Spectral Gravity Forward Modelling of Continuous 3D Mass Density Distributions [G33A-0533]

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Abstract

We generalize spectral gravity forward modelling to **any continuous 3D mass density distributions** of topographic masses. The density function is modelled by a polynomial in the radial direction, while each density polynomial coefficient is expanded into surface spherical harmonics. The method is generalized to **any integration radius**, enabling to integrate near-zone, far-zone and global topographic masses.

Method

The gravitational potential of topographic masses is given by the Newton integral

$$\boxed{V(r,\Omega) = G \iint\limits_{\Omega'} \int\limits_{r'=R}^{R+H(\Omega')} \frac{\rho(r',\Omega')}{\ell(r,\psi,r')} (r')^2 \, \mathrm{d}r' \, \mathrm{d}\Omega',} \tag{1}$$

where we assume the density ρ to be any 3D continuous function, so that it can be expressed as

$$\rho(r',\Omega') = \sum_{i=0}^{\infty} \rho_i(\Omega') \, (r')^i, \qquad \text{where} \qquad \rho_i(\Omega') = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{\rho}_{nm}^{(i)} \, \bar{Y}_{nm}(\Omega'). \tag{2}$$

Global variant

Substituting

$$\frac{1}{\ell(r,\psi,r')} = \sum_{n=0}^{\infty} \frac{(r')^n}{r^{n+1}} \frac{1}{2n+1} \sum_{m=-n}^{n} \bar{Y}_{nm}(\Omega) \, \bar{Y}_{nm}(\Omega'), \quad r > r', \tag{3}$$

into Eq. (1) and analytically evaluating the integral over the spherical radius r', we get

$$\boxed{V(r,\Omega) = \frac{GM}{R} \sum_{n=0}^{\infty} \left(\frac{R}{r}\right)^{n+1} \sum_{m=-n}^{n} \bar{V}_{nm} \, \bar{Y}_{nm}(\Omega),} \tag{4}$$

where

$$\overline{H\rho}_{nm}^{(pi)} = \frac{1}{4\pi} \iint\limits_{\Omega'} \left[\left(\frac{H(\Omega')}{R} \right)^p \rho_i(\Omega') \, R^i \right] \bar{Y}_{nm}(\Omega') \, \mathrm{d}\Omega', \tag{5}$$

$$\bar{V}_{nm} = \frac{2\pi \, R^3}{M} \sum_{p=1}^{\infty} \sum_{i=0}^{\infty} S_{npi} \, \overline{H\rho}_{nm}^{(pi)}, \qquad S_{npi} = \frac{2}{2n+1} \frac{1}{n+i+3} \binom{n+i+3}{p}. \tag{6}$$

Cap-modified variant

To spatially restrict the integration to near- or far-zone topographic masses, we employ the concept of Molodensky's truncation coefficients, here denoted as $Q_{npi}^{0,0,j}(r,\psi_0)$, where ψ_0 is the integration radius. The near- and far-zone effects on the gravitational potential ($j = \text{'In'}$ or 'Out', respectively) read

$$\boxed{V^j(r,\Omega,\psi_0) = \frac{GM}{R} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \bar{V}_{nm}^{0,0,j}(r,\psi_0,R) \, \bar{Y}_{nm}(\Omega),} \tag{7}$$

where

$$\bar{V}_{nm}^{0,0,j}(r,\psi_0,R) = \frac{2\pi \, R^3}{M} \sum_{p=1}^{\infty} \sum_{i=0}^{\infty} Q_{npi}^{0,0,j}(r,\psi_0,R) \, \overline{H\rho}_{nm}^{(pi)}. \tag{8}$$

Formally similar relations were derived for the full gravitational vector and the full gravitational tensor. Interestingly, only **three** groups of truncation coefficients and of their radial derivatives are needed to describe 10 gravitational field quantities and all their radial derivatives, $Q_{npi}^{0,0,j}(r,\psi_0,R)$, $Q_{npi}^{1,1,j}(r,\psi_0,R)$ and $Q_{npi}^{2,2,j}(r,\psi_0,R)$.

Implementation

- **Programming language**: C,
- **Parallelization**: OpenMP (shared memory architectures),
- **SIMD parallelization**: AVX, AVX2 and AVX-512,
- **Harmonic analysis**: Gauss–Legendre quadrature,
- **External C libraries**: FFTW3 (fast Fourier transform), GNU GMP and GNU MPFR (multiple-precision floating-point computations).
- **Precision (except for truncation coefficients)**: double (available also in single and quadruple precision)

The GMP and MPFR libraries are used to extend the number of significant digits (often well-beyond the quadruple precision) when computing the truncation coefficients.

The implementation will be soon available through **CHarm**, a C library for high-degree spherical harmonic transforms (visit `https://www.charmlib.org`).

