Mathematical Formulation: Direct Transcription for Drone Racing - *Srijan Dokania*

1. STATE SPACE

State Vector Definition

The state at time step k is a 6-dimensional vector:

$$\mathbf{x}_k = \begin{bmatrix} p_x \\ p_y \\ p_z \\ \psi \\ \theta \\ \phi \end{bmatrix} \in \mathbb{R}^6$$

where:

- **Position** (first 3 components): $\mathbf{p} = [p_x, p_y, p_z]^T \in \mathbb{R}^3$ represents the drone's 3D position in the world frame (meters)
- **Attitude** (last 3 components): $\boldsymbol{\eta} = [\psi, \theta, \phi]^T \in \mathbb{R}^3$ represents Euler angles (radians)
  - ψ: yaw (rotation about z-axis)
  - θ: pitch (rotation about y-axis)
  - ϕ: roll (rotation about x-axis)

State Trajectory

For N time steps, we have $N+1$ state vectors:

$$\mathbf{X} = [\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_N] \in \mathbb{R}^{(N+1)\times 6}$$

Total state variables: $6(N+1)$

---

2. CONTROL SPACE

Control Vector Definition

The control at time step k is a 4-dimensional vector:

$$\mathbf{u}_k = \begin{bmatrix} \Delta\psi \\ \Delta\theta \\ \Delta\phi \\ T \end{bmatrix} \in \mathbb{R}^4$$

where:

- **Attitude rate changes**: $\Delta\boldsymbol{\eta} = [\Delta\psi, \Delta\theta, \Delta\phi]^T \in \mathbb{R}^3$ (radians per time step)
  - Bounds: $|\Delta\psi|, |\Delta\theta|, |\Delta\phi| \leq 10^\circ = 0.1745$ rad

- **Thrust magnitude**: $T \in \mathbb{R}$ (Newtons)
    - Bounds: $5 \leq T \leq 20$ N

Control Trajectory

For N time steps, we have N control vectors:

$$\mathbf{U} = [\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{N-1}] \in \mathbb{R}^{N \times 4}$$

Total control variables: $4N$

---

3. DECISION VARIABLE VECTOR

In direct transcription, all states and controls are decision variables. They are packed into a single flat vector:

$$\mathbf{z} = [\mathbf{x}_0, \mathbf{u}_0, \mathbf{x}_1, \mathbf{u}_1, \ldots, \mathbf{x}_{N-1}, \mathbf{u}_{N-1}, \mathbf{x}_N] \in \mathbb{R}^{10N+6}$$

Dimension Calculation

$$\dim(\mathbf{z}) = \underbrace{6(N+1)}_{\text{states}} + \underbrace{4N}_{\text{controls}} = 6N + 6 + 4N = 10N + 6$$

Examples

- **For** $N = 15$: $\dim(\mathbf{z}) = 10(15) + 6 = 156$ variables
- **For** $N = 30$: $\dim(\mathbf{z}) = 10(30) + 6 = 306$ variables

More precisely:

- States: $\mathbf{x}_k \in \mathbf{z}[6k : 6(k+1)]$ for $k = 0, 1, \ldots, N$
- Controls: $\mathbf{u}_k \in \mathbf{z}[6(N+1) + 4k : 6(N+1) + 4(k+1)]$ for $k = 0, 1, \ldots, N-1$

---

4. DYNAMICS MODEL

4.1 Overview

The dynamics function maps current state and control to next state:

$$\mathbf{x}_{k+1} = f(\mathbf{x}_k, \mathbf{u}_k)$$

where $f : \mathbb{R}^6 \times \mathbb{R}^4 \rightarrow \mathbb{R}^6$

4.2 Rotation Matrix from Euler Angles

The rotation matrix $\mathbf{R} \in SO(3)$ is computed using the ZYX convention:

$$\mathbf{R}(\psi, \theta, \phi) = \mathbf{R}_z(\psi)\mathbf{R}_y(\theta)\mathbf{R}_x(\phi)$$

