Basic Statistics

- Conditional Prob: $P(A|B) = P(A \cap B)/P(B)$
- Independent $\iff P(A \cap B) = P(A)P(B)$

RV X	Prob. $f(x)$	CDF $F(x)$
Disc(pmf)	$\sum_x f(x) = 1$	$\sum_{y \le x} f(y)$
Cont(pdf)	$\int_x f(x)dx = 1$	$\int_{-\infty}^x f(y)dy$

- X is continuous RV $\Rightarrow F(X) \sim \text{Unif}(0,1)$
- $E[(X-\mu)^n]$ is the nth *central moment* of X
- $M_X(t) \equiv E[e^{tX}]$: *moment generating function*
- Under certain technical conditions,

$$E[X^k] = \frac{d^k}{dt^k} M_X(t)|_{t=0},\ k = 1, 2, \ldots$$

- $Var[X] = E[X^2] - E^2[X]$
- $E[aX+b] = aE[X] + b$, $V[aX+b] = a^2 V[X]$
- Joint CDF: $F(x,y) \equiv P(X \le x, Y \le y),\ \forall x, y$
- Joint PDF: $f(x,y) \equiv \frac{\partial^2}{\partial x \partial y} F(x,y)$
- Marginal CDF: $F_X(x) = \int F(x,y)dy$
- *Independent* $\iff f(x,y) = f_X(x) f_Y(y) \iff f(x,y) = a(x)b(y)$ and domains are independent
- Conditional PDF: $f(y|x) \equiv f(x,y)/f_X(x)$
- $E[Y|X=x] = \int y f(y|x)dy$, $E[E(Y|X)] = E[Y]$
- $E[XY] = \iint xyf(x,y)dxdy$
- $Cov(X,Y) \equiv E[(X-EX)(Y-EY)] = E[XY] - E[X]E[Y]$, $Var(X) = Cov(X,X)$
  $Cov(aX, bY) = abCov(X,Y)$
  $Cov(X \pm Y, Z) = Cov(X,Z) \pm Cov(Y,Z)$
- $Corr(X,Y) \equiv Cov(X,Y)/\sqrt{V(X)V(Y)}$
  $Corr(aX, bY) = Corr(X,Y)$
- Independent $\Rightarrow Cov(X,Y) = 0$
- $Var(X \pm Y) = Var(X) + Var(Y) \pm 2Cov(X,Y)$

Probability Distributions

- $X \sim \text{Bernoulli}(p)$:

$$f(x) = \begin{cases} p & \text{if } x = 1 \\ 1-p & \text{if } x = 0 \end{cases}$$

  $E[X] = p$, $Var(X) = p(1-p)$
- $X \sim \text{Binomial}(n,p)$:
  (# of successes in n Bern(p) trials)

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

  $E[X] = np$, $Var(X) = np(1-p)$
- $X \sim \text{Geometric}(p)$:
  (# of Bern(p) trials until a success occurs)

$$f(x) = (1-p)^{x-1} p$$

  $E[X] = 1/p$, $Var(X) = (1-p)/p^2$
- $X \sim \text{NegBin}(r,p)$: the sum of r iid Geom(p)

$$f(x) = \binom{x-1}{r-1} (1-p)^{x-r} p^r,\ x = r, r+1, \ldots$$

  $E(X) = r/p$, $Var(X) = r(1-p)/p^2$
- $X \sim \text{Poisson}(\lambda)$:

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!},\ x = 0, 1, \ldots$$

  $E[x] = \lambda = Var(X)$
- $X \sim \text{Uniform}(a,b)$:

$$f(x) = \frac{1}{b-a},\ E[X] = \frac{a+b}{2},\ V(X) = \frac{(b-a)^2}{12}$$

- $X \sim \text{Exponential}(\lambda)$: time between Poi events

$$f(x) = \lambda e^{-\lambda x},\ F(x) = 1 - e^{-\lambda x},$$

  $E[X] = 1/\lambda$, $Var(X) = 1/\lambda^2$. And

$$P(X > s+t | X > s) = P(X > t)$$

  memoryless property $\Uparrow$
- $X \sim \text{Erlang}(k, \lambda)$: the sum of k Exp(λ)

