hat{H}_{0,\text{Bayes}} = \frac{\sum_{j=0}^{\infty} b_j \ln(K+j)}{\sum_{j=0}^{\infty} b_j} \tag{14}$$

where $b_j = \binom{K+j}{j}\frac{\text{B}((K+j)\alpha^*,N)}{\text{B}(K\alpha^*,N)}$. Particular coeficients b_j can also be generated by recursive formula

$$\begin{aligned} b_0 &= 1 \\ b_j &= \frac{(K+j)}{j}\,\frac{\Gamma((K+j)\alpha^*)}{\Gamma((K+j)\alpha^*-\alpha^*)}\,\frac{\Gamma(N+(K+j)\alpha^*-\alpha^*)}{\Gamma(N+(K+j)\alpha^*)}b_{j-1}. \end{aligned} \tag{15}$$

Asymptotic properties of the Bayesian estimate for $N \to +\infty$ can be investigated via limits

$$
\begin{aligned}
&\lim_{N\to+\infty} \hat{H}_{0,\text{Bayes}} = \ln K, \\
&\lim_{N\to+\infty} (\hat{H}_{0,\text{Bayes}} - \ln K)N = K(K+1)\ln(1+1/K), \\
&\lim_{N\to+\infty} \left(\hat{H}_{0,\text{Bayes}} - \ln K - \frac{K(K+1)\ln(1+1/K)}{N}\right) N^2 = \\
&\qquad \frac{1}{2}\left(K(K+2)(K+1)\left(\ln(K+2) - \ln(K) - 2K\ln(K+1) + K\ln(K+2) + K\ln(K)\right)\right),
\end{aligned}
\tag{16}
$$

Therefore

$$
\begin{aligned}
\hat{H}_{0,\text{Bayes}} \approx{}& \ln K + \frac{K(K+1)\ln(1+1/K)}{N} + \\
& \frac{\left(K(K+2)(K+1)\left(\ln(K+2) - \ln(K) - 2K\ln(K+1) + K\ln(K+2) + K\ln(K)\right)\right)}{2N^2}
\end{aligned}
\tag{17}
$$

When K is also large, we can roughly approximate Hartley entropy as

$$
\hat{H}_{0,\text{Bayes}} \approx \ln K + \frac{K+1}{N} \tag{18}
$$

which is very similar to Miller correction (6) in the case of Shannon entropy estimation. Meanwhile formula (13) represents Bayesian estimate of H_0, formulas (4), (18), and (17) are approximations of zero, first, and second order.

Formula (14) can be also expanded to the form

$$
\hat{H}_{0,\text{Bayes}} = \ln K + \sum_{j=1}^{\infty} \frac{1}{j}\varphi(K)\left(\frac{K}{N}\right)^j \tag{19}
$$

where $\varphi(K) > 1$ for all $K \in \mathbb{N}$ and $\lim_{K\to\infty}\varphi(K) = 1$. Therefore, we obtain lower estimate

$$
\hat{H}_{0,\text{Bayes}} > \ln K + \sum_{j=1}^{\infty} \frac{1}{j}\left(\frac{K}{N}\right)^j = \ln K - \ln(1 - \frac{K}{N}) = H_{0,\text{low}} \tag{20}
$$

which exists for $K < N$.

4 Bayesian Estimation of Shannon Entropy

In the case when the number of events n is known, we can perform Bayesian estimation of Shannon entropy as

$$
\hat{H}_{1,\text{n}} = \mathrm{E}H_1(K=n) = -\sum_{j=1}^{n}\left(\frac{N_j+1}{N+n}\left(\psi(N_j+2) - \psi(N+n+1)\right)\right) \tag{21}
$$

where ψ is digamma function. But when the number of events n is unkonown, we can use K as lower estimate of n and perform final Bayesian estimation as

$$
H_{1,\text{Bayes}} = \sum_{n=K}^{\infty} \mathrm{p}\left(n \mid K, N\right) H_{1,\text{n}} \tag{22}
$$

which is also convergent sum for $N \geq K+2$.

Substituing $n = K + j$ we obtain adequating formula.

$$
\hat{H}_{1,\text{Bayes}} = \frac{\sum_{j=0}^{\infty} b_j H_{1,K+j}}{\sum_{j=0}^{\infty} b_j} \tag{23}
$$

Asymptotic expansion of (23) unfortunately depends on individual frequences N_j.

