Mathématiques appliquées à la natation (anglais)

Un court document de synthèse

August 24, 2025

Abstract

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We love how mathematics can enhance performance.

1 Movement forces and equations

The translational movement of a swimmer with mass m is considered in the longitudinal (horizontal) axis.

$$m\,\frac{\mathrm{d}v}{\mathrm{d}t} = F_{\text{prop}}(t) - F_{\text{dragged}}(v) - F_{\text{waves}}(v,h), \tag{1}$$

where v is the speed, F_{prop} the average propelling force (arms + legs), F_{dragged} the form and friction drag, and F_{waves} the wave drag (depending in particular on the depth h). The classical quadratic drag law is written:

$$F_{\text{dragged}}(v) = \tfrac{1}{2}\,\rho\,C_D\,A\,v^2, \tag{2}$$

where ρ is the density of the water, C_D a drag coefficient (size/form), and A a reference area. The power dissipated by the drag is then:

$$P_{\text{dissipated}}(v) = F_{\text{dragged}}(v)\,v = \tfrac{1}{2}\,\rho\,C_D\,A\,v^3. \tag{3}$$

The wave drag is more marked on the surface and can be modelled by:

$$F_{\text{waves}}(v,h) = k_{\text{w}}(h)\,v^n, \qquad n \in [2,3], \tag{4}$$

where $k_{\text{w}}(h)$ decreases with depth h (deep swimming reduces wave drag, but at the cost of an apnea).

2 Propellant efficiency and energy cost

External mechanical power P_{mech} is connected to metabolic power P_{met} via an overall efficiency $\eta \in (0,1)$:

$$P_{\mathrm{mech}} = \eta \, P_{\mathrm{met}}. \tag{5}$$

Under quasi-stationary regime $v \approx$ cste, we have $P_{\mathrm{mech}} \approx P_{\mathrm{dissipated}}$, hence the instantaneous energy cost:

$$P_{\mathrm{met}}(v) \approx \frac{\frac{1}{2}\,\rho\, C_D\, A\, v^3}{\eta}. \tag{6}$$

The Cost of Transport (COT) per unit distance is written:

$$\mathrm{COT}(v) = \frac{P_{\mathrm{met}}(v)}{m\, g\, v} \propto v^2, \tag{7}$$

This illustrates that moving faster becomes superlinearly expensive.

3 Simple performance model over a distance L

Over a distance of length L and a speed profile $v(t)$. Total time is:

$$T = \int_0^T \mathrm{d}t = \int_0^L \frac{\mathrm{d}x}{v(x)}. \tag{8}$$

We model energy systems (aerobic/anaerobic) with an anaerobic reservoir W' and limited aerobic power $P_{\max}$ (a "critical power" type schedule):

$$P_{\mathrm{met}}(t) = P_{\mathrm{Aero}}(t) + P_{\mathrm{ana}}(t), \tag{9}$$

$$P_{\mathrm{Aero}}(t) \le P_{\max}, \tag{10}$$

$$\int_0^T P_{\mathrm{ana}}(t)\,\mathrm{d}t \le W'. \tag{11}$$

The "classical" objective is to minimize T under these constraints and dynamics (**??**).

3.1 Optimal control of pacing

Formally, there is an optimal control problem:

$$\min_{F_{\mathrm{prop}}(t)} T \tag{12}$$

$$\text{Subject to:} \quad \dot{v} = \frac{1}{m}\Big(F_{\mathrm{prop}} - \tfrac{1}{2}\rho C_D A v^2 - k_{\mathrm{w}}(h) v^n\Big), \tag{13}$$

$$0 \le P_{\mathrm{Aero}}(t) \le P_{\max}, \quad W'(T) = W'(0) - \int_0^T P_{\mathrm{ana}}(t)\,\mathrm{d}t \ge 0, \tag{14}$$

$$v(0) = 0,\ x(0) = 0,\ x(T) = L,\ v(t) \ge 0. \tag{15}$$

Typical solutions predict a strong start, a quasi-constant phase, and then a slight decline as W' is exhausted (moderate *positive split*).

4 Dimensional analysis and scale-up

Let U be a reference speed. A Froude number $\mathrm{Fr} = \frac{U}{\sqrt{g\ell}}$ (with characteristic length ℓ) and a Reynolds number $\mathrm{Re} = \frac{\rho U \ell}{\mu}$ are defined. At the scale of the swimming pool, $\mathrm{Re} \gg 1$ and inertia dominates local viscous effects, justifying the use of quadratic drag.

5 Interaction with the wall and casting

The casting after pushing off the wall is effectively modelled by a free movement with drag:

$$m\,\dot{v} = -\tfrac{1}{2}\rho C_D A v^2, \quad v(0) = v_0. \tag{16}$$

The closed solution is given by:

$$v(t) = \frac{v_0}{1 + t/\tau}, \qquad \tau = \frac{2m}{\rho C_D A v_0}. \tag{17}$$

The distance travelled in casting is equal to:

$$s(t) = \frac{2m}{\rho C_D A}\,\ln\Big(1 + t/\tau\Big), \tag{18}$$

which shows decreasing efficiency: beyond a certain duration, remaining in casting no longer yields a "good" distance.