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!},\ for\ x, \lambda \ge 0$$

$$F(x) = 1 - \sum_{n=0}^{k-1} \frac{e^{-\lambda x}(\lambda x)^n}{n!}$$

  $E(X) = k/\lambda$, $Var(X) = k/\lambda^2$

- $X \sim \text{Beta}(a, b)$:

$$E(X) = \frac{a}{a+b},\ V(X) = \frac{ab}{(a+b)^2(a+b+1)}$$

- $X \sim \text{Gamma}(\alpha, \lambda)$:

$$E[X] = \alpha/\lambda,\ Var(X) = \alpha/\lambda^2$$

- $X \sim \text{Triangular}(a, b, c)$: $E(X) = (a+b+c)/3$

- $X \sim \text{Normal}(\mu, \sigma^2)$:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right],\ x \in \mathbb{R}$$

  $E[X] = \mu$, $V[X] = \sigma^2$, $M_X(t) = exp[\mu t + \frac{1}{2}\sigma^2 t^2]$

- $X \sim \text{LogNormal}(\mu, \sigma^2)$:

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right],\ x \in \mathbb{R}^+$$

  $E[X] = e^{\mu+\sigma^2/2}$, $V[X] = (e^{\sigma^2}-1)e^{2\mu+\sigma^2}$

- *Law of Large Numbers* (special case):

$$X_1, \ldots, X_n \text{ are iid Nor} \Rightarrow \bar{X}_n \sim \text{Nor}(\mu, \sigma^2/n)$$

- *Central Limit Theorem*: If $X_1, \ldots, X_n \overset{iid}{\sim} f(x)$,

$$Z_n \equiv \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} \text{Nor}(0, 1)$$

- $100(1-\alpha)\%$ *Confidence Intervals*:

$$\mu \in [\bar{X}_n - z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}, \bar{X}_n + z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}}] \text{ or}$$

$$\mu \in [\bar{X}_n - t_{\alpha/2,n-1}\sqrt{\frac{S^2}{n}}, \bar{X}_n + t_{\alpha/2,n-1}\sqrt{\frac{S^2}{n}}]$$

  in which, $S^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$

Simulations

- Monte Carlo Integration:

$$\hat{I}_n = \frac{b-a}{n}\sum_{i=1}^n f[a + (b-a)U_i]$$

  is an *unbiased* estimator of $\int_a^b f(x)dx$.

- Hand Simulation: #, arrival time, start time, service time, departure time, waiting time

Random Variables

- Generators We Don't Use: 1. Random Devices: tough to repeat; 2. Random Number Tables: cumbersome and slow, not random; 3. Mid-Square Method: positive serial correlation, occasionally degenerates; and 4. Fibonacci: small numbers follow small number, not possible to get $X_{i-1} \le X_{i+1} \le X_i$ or $X_i \le X_{i+1} \le X_{i-1}$.

- *Linear Congruential Generator*:

$$X_i = (\sum_{j=1}^q a_i X_{i-j}) \bmod m$$

- Generating *pseudo-random numbers* from LCG:

$$X_i = 16807 X_{i-1} \text{mod}(2^{31}-1),\ i = 1, 2, \ldots$$

  Then set $R_i = X_i/(2^{31}-1)$.

- *Tausworthe Generator*:

$$B_i = \left(\sum_{j=1}^q c_j B_{i-j}\right) \bmod 2$$

  Usual implementation:

$$B_i = (B_{i-r} + B_{i-q}) \bmod 2 \quad (0 < r < q)$$

- **Theorem 1**: When X_0 is odd and $a = 8k+3$ or $a = 8k+5$ for some k, $X_i = aX_{i-1} \bmod 2^n$ $(n > 3)$ can have cycle length of at most 2^{n-2}.

- **Theorem 2**: $X_i = (aX_{i-1} + c) \bmod m$ $(c > 0)$ has full cycle if 1. c and m are relatively prime; 2. $a-1$ is a multiple of every prime which divides m; and 3. $a-1$ is a multiple of 4 if 4 divides m.

- **Theorem 3**: $X_i = aX_{i-1} \bmod m$ wirh prime m has full period $(m-1) \Leftrightarrow m$ divides $a^{m-1}-1$ and m doesn't divide $a^i - 1$ for all $i < m-1$.

- **Theorem 4**: The k-tuples from multiplicative generators lie on parallel hyperplanes in $[0, 1]^k$.

