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SRBF_package

Gravitational potential up to its second-order derivatives in terms of spherical radial basis functions (SRBFs)

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`SRBF_package` is an enhanced version of the MATLAB-based routines developed by

Bucha, B., Janák, J., Papčo, J., Bezděk, A., 2016. High-resolution regional gravity field modelling in a mountainous area from terrestrial gravity data. Geophysical Journal International 207, 949-966, `http://doi.org/10.1093/gji/ggw311`.

The package is available as MATLAB-based functions and Fortran 95 routines accompanied by a Python wrapper. When using the package, please provide the reference in your works.

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1 Introduction

This document describes `SRBF_package`. In Section (2), some necessary preliminaries are provided (the notation, the definition of SRBFs, etc.). In Section 3, the synthesis of the gravitational potential, the gravitational vector and the gravitational tensor are introduced. Finally, in Section 4, the analysis of the gravitational potential is discussed.

1.1 List of files

The files in `SRBF_package` are organized in two folders depending on the programming language: `MATLAB` and `Fortran`.

1.1.1 `SRBF_package` for MATLAB

Below is an overview of files related to the MATLAB version of the package.

- `src/MATLAB/SRBFs_synthesis.m` – Function to perform SRBF synthesis of the gravitational potential, the gravitational vector and the gravitational tensor.
- `src/MATLAB/SRBFs_analysis.m` – Function to perform surface SRBF analysis of data given on the reference sphere.
- `src/MATLAB/DH_grid.m` – Function to generate latitudes and longitudes of the points of the Driscoll–Healy grid.
- `src/MATLAB/Test_run.m` – Script to run two test computations in MATLAB.

Attached are also the following auxiliary files.

- `src/MATLAB/GrafLab.m` – Software for surface and solid spherical harmonic synthesis up to ultra-high harmonic degrees (Bucha and Janák, 2013). Here, it is used to synthesize input data from EGM96 (see the next bullet point).
- `src/MATLAB/EGM96.mat` – Global geopotential model (Lemoine et al, 1998) expanded in terms of spherical harmonics up to degree 360. It is used as a model of the Earth's gravity field to obtain meaningful input data for the test computations.

1.1.2 `SRBF_package` for Fortran

The Fortran 95 version of `SRBF_package` is organized as follows.

- `src/Fortran/vartypes.f95` – Module defining data types.
- `src/Fortran/constants.f95` – Module defining constants.
- `src/Fortran/SRBFs_synthesis.f95` – Subroutine to perform SRBF synthesis of the gravitational potential, the gravitational vector and the gravitational tensor.

- `src/Fortran/SRBFs_analysis.f95` – Subroutine to perform surface SRBF analysis of data given on the reference sphere.

- `src/Fortran/DH_grid.f95` – Subroutine to generate latitudes and longitudes of the points of the Driscoll–Healy grid.

- `src/Fortran/sph2cart.f95` – Subroutine to transform spherical coordinates to cartesian coordinates assuming the unit sphere.

- `src/Fortran/Test_run.f95` – Program to run two test computations in Fortran.

- `src/Fortran/Input_data` – This folder containts three input data files that are necessary to run the two test computations. Details on how these files were obtained are provided in `src/Fortran/Test_run.f95` and `src/Fortran/Python/Test_run.py` (see below).

- `src/Fortran/Python/Test_run.py` – Script to run the two test computations in Python.

- `src/Fortran/Python/SRBF_package.pyf` – Signature file to create a wrapper for Python. Further details on the wrapper can be found in `src/Fortran/Python/Test_run.py`.

2 Preliminaries

2.1 Notation

The notation used throughout this document is explained in Table 1.