Experiment Setup

- **Moon's topographic masses:** MoonTopo2600p.shape [1] up to degree 360 referenced to $R = 1{,}728{,}200$ m (Fig. 1)

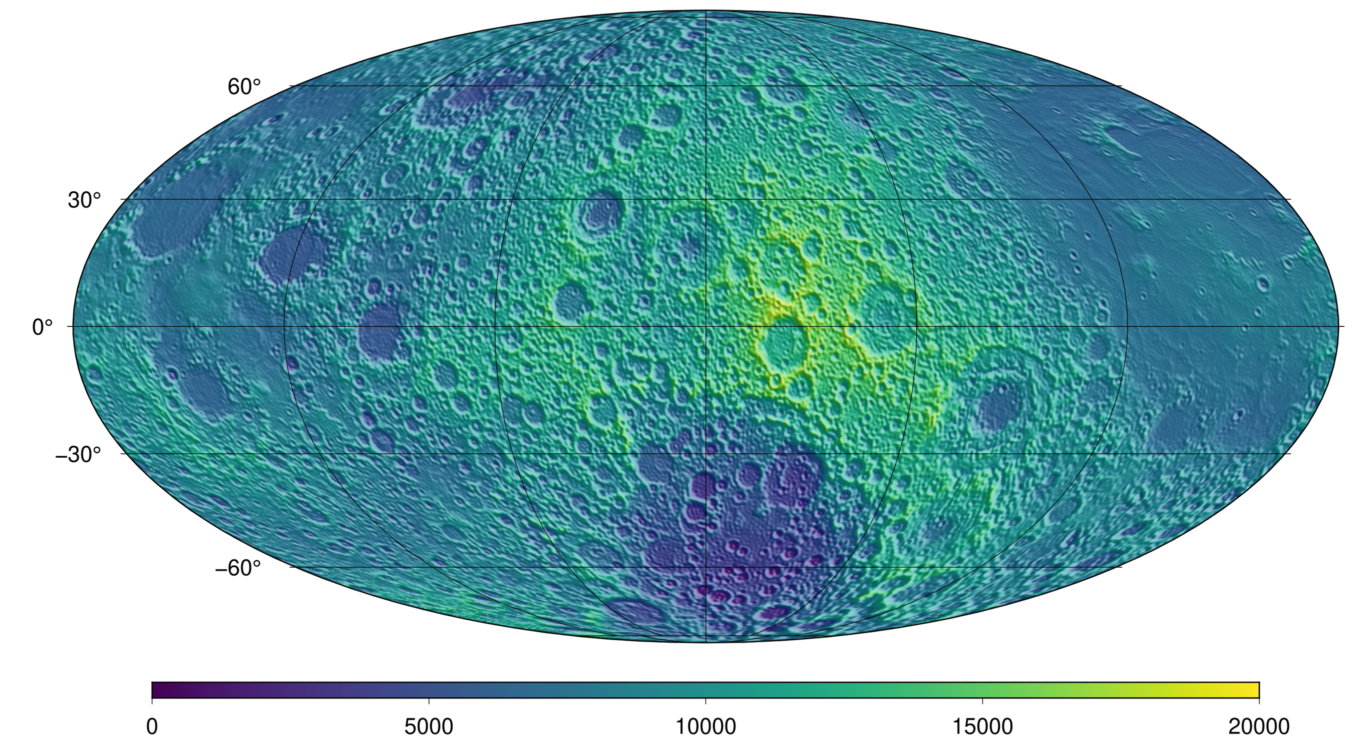

Figure 1. Moon's topographic masses (m) above the reference sphere of radius $1{,}728{,}200$ m

- **Density model:** maximum harmonic degree 180, $i_{\max} = 1$ (obtained from 3D density model due to [2]; the original model is shown in Fig. 2)