- *Runs Tests "Up and Down"*:

$$|Z_0| = \frac{|A - E[A]|}{\sqrt{\text{Var}(A)}} \le z_{\alpha/2}$$

  $A \approx \text{Nor}((2n-1)/3, (16n-29)/90)$ is the # of runs "up and down" out of n observations.

$$++, --, +, -, +, --- \ (A = 6)$$

  $+$ for increase, $-$ for decrease between two RVs.

- *Runs Test "Above and Below Mean"*:

$$|Z_0| = \frac{|B - E[B]|}{\sqrt{\text{Var}(B)}} \le z_{\alpha/2}$$

$$B \approx \text{Nor}\left(\frac{2n_1 n_2}{n} + \frac{1}{2}, \frac{2n_1 n_2(2n_1 n_2 - n)}{n^2(n-1)}\right)$$

+ for $R_i \ge 0.5$ (n_1), − for $R_i < 0.5$ (n_2).

$$- + + + + + + + - - - - + + - + - - - - - -$$
$$- - + + - - - - + + - - + - + - - + + -$$

$n_1 = 18$, $n_2 = 22$, $B = 17$.

- *Inverse Transform Method*: $X = \text{F}^{-1}[\text{U}(0,1)]$

$$a + (b-a)U \sim \text{Unif}(a,b)$$

$$a + \lfloor (b-a+1)U \rfloor \sim \text{DiscreteUnif}(a,b)$$

$$-\frac{1}{\lambda}\ln(1-U) \sim \text{Expo}(\lambda)$$

$$\frac{1}{\lambda}[-\ln(1-U)]^{1/\beta} \sim \text{Weibull}(\lambda,\beta)$$

$$\left\lceil \frac{\ln(1-U)}{\ln(1-p)} \right\rceil \sim \text{Geom}(p)$$

$$\Phi^{-1}(U) \sim \text{Nor}(0,1)$$

- *Cutpoint Method*: $I_j = \min[k : q_k > (j-1)/m]$
- *Convolution Method*: adding things up

$$\sum_{i=1}^{n} \text{Bern}(p) \sim \text{Binomial}(n,p)$$

$$U_1 + U_2 \sim \text{Triangular}(0,1,2)$$

$$-\frac{1}{\lambda}\ln(\prod_1^k U_i) \sim \text{Erlang}(k,\lambda)$$

$$\sum_{i=1}^{n} U_i \sim \text{Nor}(n/2, n/12)$$

$$\sum_{i=1}^{n} \text{Geom}(p) \sim \text{NegBin}(n,p)$$

$$\sum_{i=1}^{n} \text{Nor}^2(0,1) \sim \chi^2(n)$$

$$\sum_{i=1}^{n} \text{Cauchy}/n \sim \text{Cauchy}$$

- *Acceptance-Rejection Method*: $g(x) \equiv f(x)/t(x)$ and $0 \le g(x) \le 1$ for all x. $U \sim \text{Unif}(0,1)$, Y is a RV with pdf $h(y)$ ($h(x) \equiv t(x)/\int t(x)dx$). Then if $U \le g(Y)$, accept Y as X.
  - quickly sample from $h(y)$; and
  - $c = \int t(x)dx$ must be small.

  Half-Normal RV ($x \ge 0$):

$$f(x) = \frac{2}{\sqrt{2\pi}} e^{-x^2/2}, \quad x \ge 0$$

$$t(x) = \sqrt{\frac{2e}{\pi}} e^{-x}, \text{ and } c = 1.3155$$

$$h(x) = e^{-x}, \text{ and } g(x) = e^{-(x-1)^2/2}$$

  Poisson(λ): Generate U_i until

$$e^{-\lambda} > \prod_{i=1}^{n+1} U_i$$

  Then $X = n \sim \text{Pois}(\lambda)$. ($E[n+1] = \lambda + 1$)

- *Composition Method*: $F(x) = \sum p_j F_j(x)$
- Box-Muller Transform:

$$X_i = \sqrt{-2\ln(U_{i,1})}\cos(2\pi U_{i,2})$$

$$Y_i = \sqrt{-2\ln(U_{i,1})}\sin(2\pi U_{i,2})$$

  Then, X_i, Y_i are i.i.d. Nor(0,1); $X_i^2 + Y_i^2 \sim \chi^2(2) \sim \text{Expo}(1/2)$; $Y_i/X_i = \tan(2\pi U_{i,2}) \sim$ Cauchy $\sim t(1)$; and $Y_i^2/X_i^2 \sim F(1,1)$.