Table 1: Notation

Symbol	Definition
$\mathbf{r}_i$	Position vector of the nodal point, at which the ith SRBF is formally located
$r_i, \varphi_i, \lambda_i$	Spherical coordinates of the ith nodal point (spherical radius, spherical latitude and spherical longitude, respectively)
$\mathbf{r}$	Position vector of the evaluation point
r, φ, λ	Spherical coordinates of the evaluation point (spherical radius, spherical latitude and spherical longitude, respectively)
α_i	Azimuth between the ith nodal point and the evaluation point
ψ_i	Spherical distance between the ith nodal point and the evaluation point
I	Total number of SRBFs
n, m	Spherical harmonic degree and order, respectively
$n_{\min}$, $n_{\max}$	Minimum and maximum degree of the expansion, respectively
$P_n(\cos\psi_i)$	Un-normalized Legendre polynomial of degree n
v_i	Expansion coefficient related to the ith SRBF
ϕ_n	Shape coefficient of SRBF of degree n, $\phi_n \in [0,1]$ for all n
GM	Geocentric gravitational constant
R	Radius of the reference sphere Ω_R, $\mathbf{r_i} \in \Omega_R$ and $\mathbf{r} \in \overline{\Omega_R^{\text{ext}}}$, where $\overline{\Omega_R^{\text{ext}}} = \Omega_R \cup \Omega_R^{\text{ext}}$ with Ω_R^{ext} being the space outside Ω_R
$\bar{Y}_{nm}$	4π-fully-normalized surface spherical harmonic function of degree n and order m
$\bar{V}_{nm}$	4π-fully-normalized spherical harmonic coefficients of degree n and order m
V	Gravitational potential
$\mathbf{g}$	Gravitational vector
$\mathbf{V}$	Gravitational tensor

2.2 Spherical radial basis functions

In `SRBF_package`, the following definition of SRBFs is adopted (Freeden and Schneider, 1998),

$$\phi(\mathbf{r}, \mathbf{r}_i) = \sum_{n=n_{\min}}^{n_{\max}} \frac{2n+1}{4\pi\, R^2}\, \phi_n \left(\frac{R}{r}\right)^{n+1} P_n(\cos\psi_i)\,. \tag{1}$$

The shape coefficients ϕ_n define properties of SRBFs in the spectral and spatial domain. Examples of commonly used shape coefficients can be found, for instance, in Freeden and Schneider (1998), Schmidt et al (2007) or Eicker (2008). Note that SRBFs from Eq. (1) (i) satisfy Laplace's equation everywhere except for the origin of the coordinate system and are (ii) isotropic and (iii) band-limited.

2.3 Position of the nodal points

To synthesize (reconstruct) a function at an evaluation point $\mathbf{r}$ via `SRBF_package` (Section 3), an arbitrary location and number of nodal points $\mathbf{r}_i$ can be inserted, provided that they reside on the reference sphere Ω_R, that is, $\mathbf{r}_i \in \Omega_R$. However, some specific point distributions that provide correct/accurate approximations are usually preferred, e.g., the

Driscoll–Healy grid or the Reuter grid (e.g., Freeden and Windheuser, 1997; Schmidt et al, 2007; Eicker, 2008).

Opposed to this, to analyse a function (Section 4), that is, to estimate its expansion coefficients (v_i in Eq. 3), `SRBF_package` relies on some specific distribution of nodal points. This is done in order to ensure that expansion coefficients of a band-limited function can be recovered by exact analytical formulae, avoiding any approximations. Here, we adopted the Driscoll–Healy grid, which defines the position of points through spherical latitudes, φ, and longitudes, λ, as follows

$$\begin{aligned} \varphi_j &= -\frac{\pi}{2} + \frac{\pi\, j}{2L}\,, \qquad && j = 0, \ldots, 2L\,, \\ \lambda_k &= \frac{\pi\, k}{L}\,, && k = 0, \ldots, 2L - 1\,, \end{aligned} \tag{2}$$

where $L = n_{\max} + 1$. The total number of the nodal points is $I = (2L + 1)\, 2L$. The index i from Eq. (1) is therefore equal to $i = 1, \ldots, I$. Note that, in `SRBF_package`, the nodal points have to reside on the reference sphere Ω_R, implying that their third spherical coordinate, the radius r_i, is equal to the radius of the reference sphere, $r_i = R$ for all $i = 1, \ldots, I$. As a result, `SRBF_package` can analyse only functions given on a sphere. Subsequently, the harmonic upward continuation can be performed via synthesis.

2.4 Local north-oriented reference frame

In `SRBF_package`, the gravitational vector (Section 3.2) and the gravitational tensor (Section 3.3) are expressed in the local north-oriented reference frame (LNOF). LNOF is a right-handed orthogonal coordinate system, whose origin is at the evaluation point and its axes are defined as follows: the x-axis points to the north, the y-axis points to the west and the z-axis points radially outwards.