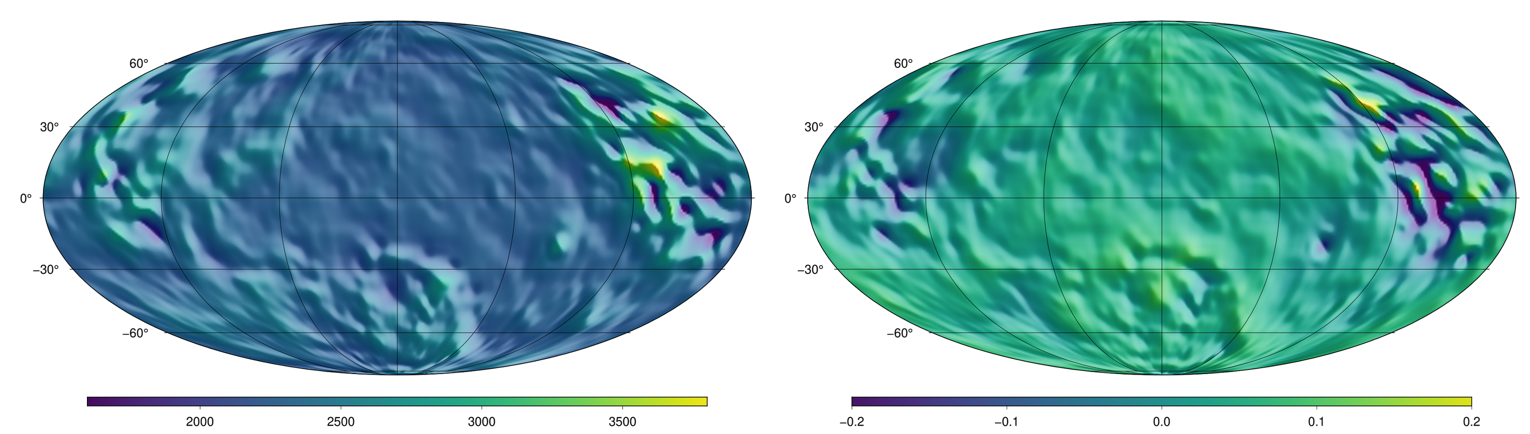

Figure 2. Surface density (*left*; $\mathrm{kg\ m^{-3}}$) and its first-order gradient (*right*; $\mathrm{kg\ m^{-3}\ km}$)

- **Evaluation points:** $5' \times 5'$ grid on a Brillouin sphere with the radius $r = 1{,}750{,}000$ m
- **Integration radius** ψ_0: $10°$
- **Maximum topography power** $p_{\max}$: 20
- **Precision to evaluate** $Q_{npi}^{0,0,j}(r,\psi_0)$: 200 bits for the significand

Results

- **Near-zone effects:** maximum degree 2160 (Fig. 3)

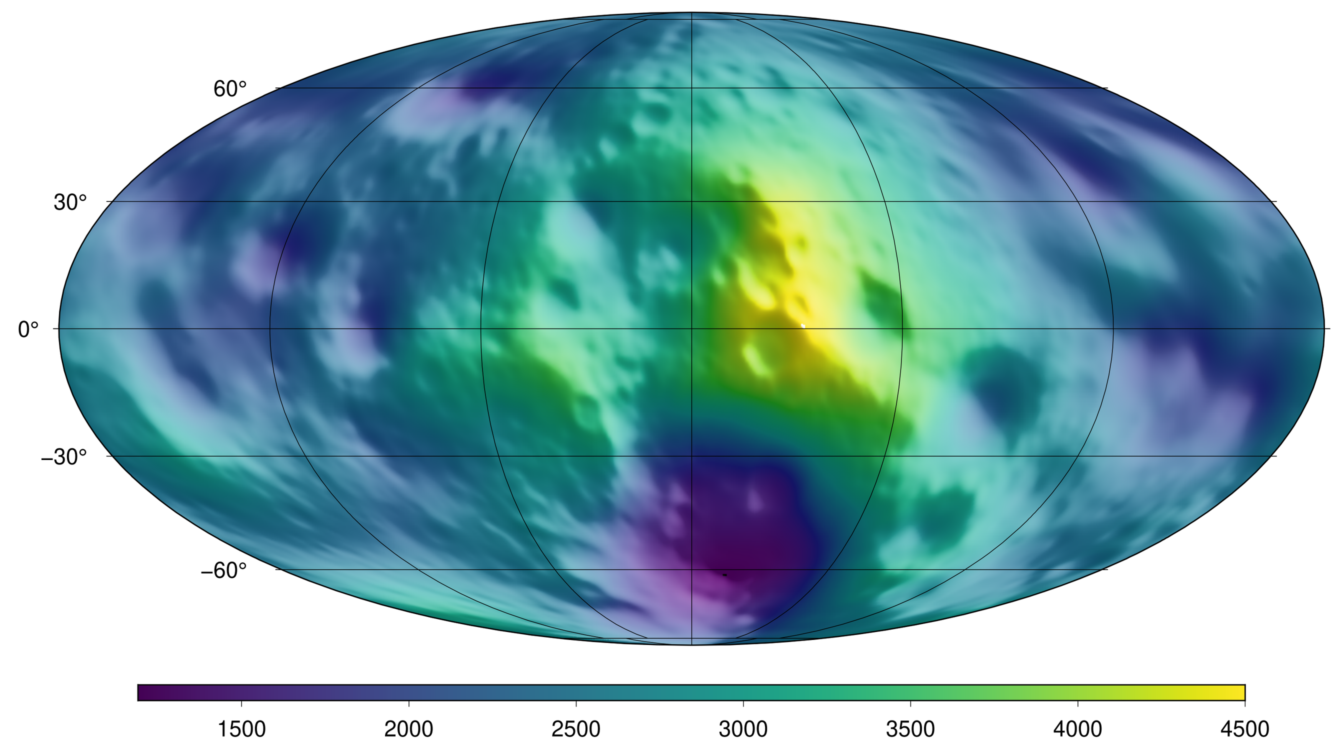

Figure 3. Gravitational potential induced by near-zone masses ($\mathrm{m^2\ s^{-2}}$)