4.3 Force Computation

Thrust Force in World Frame

The thrust vector in the body frame (FLU: Forward-Left-Up) is:

$$\mathbf{T}^{\text{body}} = \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix} \in \mathbb{R}^3$$

Transform to world frame:

$$\mathbf{T}^{\text{world}} = \mathbf{R}(\psi, \theta, \phi)\mathbf{T}^{\text{body}} = \mathbf{R} \begin{bmatrix} 0 \\ 0 \\ T \end{bmatrix}$$

Gravitational Force

$$\mathbf{W} = \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix} \in \mathbb{R}^3$$

where $m = 1.0$ kg, $g = 10.0$ m/s²

Net Force

$$\mathbf{F} = \mathbf{T}^{\text{world}} + \mathbf{W} \in \mathbb{R}^3$$

4.4 Terminal Velocity Model

Key Physics: At terminal velocity, drag force equals net force.

For quadratic drag: $\mathbf{F}_{\text{drag}} = k_d|\mathbf{v}|^2$

At equilibrium: $\mathbf{F} = \mathbf{F}_{\text{drag}}$

Solving component-wise:

$$v_{\text{terminal},i} = \text{sign}(F_i)\sqrt{\frac{|F_i| + \epsilon}{k_d}} \quad \text{for } i \in \{x, y, z\}$$

where:

- $k_d = 0.0425$ (drag coefficient from skydio2)
- $\epsilon = 10^{-6}$ (numerical regularization)

In vector form:

$$\mathbf{v}_{\text{terminal}} = \text{sign}(\mathbf{F}) \odot \sqrt{\frac{|\mathbf{F}| + \epsilon\mathbf{1}}{k_d}} \in \mathbb{R}^3$$

where $\odot$ denotes element-wise multiplication, and operations on $\mathbf{F}$ are element-wise.

4.5 Complete Dynamics Function

Given state $\mathbf{x}_k = [p_x, p_y, p_z, \psi, \theta, \phi]^T$ and control $\mathbf{u}_k = [\Delta\psi, \Delta\theta, \Delta\phi, T]^T$:

Position Dynamics

$$\mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{v}_{\text{terminal}}(\psi_k, \theta_k, \phi_k, T_k) \cdot \Delta t$$

Explicitly:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_{k+1} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}_k + \text{sign}(\mathbf{F}_k) \odot \sqrt{\frac{|\mathbf{F}_k| + \epsilon \mathbf{1}}{k_d}} \cdot \Delta t$$

where $\mathbf{F}_k = \mathbf{R}(\psi_k, \theta_k, \phi_k) \begin{bmatrix} 0 \\ 0 \\ T_k \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -mg \end{bmatrix}$

Attitude Dynamics

$$\begin{aligned} \psi_{k+1} &= \psi_k + \Delta\psi_k \\ \theta_{k+1} &= \theta_k + \Delta\theta_k \\ \phi_{k+1} &= \phi_k + \Delta\phi_k \end{aligned}$$

With angle wrapping to $[-\pi, \pi]$.

4.6 Mathematical Properties

This dynamics model is:

1. **Quasi-static**: Velocity instantly reaches equilibrium (no acceleration dynamics)
2. **Smooth**: Differentiable almost everywhere (except at $F_i = 0$, handled by ϵ)
3. **Physics-based**: Matches terminal velocity equilibrium from drag model

---

5. COST FUNCTION (OBJECTIVE)

The optimization minimizes:

$$J(\mathbf{z}) = \sum_{k=0}^{N-1} [\ell_{\text{thrust}}(\mathbf{u}_k) + \ell_{\text{angular}}(\mathbf{u}_k) + \ell_{\text{smooth}}(\mathbf{u}_k, \mathbf{u}_{k+1}) + \ell_{\text{gimbal}}(\mathbf{x}_k)]$$

where $J : \mathbb{R}^{10N+6} \to \mathbb{R}$

5.1 Thrust Deviation Cost

Penalizes deviation from hover thrust:

$$\ell_{\text{thrust}}(\mathbf{u}_k) = w_{\text{thrust}}(T_k - T_{\text{hover}})^2$$

where:

- $T_{\text{hover}} = 10$ N (equilibrium thrust for 1 kg drone)

- $w_{\text{thrust}} \in \{0.05, 0.1\}$ (racing vs normal)

5.2 Angular Velocity Cost

Penalizes large attitude changes:

$$\begin{aligned}\ell_{\text{angular}}(\mathbf{u}_k) &= w_{\text{angular}} \left((\Delta\psi_k)^2 + (\Delta\theta_k)^2 + (\Delta\phi_k)^2\right) \\ &= w_{\text{angular}} \|\Delta\boldsymbol{\eta}_k\|_2^2\end{aligned}$$

where:

- $w_{\text{angular}} \in \{0.5, 1.0\}$ (racing vs normal)

5.3 Control Smoothness Cost

Penalizes control jerk (second derivative of state):

$$\ell_{\text{smooth}}(\mathbf{u}_k, \mathbf{u}_{k+1}) = \begin{cases} w_{\text{smoothness}} \|\mathbf{u}_{k+1} - \mathbf{u}_k\|_2^2 & \text{if } k > 0 \\ 0 & \text{if } k = 0 \end{cases}$$

Explicitly:

$$\ell_{\text{smooth}} = w_{\text{smoothness}} \sum_{i=1}^{4} (u_{k+1,i} - u_{k,i})^2$$

where:

- $w_{\text{smoothness}} \in \{5.0, 10.0\}$ (normal vs racing)

Motivation: Minimizes jerk, leading to smooth trajectories.

5.4 Gimbal Lock Penalty

Soft constraint to avoid gimbal lock singularity:

$$\ell_{\text{gimbal}}(\mathbf{x}_k) = \begin{cases} 1000 \cdot (\|\theta_k\| - \theta_{\text{limit}})^2 & \text{if } |\theta_k| > \theta_{\text{limit}} \\ 0 & \text{otherwise} \end{cases}$$

where $\theta_{\text{limit}} = 50^\circ = \frac{5\pi}{18}$ rad

Motivation: Prevents numerical issues near $\theta = \pm 90^\circ$ where Euler angles are singular.

5.5 Total Cost

Combining all terms:

$$\boxed{J(\mathbf{z}) = \sum_{k=0}^{N-1} \left[w_{\text{thrust}}(T_k - 10)^2 + w_{\text{angular}} \|\Delta\boldsymbol{\eta}_k\|_2^2 + w_{\text{smoothness}} \|\mathbf{u}_{k+1} - \mathbf{u}_k\|_2^2 + \ell_{\text{gimbal}}(\theta_k) \right]}$$

Typical Weight Values:

Scenario	w_{thrust}	w_{angular}	$w_{\text{smoothness}}$
Normal	0.1	1.0	5.0
Racing	0.05	0.5	10.0

6. CONSTRAINTS

6.1 Dynamics Constraints (Equality)

Enforce that the trajectory satisfies the dynamics model:

$$\mathbf{c}_{\text{dyn}}(\mathbf{z}) = \begin{bmatrix} \mathbf{x}_1 - f(\mathbf{x}_0, \mathbf{u}_0) \\ \mathbf{x}_2 - f(\mathbf{x}_1, \mathbf{u}_1) \\ \vdots \\ \mathbf{x}_N - f(\mathbf{x}_{N-1}, \mathbf{u}_{N-1}) \end{bmatrix} = \mathbf{0} \in \mathbb{R}^{6N}$$

Implementation: For each $k = 0, \ldots, N-1$, add 6 constraints:

- 3 position constraints: $\mathbf{p}_{k+1} = \mathbf{p}_k + \mathbf{v}_{\text{terminal}}\Delta t$
- 3 attitude constraints: $\boldsymbol{\eta}_{k+1} = \boldsymbol{\eta}_k + \Delta\boldsymbol{\eta}_k$ (with angle wrapping)

Total: $6N$ equality constraints

6.2 Boundary Constraints (Equality)

Initial State

Fix the starting pose:

$$\mathbf{c}_{\text{start}} = \mathbf{x}_0 - \mathbf{x}_{\text{start}} = \mathbf{0} \in \mathbb{R}^6$$