3 Synthesis

3.1 Gravitational potential

The gravitational potential V is synthesized from a set of expansion coefficients v_i by the relation

$$V(\mathbf{r}) = \sum_{i=1}^{I} v_i \, \phi(\mathbf{r}, \mathbf{r}_i) \, . \tag{3}$$

Note that if (i) $\phi_n = 1$ for all $n = n_{\min}, \ldots, n_{\max}$ and $\phi_n = 0$ otherwise (see Eq. 1), and (ii) the number and the location of nodal points $\mathbf{r}_i$ are chosen properly, then Eq. (3) is equal to the following spherical harmonic expansion (e.g., Freeden and Schneider, 1998; Schmidt et al, 2005; Eicker, 2008)

$$V(\mathbf{r}) = \frac{GM}{R} \sum_{n=n_{\min}}^{n_{\max}} \left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \bar{V}_{nm} \, \bar{Y}_{nm}(\varphi, \lambda) \, , \tag{4}$$

where GM is the geocentric gravitational constant, m is spherical harmonic order, $\bar{V}_{nm}$ are spherical harmonic coefficients of the gravitational potential and $\bar{Y}_{nm}$ are 4π-fully-normalized surface spherical harmonics (see, e.g., Hoffmann-Wellenhof and Moritz, 2005). Numerically, the equality of the two particular expansions can be verified by running the test examples that are attached to the package.

By using appropriate values of the minimum and maximum harmonic degrees in Eq. (1) ($n_{\min}$ and $n_{\max}$, respectively), some specific spectral (frequency) band of the gravitational potential can be extracted, similarly as can be done in Eq. (4) with spherical harmonics.

Finally, we note that Eq. (3) can be used to reconstruct not only the gravitational potential, but also many other potential fields on and above some reference sphere Ω_R, such as the disturbing gravitational potential or the magnetic potential. Furthermore, scalar functions that are given on a sphere but are not defined above it can also be synthesized by Eq. (3), simply by assuming that the evaluation point may reside only on the same sphere as the nodal points, that is, $\mathbf{r} \in \Omega_R$ and $\mathbf{r}_i \in \Omega_R$. These functions even do not have to be harmonic. Naturally, each function has then a different set of expansion coefficients in general. In the following, we always assume that V is the gravitational potential and is evaluated on or above the reference sphere Ω_R.

3.2 Gravitational vector in LNOF

The gravitational vector can be obtained from Eq. (3) by applying the gradient operator expressed in LNOF (see, e.g., Eq. 36 of Bucha et al, 2019),

$$\mathbf{g}(\mathbf{r}) = \nabla V(\mathbf{r}) = \sum_{i=1}^{I} v_i \, \nabla \phi(\mathbf{r}, \mathbf{r}_i) = \begin{bmatrix} g^{\mathrm{x}}(\mathbf{r}) \\ g^{\mathrm{y}}(\mathbf{r}) \\ g^{\mathrm{z}}(\mathbf{r}) \end{bmatrix} \, , \tag{5}$$

where

$$g^{\mathrm{x}}(\mathbf{r}) = -\sum_{i=1}^{I} v_i \, \cos\alpha_i \, \phi^{1,1}(\mathbf{r}, \mathbf{r}_i) \,, \tag{6}$$

$$g^{\mathrm{y}}(\mathbf{r}) = \sum_{i=1}^{I} v_i \, \sin\alpha_i \, \phi^{1,1}(\mathbf{r}, \mathbf{r}_i) \,, \tag{7}$$

$$g^{\mathrm{z}}(\mathbf{r}) = \sum_{i=1}^{I} v_i \, \phi^{1,0}(\mathbf{r}, \mathbf{r}_i) \tag{8}$$

and

$$\phi^{1,0}(\mathbf{r}, \mathbf{r}_i) = -\frac{1}{R} \sum_{n=n_{\min}}^{n_{\max}} \frac{2n+1}{4\pi\, R^2} \, \phi_n \, (n+1) \left(\frac{R}{r}\right)^{n+2} P_n(\cos\psi_i) \,, \tag{9}$$

$$\phi^{1,1}(\mathbf{r}, \mathbf{r}_i) = -\frac{1}{R} \sum_{n=n_{\min}}^{n_{\max}} \frac{2n+1}{4\pi\, R^2} \, \phi_n \left(\frac{R}{r}\right)^{n+2} \sin\psi_i \, \frac{\partial P_n(\cos\psi_i)}{\partial \cos\psi_i} \,. \tag{10}$$

