Mathematics 2, Homework 3 part 2 **April 26th 2024**

Duals and duals and duals

Consider the following two LPs.

$$\begin{aligned} \max\ & c^T x \\ & Ax \leq b \\ & x \geq 0 \end{aligned} \qquad \text{(P)} \qquad\qquad \begin{aligned} \max\ & b^T y \\ & A^T y \geq c \\ & y \geq 0 \end{aligned} \qquad \text{(D)}$$

The *standard form* of a LP is the pattern (maximization, less-than constraints, nonnegativity) of (P). We have shown that the problem (D) is the *dual* of (P).

1. Show that (P) is the dual of (D). You can do that by rewriting (P) in standard form and dualizing it according to the above pattern.
2. Show that problem [1, problem (D)] is the dual of [1, problem (P)]. You can rewrite one of them in standard form and dualize it.

Bounding the unbounded

Observe the following two obvious examples. Maximizing the sum of coordinates in the positive quadrant (of $\mathbb{R}^2$) is an unbounded linear program. And minimizing the sum of coordinates in the positive quadrant is a linear problem that has an optimal solution. Both problems share the same set of feasible solutions which is by itself unbounded.

There is nothing we can do in the former case. However, in the latter we would like to a-priori bound the coordinates so that (i) we preserve the feasibility of the problem and also (ii) we do not alter the optimum. This can be done:

Proposition 1 *Observe the linear program*

$$\begin{aligned} \min\ & c^T x \\ & Ax = b \\ & x \geq 0, \end{aligned} \qquad \text{(P1)}$$

where A is an $n \times m$ matrix, c, x are m-dimensional, and b is an n-dimensional vector. Assume also that the coefficients so A, b, c are integers and are (absolutely) bounded by U. Let $M = (nU)^n$.

If (LP) *is feasible, then there exists a feasible solution with all coordinates bounded by M. Also if* (LP) *has an optimal solution, then there exists an optimal solution with all coordinates bounded by M.*

3. Prove Proposition 1.

(a) This seems a Herculean task without proper guidance. So let us try to give a few directions.

(b) Observe first that we have equality constraints in (P1). We know that inequality constraints can be transformed to equalities using slack/surplus variables. So (P1) is as general as it gets.

(b) Estimate the determinant of a square matrix in terms of its coefficients and its dimension. Try not to overcomplicate, the first bound that you find will probably suffice.

(c) Get familiar with Cramer's rule. What are the exact conditions that allow its application?

(d) Which linear systems of equalities have unique solutions? Is there always a square matrix involved?

(d) Look for a feasible solution with as many 0 coordinates as possible. Or more precisely, let x° be a feasible solution whose *support*[1] S is *containmentwise*[2] minimal. Show that x° is unique. (Assuming there is an alternative $x^{\circ\prime}$ this can be established by observing the line through x° and $x^{\circ\prime}$ towards the boundary of the feasibility region.)

(e) Repeat the same argument with an optimal solution (assuming one exists) x^* whose support is *containmentwise* minimal. It should be unique as well.

(f) Finally focus on nonzero coordinates of x° (and similarly x^*) and rewrite the constraints.

(g) Can you prove an analogous lower bound for the smallest nonzero coordinates of said x° (and similarly x^*)?

Implement the interior-point algorithm

Let us be precise and not as enthusiastic. Consider the problem

$$\begin{aligned} Ax &= b \\ A^T y + s &= c \\ x > 0, s &> 0 \end{aligned} \tag{P_μ}$$

Assuming that you are given a solution (x, y, s, μ) to (P_μ), compute the next solution (x', y', s', μ') to $(P_{\mu'})$, where

$$\mu' = \left(1 - \frac{1}{6\sqrt{m}}\right)\mu.$$

4. Implement an algorithm computing the next-step solution (x', y', s', μ') from (x, y, s, μ). You are of course allowed to use library routines for solving linear systems.

[1] Support of a vector is the set of nonzero indices/coordinates.

[2] We compare supports by subset relation.

Observe the following linear problem.

$$\begin{aligned}
\min\ & -3x_1 - 4x_2 \\
& 3x_1 + 3x_2 + 3x_3 = 4 \\
& 3x_1 + x_2 + x_4 = 3 \\
& x_1 + 4x_2 + x_5 = 4 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0,
\end{aligned} \tag{problem X}$$

5. Compute the dual problem of (problem X).

6. Show that the vectors

$$\begin{aligned}
x &= \left(\frac{2}{5}, \frac{8}{15}, \frac{2}{5}, \frac{19}{15}, \frac{22}{15}\right)^T \\
y &= \left(-\frac{4}{5}, -\frac{4}{5}, -\frac{2}{3}\right)^T \\
s &= \left(\frac{37}{15}, \frac{28}{15}, \frac{12}{5}, \frac{4}{5}, \frac{2}{3}\right)^T
\end{aligned}$$

are strictly feasible solutions of both (problem X) and its dual.

7. Show that above vectors form a good starting solution for a corresponding problem (P_μ) (What is your initial choice of μ? Is $\mu = 1$ an appropriate possibility?

8. Iterate your next-step algorithm until it converges (or at least stabilizes on most of the digits).

9. Heuristically decide when to stop. If your current μ is sufficiently small, then for every i the product $x_i s_i$ is small. For every i make a decision and set either $x_i = 0$ or $s_i = 0$, depending on which of x_i, s_i is closer to zero. This gives an extra collection of m scalar equations.

10. Test whether your choice is correct. Exactly compute the solution of

$$\begin{aligned}
Ax &= b \\
A^T y + s &= c
\end{aligned}$$

assuming the above m scalar equations, and test if the solutions are indeed feasible and optimal.

Commercial solver

Look for commercial (open-source) optimization solvers that allow the use of both simplex and interior-point methods.

12. Test the solver of your choice on (problem X). Use both the combinatorial (simplex) method and interior-point method.

References

[1] Kurt Mehlhorn and Sanjeev Saxena. A still simpler way of introducing the interior-point method for linear programming (ver. 8dec21). *CoRR*, abs/1510.03339, 2015.

Upload your solution in a single .zip archive which contains the source-code (I would be most pleased with a Jupyter notebook in Python - nevertheless you can use any programming platform/language according to your personal preferences) and a .pdf.