- Polar Method: Z_1, Z_2 are i.i.d. Nor(0,1) if

$$V_i = 2U_i - 1, \ i = 1,2 \text{ and } W = V_1^2 + V_2^2$$

  If $W \le 1$, let $Y = \sqrt{-2(\ln(W))/W}$, and accept $Z_i \leftarrow V_i Y$. Otherwise, reject and repeat.

- $Y = \min\{X_1, \ldots, X_n\}$
  ($Y \sim \text{Exp}(n\lambda)$ if $X \sim \text{Exp}(\lambda)$)

$$P(Y \le y) = 1 - [P(X > y)]^n$$

  $Z = \max\{X_1, \ldots, X_n\}$

$$P(Z \le z) = [P(X \le z)]^n$$

- Other Quickies:

$$\mathrm{Nor}(0,1)/\sqrt{\chi^2(n)/n} \sim \mathrm{t}(n)$$

$$(\chi^2(n)/n)/(\chi^2(m)/m) \sim \mathrm{F}(n,m)$$

- Multivariate Normal Distribution:

$$\boldsymbol{X} \sim \mathrm{Nor}_k(\boldsymbol{\mu}, \Sigma = CC^T)$$

  C is Cholesky matrix.

- Nonhomogeneous Poisson Process (Thinning Algorithm): generate potential arrivals with rate $\lambda^* \equiv \sup \lambda(t)$ and accept a potential arrival at time t with probability $\lambda(t)/\lambda^*$. ($T_i = T_{i-1} - \ln(U_i)/\lambda$)

- First-Order Moving Average Process:

$$Y_i = \epsilon_i + \theta\epsilon_{i-1}$$

  with $\mathrm{Var}(Y_i) = 1+\theta^2$ and $\mathrm{Cov}(Y_i, Y_{i+1}) = \theta$.

- First-Order Autoregressive Process:

$$Y_i = \phi Y_{i-1} + \epsilon_i$$

  with $Y_0 \sim \mathrm{Nor}(0,1)$ and $\epsilon_i \sim \mathrm{Nor}(0, 1-\phi^2)$. The covariance function $\mathrm{Cov}(Y_i, Y_{i+k}) = \phi^{|k|}$.

- M/M/1 Queue:

$$W_{i+1} = \max\{W_i + S_i - I_{i+1}, 0\}$$

  with W_i as waiting time, S_i as service time, and I_i as interarrival time.

- Standard Brownian Motion: 1. $W(0) = 0$; 2. $W(t) \sim \mathrm{Nor}(0,t)$; and 3. $W(t+h) - W(t)$ is independent and only depends on h.

$$W(\frac{i}{n}) = W(\frac{i-1}{n}) + \frac{Y_i}{\sqrt{n}},\ [Y_i \sim \mathrm{Nor}(0,1)]$$

  Continuous but not derivable, $\int_0^1 W(t)dt \sim \mathrm{Nor}(0, 1/3)$, $\mathrm{Cov}(W(s), W(t)) = \min(s,t)$.

- Geometric Brownian Motion ($\mathrm{E}[(S(T)-k)^+]$):

$$S(t) = \exp\{(\mu - \sigma^2/2)t + \sigma W(t)\}, t \geq 0$$

Input Analysis

- Sample Variance: $S^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2$

- Test Independence: X_i vs X_{i+1} or
  von Neumann's test: $|U_n| \leq z_{\alpha/2}$

$$U_n = \sqrt{\frac{n^2-1}{n-2}} \times \left[\hat{\rho}_1 + \frac{(x_1-\bar{x})^2 + (x_n - \bar{x})^2}{2\sum_{i=1}^n (x_i - \bar{x})^2}\right]$$

$$\hat{\rho}_1 = \frac{\sum_{i=1}^{n-1}(x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

- Parameters: *Location*, *Shape*, and *Scale*.