3.3 Gravitational tensor in LNOF

Similarly, the gravitational tensor is given as (see the differential operators in Šprlák et al 2015 or Bucha et al 2019)

$$\mathbf{V}(\mathbf{r}) = \nabla \otimes \nabla V(\mathbf{r}) = \sum_{i=1}^{I} v_i \, \nabla \otimes \nabla \phi(\mathbf{r}, \mathbf{r}_i) = \begin{bmatrix} V^{\mathrm{xx}}(\mathbf{r}) & V^{\mathrm{xy}}(\mathbf{r}) & V^{\mathrm{xz}}(\mathbf{r}) \\ V^{\mathrm{yx}}(\mathbf{r}) & V^{\mathrm{yy}}(\mathbf{r}) & V^{\mathrm{yz}}(\mathbf{r}) \\ V^{\mathrm{zx}}(\mathbf{r}) & V^{\mathrm{zy}}(\mathbf{r}) & V^{\mathrm{zz}}(\mathbf{r}) \end{bmatrix} \,, \tag{11}$$

where

$$V_{\mathrm{xx}}(\mathbf{r}) = \sum_{i=1}^{I} v_i \left(-\frac{1}{2} \phi^{2,0}(\mathbf{r}, \mathbf{r}_i) + \cos 2\alpha_i \, \phi^{2,2}(\mathbf{r}, \mathbf{r}_i) \right) \,, \tag{12}$$

$$V_{\mathrm{xy}}(\mathbf{r}) = -\sum_{i=1}^{I} v_i \, \sin 2\alpha_i \, \phi^{2,2}(\mathbf{r}, \mathbf{r}_i) \,, \tag{13}$$

$$V_{\mathrm{xz}}(\mathbf{r}) = \sum_{i=1}^{I} v_i \, \cos\alpha_i \, \phi^{2,1}(\mathbf{r}, \mathbf{r}_i) \,, \tag{14}$$

$$V_{\mathrm{yy}}(\mathbf{r}) = \sum_{i=1}^{I} v_i \left(-\frac{1}{2} \phi^{2,0}(\mathbf{r}, \mathbf{r}_i) - \cos 2\alpha_i \, \phi^{2,2}(\mathbf{r}, \mathbf{r}_i) \right) \,, \tag{15}$$

$$V_{\mathrm{yz}}(\mathbf{r}) = -\sum_{i=1}^{I} v_i \, \sin\alpha_i \, \phi^{2,1}(\mathbf{r},\mathbf{r}_i)\,, \tag{16}$$

$$V_{\mathrm{zz}}(\mathbf{r}) = \sum_{i=1}^{I} v_i \, \phi^{2,0}(\mathbf{r},\mathbf{r}_i) \tag{17}$$

and

$$\phi^{2,0}(\mathbf{r},\mathbf{r}_i) = \frac{1}{R^2} \sum_{n=n_{\min}}^{n_{\max}} \frac{2n+1}{4\pi\, R^2}\, \phi_n \,(n+1)\,(n+2)\, \left(\frac{R}{r}\right)^{n+3} P_n(\cos\psi_i)\,, \tag{18}$$

$$\phi^{2,1}(\mathbf{r},\mathbf{r}_i) = -\frac{1}{R^2} \sum_{n=n_{\min}}^{n_{\max}} \frac{2n+1}{4\pi\, R^2}\, \phi_n \,(n+2)\, \left(\frac{R}{r}\right)^{n+3} \sin\psi_i \, \frac{\partial P_n(\cos\psi_i)}{\partial \cos\psi}\,, \tag{19}$$

$$\phi^{2,2}(\mathbf{r},\mathbf{r}_i) = \frac{1}{2R^2} \sum_{n=n_{\min}}^{n_{\max}} \frac{2n+1}{4\pi\, R^2}\, \phi_n \, \left(\frac{R}{r}\right)^{n+3} \sin^2\psi_i \, \frac{\partial^2 P_n(\cos\psi_i)}{\partial(\cos\psi_i)^2}\,. \tag{20}$$

These relations for the gravitational tensor are more efficient than that from Appendix of Bucha et al (2016). This is because only three radial basis functions ($\phi^{2,0}$, $\phi^{2,1}$, $\phi^{2,2}$) are needed to compute six unique elements of the gravitational tensor. This may significantly improve computational speed, since the evaluation of band-limited spherical radial basis functions (the sum over harmonic degrees) is the most time-consuming part of the synthesis.