5 Revisited Box Counting Method

Let $\mathbb{F} \subset \mathbb{R}^m$ be set of N points placed into m-dimensional rectangular grid of element size $a > 0$. Let $\hat{H}_{0,\mathrm{Bayes}}$ be unbiased estimate of Hartley entropy H_0. Fitting linear model

$$\hat{H}_{0,\mathrm{Bayes}} = A - D_0 \ln a \tag{24}$$

via method of least squares is called Revisited Box Counting.

Revisited Box Counting can be modified by:

- using $\hat{H}_{1,\mathrm{Bayes}}$ instead of $\hat{H}_{0,\mathrm{Bayes}}$ comes to estimation of information dimension according to model

$$\hat{H}_{1,\mathrm{Bayes}} = A - D_1 \ln a \tag{25}$$

- using nontrivial approximations of $\hat{H}_{0,\mathrm{Bayes}}$, namely : $\hat{H}_{0,1}, \hat{H}_{0,2}, \hat{H}_{0,\mathrm{low}}$ instead of $\hat{H}_{0,\mathrm{Bayes}}$

Remark:
Using of $\hat{H}_{0,0} \equiv \hat{H}_{0,\mathrm{naive}}$ instead of $\hat{H}_{0,\mathrm{Bayes}}$ comes back to traditional Box Counting.

6 Experimental Part

Revisited Box Counting technique will be tested on models of deterministic self-similar 2D fractal sets. They are generated by recursive expansion of binary matrix $\mathbb{G}_{u,v} \in \{0,1\}^{v \times v}$ where u is number of nonzero elements (units), $v > 1$ is matrix dimension, and $v < u < v^2$.

Recursive expansion of $\mathbb{G}_{u,v}$ generates binary matrix which represents fractal set $\mathbb{F}_{u,v}$ of similarity dimension $D_\mathrm{S} = D_\mathrm{H} = D_0 = D_1 = \frac{\log u}{\log v}$. Depth h of recursion depends on v and competer memory size.

Four testing sets: $\mathbb{F}_{3,2}$, $\mathbb{F}_{4,3}$, $\mathbb{F}_{5,3}$, $\mathbb{F}_{8,3}$ were generated by matrices:

- $\mathbb{G}_{3,2} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ for $h = 11$
- $\mathbb{G}_{4,3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ for $h = 7$
- $\mathbb{G}_{5,3} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ for $h = 7$
- $\mathbb{G}_{8,3} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ for $h = 7$

Sets $\mathbb{G}_{3,2}, \mathbb{G}_{8,3}$ corresponds to Sierpinski gasket and carpet.

Dimensions of $\mathbb{F}_{3,2}$, $\mathbb{F}_{4,3}$, $\mathbb{F}_{5,3}$, $\mathbb{F}_{8,3}$ are

$$\begin{aligned} \dim(\mathbb{F}_{3,2}) &= \frac{\log 3}{\log 2} = 1.5850, \\ \dim(\mathbb{F}_{4,3}) &= \frac{\log 4}{\log 3} = 1.2619, \\ \dim(\mathbb{F}_{5,3}) &= \frac{\log 5}{\log 3} = 1.4650, \\ \dim(\mathbb{F}_{8,3}) &= \frac{\log 8}{\log 3} = 1.8928. \end{aligned} \tag{26}$$

Agequate points sets with given depth h were generated, first. Then, they were randomly rotated around origin and finally they were randomly shifted. After these operations the grid of size a were put on the data points and entropy estimates were calculated. Due to physical interpretation of entropy, the estimates were averaged over 10 realizations and mean values of entropy were calculated.

Various estimates of Hartley entropy for grid size $a = 2, 3, 4, ..., 30$ are depicted on Fig 1. Hartley entropy estimates are similar each other except underbiased naive estimate. Estimates $H_{0,\mathrm{BAYES}}$ and $H_{0,\mathrm{low}}$ are very similar in these four cases. Corresponding estimates of Shannon entropy are depicted on Fig. 2 in the same range.

As seen in the Fig. 1 and 2, too small grid size $a \leq 20$ comes to underestimation of $H_{0,\mathrm{naive}}, H_{1,\mathrm{naive}}$, but the other estimates are unfortunatelly overestimated. Therefore, revisited Box Counting was applied in the range $30 \leq a \leq 100$.