- Method of Moments:

$$\mathrm{E}[X^k] = \frac{1}{n}\sum_{i=1}^n x_i^k \quad \Rightarrow$$

$$\mathrm{E}[X] = \mu, \quad \mathrm{E}[X^2] = \mu^2 + \sigma^2$$

- Maximum Likelihood Estimation:

$$L(\theta) = \prod_{i=1}^n f(\boldsymbol{x}; \theta) \leq L(\hat{\theta}) \text{ for all } \theta$$

  Poisson: $\hat{\lambda} = \bar{x}$ ($\lambda \in \bar{x} \pm z_{\alpha/2}\sqrt{\bar{x}/n}$)
  Uniform(a,b): $\hat{a} = \min x_i$, $\hat{b} = \max x_i$
  Exponential: $\hat{\lambda} = 1/\bar{x}$; Poisson: $\hat{\lambda} = \sum_{i=1}^n x_i/n$
  Shifted EXPO: $\hat{\gamma} = \min x_i$, $\hat{\beta} = \bar{x} - \min x_i$
  Normal: $\hat{\mu} = \bar{x}$, $\hat{\sigma}^2 = \sum(x_i - \bar{x})^2/n = \frac{n-1}{n}S_n^2$
  LogNor: $\hat{\mu} = \sum \ln x_i/n$, $\hat{\sigma}^2 = \sum(\ln x_i - \hat{\mu})^2/n$
  Weibull: $\hat{\lambda} = \left(\sum_{i=1}^n x_i^{\hat{\alpha}}/n\right)^{-1/\hat{\alpha}}$
  $(\sum x_i^{\hat{\alpha}} \ln x_i)/(\sum x_i^{\hat{\alpha}}) - 1/\hat{\alpha} = \sum \ln x_i/n$

- MLEs are "nice" because: 1. asymptotically ($n \to \infty$) unbiased; 2. asymptotically normal; and 3. invariant.

- Test Goodness-of-Fit: Power $= 1-\beta$, p-value
  H_0: $x_1, \ldots, x_n$ are from $\hat{f}(x) \equiv f(x; \hat{\theta})$
  $\alpha =$ Type I Error $= \Pr(\text{reject } H_0 | H_0 \text{ true})$
  $\beta =$ Type II Error $= \Pr(\text{accept } H_0 | H_0 \text{ false})$

- `Q-Q Plot`: fitted quantiles vs. sample quantiles;
  `P-P Plot`: fitted CDF vs. empirical CDF.

- χ^2 *Goodness-of-Fit Test*: (s is # of parameters)

$$\chi_0^2 \equiv \sum_{i=1}^k \frac{(O_i - E_i)^2}{E_i} \leq \chi^2_{\alpha, k-s-1}$$

  Observed and expected # of RVs in ith interval.
  (Make sure $E_i \geq 5$ by combining intervals)

- Kolmogorov-Smirnov Test: $D_n \le d_{n,\alpha}$

$$\max\left\{\max\left[\frac{i}{n} - \hat{F}(x_i)\right], \max\left[\hat{F}(x_i) - \frac{i-1}{n}\right]\right\}$$

$D'_n \le c_\alpha$, find D'_n and c_α from tables above.

Case	Adjusted Test Statistic
All parameters known	$\left(\sqrt{n} + 0.12 + \frac{0.11}{\sqrt{n}}\right) D_n$
$\text{Nor}(\bar{X}_n, S_n^2)$	$\left(\sqrt{n} - 0.01 + \frac{0.85}{\sqrt{n}}\right) D_n$
$\text{Expo}(1/\bar{X}_n)$	$\left(D_n - \frac{0.2}{n}\right)\left(\sqrt{n} + 0.26 + \frac{0.5}{\sqrt{n}}\right)$

Type I error α				
0.150	0.100	0.050	0.025	0.001
1.138	1.224	1.358	1.480	1.628
0.775	0.819	0.895	0.955	1.035
0.926	0.990	1.094	1.190	1.308

- Anderson-Darling Test:
- No Data: 1. interview "experts"; 2. Distribution Viewer in ExpertFit; 3. interarrival / EXPO; 4. # of "random" events in an interval / POIS; 5. sum of independent "pieces" / NORM; 6. bounded task times / BETA; and 7. unbounded task times / LOGNORMAL or WEIBULL.