6 Estimation of parameters from data

Suppose speed-time measurements are available $\{t_i, v_i\}$. A least-squares estimator for $C_D A$ (in casting) is obtained by linearising $\dot{v} = -kv^2$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{v}\right) = k, \quad k = \frac{\rho C_D A}{2m}. \tag{19}$$

A linear regression of $1/v$ based on t then gives k and thus $C_D A$.

6.1 Sensor filtering

Inertial Measurement Units (IMU) and aquatic GPS produce noisy signals. A discrete Kalman filter for speed v can be written:

$$v_{k+1} = v_k + \Delta t\, a_k + w_k, \tag{20}$$

$$z_k = v_k + r_k, \tag{21}$$

where a_k is the measured acceleration, z_k is an observation (e.g., derived from distance), $w_k \sim \mathcal{N}(0, Q)$ and $r_k \sim \mathcal{N}(0, R)$. Kalman's gain K_k weights the confidence between the model and measurement to provide a smooth estimate $\hat{v}_k$.

7 Minimum numerical example

Consider a swimmer with $m = 75\,\mathrm{kg}$, $A = 0.5\,\mathrm{m}^2$, $C_D = 0.9$, $\rho = 1000\,\mathrm{kg\,m^{-3}}$, $\eta = 0.2$. At $v = 2\,\mathrm{m\,s^{-1}}$:

$$F_{\text{dragged}} = \tfrac{1}{2}\rho C_D A v^2 = \tfrac{1}{2}\cdot 1000\cdot 0.9\cdot 0.5\cdot 4 = 900\,\mathrm{N}, \tag{22}$$

$$P_{\text{dissipated}} = Fv = 1800\,\mathrm{W}, \qquad P_{\text{met}} \approx 9000\,\mathrm{W}. \tag{23}$$

These orders of magnitude justify the role of technique (reducing $C_D A$) and efficiency η.

8 Set of typical parameters

Table 1: Examples of parameters (order of magnitude).

Parameter	Symbol	Typical value
Mass	m	60–85 kg
Reference area	A	0.4–0.7 m^2
Drag coefficient	C_D	0.7–1.1
Water density	ρ	1000 $\mathrm{kg\,m^{-3}}$
Overall efficiency	η	0.15–0.25
Max aerobic power	$P_{\max}$	300–500 W
Anaerobic tank	W'	10–25 kJ

9 Figure of forces (side view)

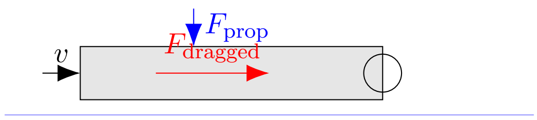

Figure 1: Main forces modelled along the swimming axis.

10 Effect of the technique: alignment and roll

A reduction of the projected area A and C_D (better alignment, controlled roll, more effective hand trajectories) decreases $P_{\text{dissipated}} \propto v^3$. Mathematically, a small change δC_D implies:

$$\frac{\delta P_{\text{dissipated}}}{P_{\text{dissipated}}} = \frac{\delta C_D}{C_D}. \tag{24}$$

At a specified speed, *each* percentage point gained on C_D translates directly into energy saving.

11 Waves and gesture frequency

Let f be the stroke frequency (arms) and S the stroke length. The kinematic identity is $v = f\,S$. Optimizing performance involves seeking an operating point (f, S) where metabolic power remains sustainable while maximizing v:

$$\max_{f,S} \ fS \quad \text{S.c.} \quad P_{\text{met}}(f, S) \leq P_{\text{tol}}, \tag{25}$$

where P_{tol} is an individual tolerance related to training. Empirically, S often decreases with f (fatigue, coordination), hence a compromise is needed.

12 Uncertainty and Variability

Real parameters vary between individuals and during a race. A Bayesian framework allows expressing uncertainty about $\theta = (C_D, A, \eta, \ldots)$ via a prior distribution $p(\theta)$ and a likelihood $p(\text{data} \mid \theta)$, producing a posterior distribution $p(\theta \mid \text{data})$. Decisions (pacing, technique) can then be aimed at *minimum risk* (minimization of the expected loss) rather than point optimization.

13 Conclusion

Mathematics offers a coherent toolbox to understand and improve swimming: non-linear dynamics, energy-intensive optimization, statistical estimation and signal processing. Even simple models, properly configured, are often enough to guide training: reduce drag (technical), choose sustainable pacing (optimal control) and use data (sensors) to customize parameters. To go beyond simplified models, a Bayesian approach is particularly relevant. It not only quantifies the uncertainty inherent in individual parameters and running conditions, but also makes more robust decisions. By moving from one-off optimisation to risk minimisation, swimmers and coaches can refine pacing and technical strategies, leading to personalized and adaptive action plans that maximize the chances of success in the face of real-world uncertainty.

Practical note. This text is deliberately synthetic and modular. Each section can be extended (experimental validation, variations by stroke type — freestyle, backstroke, breaststroke, butterfly — effects of turns, open water drafting, etc.).

References

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