# Spatiotemporal-Weather Gaussian Process Kernel with Nonstationary Matérn Kernel

## 1 Mathematical Formulation

### 1.1 Spatial Kernel: Nonstationary Matérn Kernel

We define the **local covariance matrix**:

$$\Sigma(x) = \begin{bmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{bmatrix}$$

Given two points $x, x'$, we compute:

$$M = \frac{1}{2}\left(\Sigma(x) + \Sigma(x')\right)$$

$$Q_{ij} = (x - x')^T M^{-1} (x - x')$$

The **Matérn covariance** function is:

$$K_S(x, x') = \sigma_f^2 \cdot \frac{(\det \Sigma(x))^{1/4} (\det \Sigma(x'))^{1/4}}{(\det M)^{1/2}} \cdot \frac{1}{\Gamma(\nu) 2^{\nu-1}} \cdot \left(\sqrt{2\nu Q_{ij}}\right)^{\nu} K_\nu\left(\sqrt{2\nu Q_{ij}}\right)$$

### 1.2 Temporal Kernel: Exponential Kernel

$$K_T(t, t') = \sigma_T^2 \exp\left(-\frac{|t - t'|}{\ell_T}\right)$$

### 1.3 Weather Kernel: Periodic Kernel

$$K_W(w, w') = \sigma_W^2 \exp\left(-\frac{2\sin^2\left(\frac{\pi|w - w'|}{p}\right)}{\ell_W^2}\right)$$

### 1.4 Final Spatiotemporal-Weather Kernel

We compute the full kernel as:

$$K((x, t, w), (x', t', w')) = K_T(t, t') \cdot \left(K_S(x, x') \cdot K_W(w, w')\right)$$

## 2 RKHS Weights Computation

To estimate function values in the Reproducing Kernel Hilbert Space (RKHS), we solve:

$$\boldsymbol{\alpha} = K^{-1}\mathbf{y}$$

Using Cholesky factorization:

$$K = LL^T$$

Solving for $\boldsymbol{\alpha}$:

$$L\mathbf{u} = \mathbf{y}$$

$$L^T\boldsymbol{\alpha} = \mathbf{u}$$

## 3 Final RKHS Function Representation

The function reconstruction in RKHS is given by:

$$f(x,t,w) = \sum_{i=1}^{N} \alpha_i K((x,t,w),(x_i,t_i,w_i))$$