Output Analysis

- Covariance Function: $R_k \equiv \text{Cov}(Y_1, Y_{1+k})$

$$\text{Var}(\bar{Y}_n) = \frac{1}{n}\left[R_0 + 2\sum_{k=1}^{n-1}\left(1 - \frac{k}{n}\right) R_k\right]$$

- Types of Simulations: **Finite-Horizon**, terminate at a specific time or event; **Steady-State**, long-run behaviour of a system
- Finite-Horizon:

$$S_Z^2 \equiv \frac{1}{b-1}\sum_{i=1}^{b}(Z_i - \bar{Z}_b)^2$$

If # of observations per replication, m, is large:

$$S_Z^2 \approx \frac{\text{Var}(Z_i)\chi^2(b-1)}{b-1}, \theta \in \bar{Z}_b \pm t_{\alpha/2,b-1}\sqrt{S_Z^2/b}$$

- How to detect initialization bias: attempt to detect the bias visually; or conduct statistical tests.
- How to deal with: truncate the output by allowing "warm up"; or make a very long run to overwhelm the effects of initialization bias.
- Steady-State Analysis:

$$\sigma^2 = \lim_{n\to\infty} n\text{Var}(\bar{Y}_n) = \sum_{k=-\infty}^{\infty} R_k$$

In `MA(1)` process, $\sigma^2 = (1+\theta)^2$, and
In `AR(1)` process, $\sigma^2 = (1+\phi)/(1-\phi)$

- Batch Means (divided $n = bm$ into b groups):

$$\hat{V}_B \equiv \frac{m}{b-1}\sum_{i=1}^{b}(\bar{Y}_{i,m} - \bar{Y}_n)^2 \approx \frac{\sigma^2\chi^2(b-1)}{b-1}$$

$$\mu \in \bar{Y}_n \pm t_{\alpha/2,b-1}\sqrt{\hat{V}_B/n}$$

taking $b \approx 30$ and concentrating on increasing the batch size m as much as possible.
But problems can come up if the Y_j's are not stationary, if the batch means are not normal, or if the batch means are not independent.

- Overlapping BM (for large m and n/m):

$$\bar{Y}_{i,m}^O = \frac{1}{m}\sum_{j=i}^{i+m-1} Y_j, \ \forall i = 1, \ldots, n-m+1$$

$$\hat{V}_O = \frac{m}{n-m+1}\sum_{i=1}^{n-m+1}(\bar{Y}_{i,m}^O - \bar{Y}_n)^2$$

$$\mu \in \bar{Y}_n \pm t_{\alpha/2,\frac{3}{2}(b-1)}\sqrt{\hat{V}_O/n}$$

- Other Methods: Spectral Estimation; Regeneration; Standardized Time Series.
- Classical Confidence Intervals:

$$\bar{Z}_{i,b_i} \equiv \frac{1}{b_i}\sum_{j=1}^{b_i} Z_{i,j}, \ i = 1, 2$$

$$S_i^2 \equiv \frac{1}{b_i - 1}\sum_{j=1}^{b_i}(Z_{i,j} - \bar{Z}_{i,b_i})^2, \ i = 1, 2$$

$$\mu_1 - \mu_2 \in \bar{Z}_{1,b_1} - \bar{Z}_{2,b_2} \pm t_{\alpha/2,\nu}\sqrt{\frac{S_1^2}{b_1} + \frac{S_2^2}{b_2}}$$

- Common Random Numbers: the same PRNs
- Antithetic Variates: negative related PRNs
- Ranking and Selection Procedures:

Arena

- *Event-Scheduling Approach*: Concentrate on the events and how they affect the system state
- *Process-Interaction Approach*: Concentrate on a generic customer (entity) and the sequence of events and activities it undergoes as it progresses through the system
- *Future Events List* (FEL): list of all activities' scheduled times of completion in chronological order
- Simulated Time: `TNOW`
  Number in Queue: `NQ(Process.Queue)`
  `Name.`: `NumberIn`, `NumberOut`, `WIP`, `WaitTime`
- Discrete Probaility: `DISC`(CumulativeP, V)
  `CONT(P,V)`, `POIS(`λ`)`, `EXPO(`λ`)`, `NORM(`μ, σ`)`, `TRIA(a,b,c)`, `UNIF(a,b)`, `WEIB(`β, α`)`
- **Basic Process**: Create, Process, Decide, Dispose, Assign, Record, Batch, Separate
- **Advanced Process**: Delay, Dropoff, Hold, Match, Pickup, ReadWrite, Release, Remove, Seize, Search, Signal, Store, Unstore, Adjust Variable
- **Advanced Transfer**: Enter, Leave, PickStation, Route, Station, Access, Convet, Exit, Start, Stop, Activate, Allocate, Free, Halt, Move, Request, Transport