Explicitly:

$$\begin{bmatrix} p_{x,0} - p_{x,\text{start}} \\ p_{y,0} - p_{y,\text{start}} \\ p_{z,0} - p_{z,\text{start}} \\ \angle(\psi_0, \psi_{\text{start}}) \\ \angle(\theta_0, \theta_{\text{start}}) \\ \angle(\phi_0, \phi_{\text{start}}) \end{bmatrix} = \mathbf{0}$$

where $\angle(\alpha, \beta)$ denotes the wrapped angle difference.

Count: 6 constraints

Goal Position

Fix the final position (but allow free final orientation):

$$\mathbf{c}_{\text{goal}} = \mathbf{p}_N - \mathbf{p}_{\text{goal}} = \mathbf{0} \in \mathbb{R}^3$$

Count: 3 constraints

Total Boundary Constraints

$$\mathbf{c}_{\text{boundary}}(\mathbf{z}) = \begin{bmatrix} \mathbf{x}_0 - \mathbf{x}_{\text{start}} \\ \mathbf{p}_N - \mathbf{p}_{\text{goal}} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^9$$

Total: 9 equality constraints

6.3 Hoop Waypoint Constraints (Equality)

For drone racing, enforce passage through hoops:

$$\mathbf{c}_{\text{hoop}}(\mathbf{z}) = \begin{bmatrix} \mathbf{p}_{k_1} - \mathbf{p}_{\text{hoop}_1} \\ \mathbf{p}_{k_2} - \mathbf{p}_{\text{hoop}_2} \\ \vdots \\ \mathbf{p}_{k_H} - \mathbf{p}_{\text{hoop}_H} \end{bmatrix} = \mathbf{0} \in \mathbb{R}^{3H}$$

where:

- H is the number of hoops
- k_h is the knot index assigned to hoop h

Total: $3H$ equality constraints (0 if no hoops)

6.4 Collision Avoidance Constraints (Inequality)

Maintain safety margin from obstacles:

$$\mathbf{c}_{\text{collision}}(\mathbf{z}) \leq \mathbf{0}$$

Sphere Obstacles

For sphere with center $\mathbf{c}_{\text{obs}}$ and radius r_{obs}:

$$c_{\text{sphere},k} = (r_{\text{obs}} + r_{\text{safety}}) - \|\mathbf{p}_k - \mathbf{c}_{\text{obs}}\|_2 \leq 0$$

Equivalent to: $\|\mathbf{p}_k - \mathbf{c}_{\text{obs}}\|_2 \geq r_{\text{obs}} + r_{\text{safety}}$

Parameters

- $r_{\text{safety}} = 0.5$ m (safety margin)
- Subsampling: check every 2nd knot point for efficiency

Total: M inequality constraints, where M depends on number of obstacles and subsampling

---

7. COMPLETE OPTIMIZATION PROBLEM

7.1 Standard Form

$$\boxed{\begin{array}{lll} \displaystyle\min_{\mathbf{z}\in\mathbb{R}^{10N+6}} & J(\mathbf{z}) & \\ \text{subject to:} & \mathbf{c}_{\text{dyn}}(\mathbf{z}) = \mathbf{0} & (\mathbb{R}^{6N}) \\ & \mathbf{c}_{\text{boundary}}(\mathbf{z}) = \mathbf{0} & (\mathbb{R}^{9}) \\ & \mathbf{c}_{\text{hoop}}(\mathbf{z}) = \mathbf{0} & (\mathbb{R}^{3H}) \\ & \mathbf{c}_{\text{collision}}(\mathbf{z}) \leq \mathbf{0} & (\mathbb{R}^{M}) \end{array}}$$

7.2 Constraint Summary

Constraint Type	**Dimension**	**Type**	**Count**
Dynamics	$\mathbb{R}^{6N}$	Equality	$6N$
Boundary (start)	$\mathbb{R}^{6}$	Equality	6
Boundary (goal)	$\mathbb{R}^{3}$	Equality	3
Hoops	$\mathbb{R}^{3H}$	Equality	$3H$
Collision	$\mathbb{R}^{M}$	Inequality	M
Total Equality	-	-	$6N + 9 + 3H$

---

8. INITIALIZATION STRATEGY

The initial guess $\mathbf{z}_0$ is critical for convergence.