Using least square method we obtained various estimates of D_0 and D_1 and compared them with similarity dimension. Results of estimation are collected in Tabs. 1 - 8, where $\mathrm{E}D$ is point estimate of given dimension, s_D is its standard deviation, and p_{value} is probability from t-test of hypothesis

$$\mathrm{H}_0 : \mathrm{E}D = D_{\mathrm{S}} \tag{27}$$

7 Conclusion

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References

[1] todo t.*todo.* todo.

[2] Harris, B., *The statistical estimation of entropy in the non-parametric case.* MRC Technical Summary Report, 1975

[3] Wendel, J., *Note on the gamma function.* Amer. Math. Monthly, 55(1948), 563-564.

A Appendix

A.1 Derivation of $\hat{\mathrm{p}}(K|n,N)$ for Hartley Entropy

Let $\mathbb{Q}_n = \{\vec{q} \in (\mathbb{R}_0^+)^n | \sum_{j=1}^n q_j = 1\}$ be a support set of a Dirichlet-distributed random variable $\vec{p} \in \mathbb{Q}_n$ with parameters α_j, for $j = 1, ..., n$. The conditional probability of an integer K satisfying $1 \leq K \leq \min(n, N)$ is

$$\mathrm{p}(K \mid n, N) = \mathrm{prob}\left(\sum_{N_j > 0} 1 = K \,\middle|\, n, \sum_{j=1}^{n} N_j = N \right). \tag{28}$$

The vector of N_j can be reorganized to begin with positive values:

$$\mathrm{p}(K \mid n, N) = \binom{n}{K} \mathrm{prob}\left(\forall j = 1, ..., n : N_j > 0 \Leftrightarrow j \leq K \,\middle|\, n, \sum_{j=1}^{K} N_j = N \right). \tag{29}$$

Let $\mathbb{D}_{K,N} = \{\vec{x} \in \mathbb{N}^K | \sum_{j=1}^K x_j = N\}$ be the domain of $\vec{N} = (N_1, ..., N_K) \in \mathbb{D}_{K,N}$. Using the mean value of a multinomial distribution over $\mathbb{Q}_n$, we obtain an unbiased estimate of $\mathrm{p}(K \mid n, N)$ as

$$\hat{\mathrm{p}}(K \mid n, N) = \binom{n}{K} \mathrm{E}\left(\sum_{\vec{N} \in \mathbb{D}_{K,N}} \binom{N}{N_1, ..., N_K} \prod_{j=1}^{K} p_j^{N_j} \prod_{j=k+1}^{n} p_j^0 \right) = \binom{n}{K} \sum_{\vec{N} \in \mathbb{D}_{K,N}} \binom{N}{N_1, ..., N_K} \mathrm{E}\left(\prod_{j=1}^{K} p_j^{N_j} \right). \tag{30}$$

Using the generalized Beta function

$$B(\vec{x}) = \int_{\vec{p} \in \mathbb{Q}_m} \prod_{j=1}^{m} p_j^{x_j - 1} \mathrm{d}\vec{p} = \frac{\prod_{j=1}^m \Gamma(x_j)}{\Gamma(\sum_{j=1}^m x_j)}, \tag{31}$$

we can calculate

$$\mathrm{E}\left(\prod_{j=1}^{K} p_j^{N_j} \right) = \frac{\int_{\vec{p} \in \mathbb{Q}_n} B(\vec{\alpha})^{-1} \prod_{j=1}^K p_j^{N_j + \alpha_j - 1} \prod_{j=K+1}^n p_j^{\alpha_j - 1} \mathrm{d}\vec{p}}{\int_{\vec{p} \in \mathbb{Q}_n} B(\vec{\alpha})^{-1} \prod_{j=1}^n p_j^{\alpha_j - 1} \mathrm{d}\vec{p}} = \frac{\Gamma(\alpha)}{\Gamma(N + \alpha)} \prod_{j=1}^{K} \frac{\Gamma(N_j + \alpha_j)}{\Gamma(\alpha_j)}, \tag{32}$$

where α is the sum of all α_j. Therefore

$$\hat{\mathrm{p}}(K|n, N) = \binom{n}{K} \sum_{\vec{N} \in \mathbb{D}_{K,N}} \frac{N!}{\prod_{j=1}^K N_j!} \frac{\Gamma(\alpha)}{\Gamma(N + \alpha)} \frac{\prod_{j=1}^K \Gamma(N_j + \alpha_j)}{\prod_{j=1}^K \Gamma(\alpha_j)} = \binom{n}{K} \frac{\Gamma(N+1)\Gamma(\alpha)}{\Gamma(N + \alpha)} \sum_{\vec{N} \in \mathbb{D}_{K,N}} \prod_{j=1}^{K} \frac{\Gamma(N_j + \alpha_j)}{\Gamma(N_j + 1)\Gamma(\alpha_j)} \tag{33}$$