- **Far-zone effects:** maximum degree 2160 (Fig. 4)

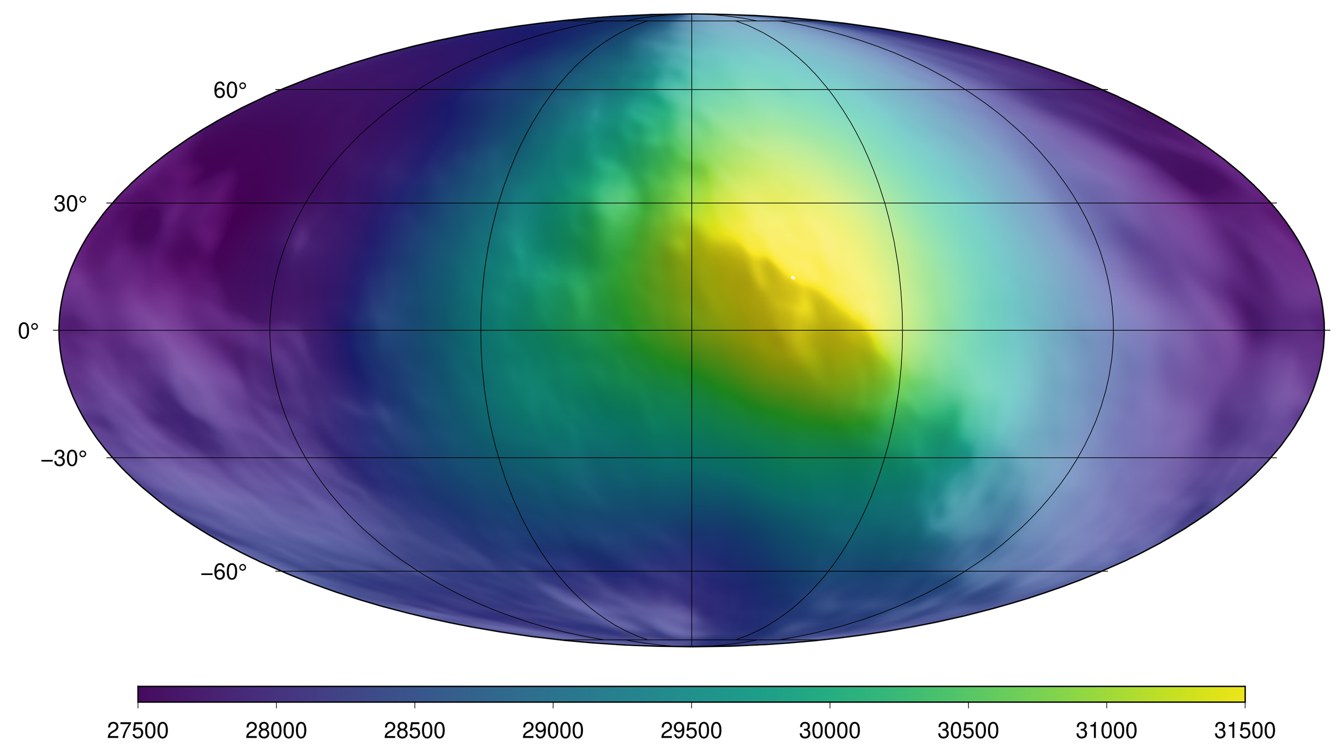

Figure 4. Gravitational potential induced far-zone masses ($\mathrm{m^2\ s^{-2}}$)

Only **less than 1.5 minutes** were needed to compute one of the two gravitational effects (inluding 40 harmonic analyses and 40 harmonic syntheses, I/Os, etc.). The computations were conducted on an ordinary PC with 6 CPU cores clocked at 3.40GHz.

Summary

- Spectral gravity forward modelling of 3D density distributions developed
- Implemented in CHarm
- Can be used for evaluation points on irregular surface (e.g., the Earth's surface)

References

[1] M. A. Wieczorek, "Gravity and topography of the terrestrial planets," in *Treatise on Geophysics* (G. Schubert, ed.), ch. 10.5, pp. 153–193, Elsevier, 2 ed., 2015.

[2] S. Goossens, T. J. Sabaka, M. A. Wieczorek, G. A. Neumann, E. Mazarico, F. G. Lemoine, J. B. Nicholas, D. E. Smith, and M. T. Zuber, "High-resolution gravity field models from GRAIL data and implications for models of the density structure of the Moon's crust," *Journal of Geophysical Research: Planets*, vol. 125, p. e2019JE006086, 2020.