8.1 State Initialization from RRT

Given RRT path of length L, resample to $N+1$ knot points via linear interpolation:

For knot $i \in \{0, 1, \ldots, N\}$:

$$\alpha_i = \frac{i(L-1)}{N}, \quad j_{\text{low}} = \lfloor \alpha_i \rfloor, \quad j_{\text{high}} = \lceil \alpha_i \rceil$$

$$\lambda = \alpha_i - j_{\text{low}}$$

Position interpolation:

$$\mathbf{p}_i = (1-\lambda)\mathbf{p}_{\text{RRT},j_{\text{low}}} + \lambda \mathbf{p}_{\text{RRT},j_{\text{high}}}$$

Attitude interpolation (with angle wrapping):

$$\begin{aligned} \psi_i &= \psi_{j_{\text{low}}} + \lambda \cdot \angle(\psi_{j_{\text{high}}}, \psi_{j_{\text{low}}}) \\ \theta_i &= \theta_{j_{\text{low}}} + \lambda \cdot \angle(\theta_{j_{\text{high}}}, \theta_{j_{\text{low}}}) \\ \phi_i &= \phi_{j_{\text{low}}} + \lambda \cdot \angle(\phi_{j_{\text{high}}}, \phi_{j_{\text{low}}}) \end{aligned}$$

8.2 Control Initialization

Compute controls from state differences:

$$
\begin{aligned}
\Delta\psi_k &= \angle(\psi_{k+1}, \psi_k) \\
\Delta\theta_k &= \angle(\theta_{k+1}, \theta_k) \\
\Delta\phi_k &= \angle(\phi_{k+1}, \phi_k) \\
T_k &= 10 \text{ N}
\end{aligned}
$$

Then clip to bounds:

$$\Delta\boldsymbol{\eta}_k \in [-10^\circ, +10^\circ]^3, \quad T_k \in [5, 20]$$

---

9. SOLVER CONFIGURATION

9.1 Primary Solver: SLSQP

Sequential Least Squares Programming

- Method: Gradient-based, handles both equality and inequality constraints
- Max iterations: 50
- Convergence tolerance: 10^{-5}

9.2 Fallback Solver: trust-constr

Trust Region Constrained Algorithm

- Activated if SLSQP fails
- Max iterations: 30
- Initializes from SLSQP result (warm start)

9.3 Acceptance Criteria

Solution accepted if:

$$\|\mathbf{c}_{\text{eq}}\|_\infty < 0.01 \quad \text{OR} \quad (\text{success} \wedge \|\mathbf{c}_{\text{eq}}\|_\infty < 0.1 \wedge J(\mathbf{z}) < 100)$$

11. DIMENSIONAL ANALYSIS SUMMARY

Quantity	Dimension	Description
State $\mathbf{x}$	$\mathbb{R}^6$	Position ($\mathbb{R}^3$) + Attitude ($\mathbb{R}^3$)
Control $\mathbf{u}$	$\mathbb{R}^4$	Attitude rates ($\mathbb{R}^3$) + Thrust ($\mathbb{R}$)
Decision vector $\mathbf{z}$	$\mathbb{R}^{10N+6}$	Flat concatenation of states and controls
Cost J	$\mathbb{R}$	Scalar objective
Dynamics f	$\mathbb{R}^6 \times \mathbb{R}^4 \to \mathbb{R}^6$	State transition function
Terminal velocity $\mathbf{v}$	$\mathbb{R}^3$	Velocity vector
Rotation matrix $\mathbf{R}$	$SO(3) \subset \mathbb{R}^{3\times 3}$	Orthogonal matrix
Force $\mathbf{F}$	$\mathbb{R}^3$	Net force vector