In this particular paper, we assume $\alpha_j = \alpha^*, \forall j = 1, ..., n$ which results in easier form of the equation 33:

$$\hat{\mathrm{p}}(K \mid n, N) = \binom{n}{K} \frac{\Gamma(N+1)\Gamma(n\alpha^*)}{\Gamma(N + n\alpha^*)} \sum_{\vec{N} \in \mathbb{D}_{K,N}} \prod_{j=1}^{K} \frac{\Gamma(N_j + \alpha^*)}{\Gamma(N_j + 1)\Gamma(\alpha^*)} \tag{34}$$

A.2 Proofs of convergence

A.2.1 Convergence of $\sum_{n=K}^{\infty} \hat{\mathrm{p}}\left(K \mid n, N\right)$

$$\hat{\mathrm{p}}\left(K \mid n, N\right) = \tag{35}$$

can be transformed to b_j which convergence is proved in the next paragraph. (TODO: transformation)

A.2.2 Convergence of $\sum_{j=0}^{\infty} b_j$

The ratio of coeficients b_j could be expressed as:

$$\frac{b_j}{b_{j-1}} \frac{\ln(K+j)}{\ln(K+j-1)} = \frac{(K+j)}{j} \frac{\ln(K+j)}{\ln(K+j-1)} \frac{\Gamma((K+j)\alpha^*)}{\Gamma((K+j-1)\alpha^*)} \frac{\Gamma(N + (K+j-1)\alpha^*)}{\Gamma(N + (K+j)\alpha^*)}. \tag{36}$$

Wendel [3] proved inequality:

$$\forall \Delta \in \langle 0; 1 \rangle, \forall x > 0 : \frac{\Gamma(x + \Delta)}{\Gamma(x)} \leq x^{\Delta}, \tag{37}$$

we should realize that $\alpha \in \langle 0;1\rangle$ and than see the similarity leading to

$$\frac{b_j}{b_{j-1}}\frac{\ln(K+j)}{\ln(K+j-1)} \le \frac{K+j}{j}\frac{\ln(K+j)}{\ln(K+j-1)}\left(\frac{(K+j-1)\alpha^*}{(K+j-1)\alpha^*+N}\right)^{\alpha^*} \tag{38}$$

The Raabe criterion says that series of complex numbers $\sum_n a_n$ converges absolutely if $a_n \neq 0$ and there exists $\lim_{n\to\infty}\left(n\left(\frac{|a_n|}{|a_{n+1}|}-1\right)\right) > 1$

$$\lim_{j\to\infty} j\left(\frac{b_{j-1}}{b_j}\frac{\ln(K+j-1)}{\ln(K+j)}-1\right) \ge \lim_{j\to\infty} j\left(\frac{j}{K+j}\frac{\ln(K+j-1)}{\ln(K+j)}\left(\frac{(K+j-1)\alpha^*+N}{(K+j-1)\alpha^*}\right)^{\alpha^*}-1\right) = L. \tag{39}$$

Substitution $x : K+j \Leftrightarrow j = x-K$ leads to:

$$\begin{aligned}
&\lim_{x\to\infty}\left((x-K)\left(\frac{x-K}{x}\frac{\ln(x-1)}{\ln(x)}\left(\frac{(x-1)\alpha^*+N}{(x-1)\alpha^*}\right)^{\alpha^*}-1\right)\right)\\
&= \lim_{x\to\infty}\left(x\left(\frac{x-K}{x}\frac{\ln(x-1)}{\ln(x)}\left(\frac{(x-1)\alpha^*+N}{(x-1)\alpha^*}\right)^{\alpha^*}-1\right)\right)\\
= \lim_{x\to\infty}&\left((x-K)\frac{\ln(x-1)}{\ln(x)}\left(\frac{(x-1)\alpha^*+N}{(x-1)\alpha^*}\right)^{\alpha^*}-x\right) = -K + \lim_{x\to\infty}\left(x\frac{\ln(x-1)}{\ln(x)}\left(\frac{(x-1)\alpha^*+N}{(x-1)\alpha^*}\right)^{\alpha^*}-x\right)\\
&-K + \lim_{x\to\infty}\left(x\left(\frac{1+\frac{N-\alpha^*}{x\alpha^*}}{1-\frac{1}{x}}\right)^{\alpha^*}-x\right) = -K + \lim_{x\to\infty}\left(x\left(1+\frac{N-\alpha^*}{x\alpha^*}\right)^{\alpha^*}-x\left(1-\frac{1}{x}\right)^{\alpha^*}\right)
\end{aligned} \tag{40}$$