4 Analysis

In `SRBF_package`, the analysis is performed on the reference sphere Ω_R, at which the data have to be given. We adopted the exact quadrature due to Driscoll and Healy (1994). The latitudes and the longitudes of the Driscoll–Healy grid points are defined in Section 2.3 and the corresponding weights read

$$w_j = \frac{2\pi\, R^2}{L^2}\, \sin\vartheta_j \sum_{k=0}^{L-1} \frac{\sin\left[(2k+1)\,\vartheta_j\right]}{2k+1}\,, \tag{21}$$

where $\vartheta_j = \pi/2 - \varphi_j$ is the co-latitude. Then,

$$\tilde{v}_{jk} = w_j\, V(\mathbf{r}_{jk})\,. \tag{22}$$

Finally, the expansion coefficients v_i from Section 3 are obtained after some suitable rearrangement of $\tilde{v}_{jk}$ to a single-indexed variable v_i, $i = 1, \ldots, I$ (cf. Section 2.3), here as $\tilde{v}_{00},\ \tilde{v}_{10},\ \ldots, \tilde{v}_{2L+1,0},\ \tilde{v}_{10},\ \tilde{v}_{11},\ \ldots, \tilde{v}_{2L+1,1},\ \ldots,\ \tilde{v}_{2L+1,2L}$.

References

Bucha B, Janák J (2013) A MATLAB-based graphical user interface program for computing functionals of the geopotential up to ultra-high degrees and orders. Computers and Geosciences 56:186–196, doi: 10.1016/j.cageo.2013.03.012

Bucha B, Janák J, Papčo J, Bezděk A (2016) High-resolution regional gravity field modelling in a mountainous area from terrestrial gravity data. Geophysical Journal International 207:949–966, doi: 10.1093/gji/ggw311

Bucha B, Hirt C, Kuhn M (2019) Cap integration in spectral gravity forward modelling up to the full gravity tensor. Journal of Geodesy doi: 10.1007/s00190-019-01277-3

Driscoll JR, Healy DM (1994) Computing Fourier transforms and convolutions on the 2-sphere. Advances in Applied Mathematics 15:202–250

Eicker A (2008) Gravity field refinement by radial basis functions from in-situ satellite data. PhD thesis, Universität Bonn, Bonn, Germany, 137 pp

Freeden W, Schneider F (1998) An integrated wavelet concept of physical geodesy. Journal of Geodesy 72:259–281

Freeden W, Windheuser U (1997) Combined spherical harmonic and wavelet expansion – a future concept in Earth's gravitational determination. Applied and Computational Harmonic Analysis 4:1–37

Hoffmann-Wellenhof B, Moritz H (2005) Physical Geodesy. Springer, Wien, New York, 403 pp

Lemoine FG, Kenyon SC, Factor JK, Trimmer RG, Pavlis NK, Chinn DS, Cox CM, Klosko SM, Luthcke SB, Torrence MH, Wang YM, Williamson RG, Pavlis EC, Rapp RH, Olson TR (1998) The development of the joint NASA GSFC and the National Imagery and Mapping Agency (NIMA) geopotential model EGM96. Tech. Rep. NASA/TP1998206861, National Aeronautics and Space Administration, USA, 582 pp

Schmidt M, Fabert O, Shum CK (2005) On the estimation of a multi-resolution representation of the gravity field based on spherical harmonics and wavelets. Journal of Geodynamics 39:512–526, doi: 10.1016/j.jog.2005.04.007

Schmidt M, Fengler M, Mayer-Gürr T, Eicker A, Kusche J, Sánchez L, Han SC (2007) Regional gravity modeling in terms of spherical base functions. Journal of Geodesy 81:17–38, doi: 10.1007/s00190-006-0101-5

Šprlák M, Hamáčková E, Novák P (2015) Alternative validation method of satellite gradiometric data by integral transform of satellite altimetry data. Journal of Geodesy 89:757–773, doi: 10.1007/s00190-015-0813-5