Substituting $h = \frac{1}{x} \to 0^+$:

$$\begin{aligned}
&= -K + \lim_{h\to 0^+}\frac{\left(1+\frac{N-\alpha^*}{\alpha^*}h\right)^{\alpha^*}-(1-h)^{\alpha^*}}{h}\\
&\overset{L'H}{=} -K + \lim_{h\to 0^+}\left(\left(1+\frac{N-\alpha^*}{\alpha^*}h\right)^{\alpha^*-1}(N-\alpha^*)+\alpha^*(1-h)^{\alpha^*-1}\right)\\
&= -K + N - \alpha^* + \alpha^*\\
L &\overset{!}{>} 1 => -K+N > 1 => K \le N-2
\end{aligned} \tag{41}$$

Thus the series $\sum_{j=0}^{\infty} b_j$ absolutely converges when $K \le N-2$.

A.2.3 Convergence of $\sum_{j=0}^{\infty} b_j \ln(K+j)$

A.2.4 Convergence of $\frac{\sum_{j=0}^{\infty} b_j \ln(K+j)}{\sum_{j=0}^{\infty} b_j}$

As both series $\sum_{j=0}^{\infty} b_j \ln(K+j)$ and $\sum_{j=0}^{\infty} b_j$ converge absolutely, their fraction converges as well.

A.3 Matlab library

A.3.1 Bayesian estimate $H_{0,\mathbf{Bayes}}$

```
function H0=HARTLEYBAYES(N,k,nmax)
    nstar=1e7;
    tol=1e200;
    if nargin == 1
        nmax=length(N);
        k=sum(N>0);
        Ntotal=sum(N);
    else
        Ntotal=N;
```

```
    end
    if nargin==2
        nmax=nstar;
    end
    if k>Ntotal-2 && nmax==nstar
        H0=NaN;
        return
    end
    bay=1;
    bay0=log(k);
    b=1;
    H0=bay0/bay;
    for j=1:nmax-k
        H0old=H0;
        b=b/j*(k+j)*(k+j-1)/(k+Ntotal+j-1);
        bay=bay+b;
        bay0=bay0+b*log(k+j);
        H0=bay0/bay;
        if bay>tol
            bay0=bay0/bay;
            b=b/bay;
            bay=1;
        end
        if abs(H0-H0old)< 1e-8
            break
        end
    end
end
```

A.3.2 Bayesian estimate $H_{1,\mathbf{Bayes}}$

```
function H1=SHANNONBAYES(N)
    nstar=1e7;
    tol=1e200;
    Ntotal=sum(N);
    nmax=length(N);
    k=sum(N>0);
    N=N(N>0);
    if k>Ntotal-2 && nmax==nstar
        H1=NaN;
        return
    end
    bay=1;
    bay1=SHANNONFIXED(N);
    b=1;
    H1=bay1/bay;
    for j=1:nmax-k
        H1old=H1;
        b=b/j*(k+j)*(k+j-1)/(k+Ntotal+j-1);
        bay=bay+b;
        N=[N,0];
        bay1=bay1+b*SHANNONFIXED(N);
        H1=bay1/bay;
        if bay>tol
            bay1=bay1/bay;
            b=b/bay;
            bay=1;
        end
        if abs(H1-H1old)< 1e-8
            break
```

```
        end
    end
end
```

A.3.3 Bayesian estimate $H_{1,\mathbf{n}}$

```
function H1=SHANNONFIXED(N)
        Ntotal=sum(N);
        n=length(N);
        H1=(N+1)/(Ntotal+n)*(psi(Ntotal+n+1)-psi(N+2))';
end
```