Derivation of the Variance-Covariance Matrix of $\widehat{\boldsymbol{\beta}}$, i.e., Var($\widehat{\boldsymbol{\beta}}$), in Matrix Form

What do we know?

Population equation: $y = X\beta + u$

Sample equation: $y = X\hat{\beta} + \hat{u}$

Regression assumptions, MLR.1 – MLR.6

OLS Estimator: $\hat{\beta} = (X'X)^{-1}X'y$

$var - cov\ matrix\ of\ true\ errors = Var(u) = E(uu') = \sigma^2 I_n$

$u \sim Normal(0, \sigma^2 I_n)$

$E(u) = 0\ and\ E(y) = X\beta$

$Var(\hat{\beta}) = E\left[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'\right]$

1. $\hat{\beta} = (X'X)^{-1}X'y$
2. $\hat{\beta} = (X'X)^{-1}X'(X\beta + u)$
3. $\hat{\beta} = (X'X)^{-1}X'X\beta + (X'X)^{-1}X'u$
4. $\hat{\beta} = I\beta + (X'X)^{-1}X'u$
5. $\hat{\beta} = \beta + (X'X)^{-1}X'u$
6. $\hat{\beta} - \beta = (X'X)^{-1}X'u$
7. $(\hat{\beta} - \beta)(\hat{\beta} - \beta)' = [(X'X)^{-1}X'u][(X'X)^{-1}X'u]'$
8. $(\hat{\beta} - \beta)(\hat{\beta} - \beta)' = [(X'X)^{-1}X'u][u'X((X'X)^{-1})']$
9. $(\hat{\beta} - \beta)(\hat{\beta} - \beta)' = [(X'X)^{-1}X'u][u'X((X'X)')^{-1}]$
10. $(\hat{\beta} - \beta)(\hat{\beta} - \beta)' = [(X'X)^{-1}X'u][u'X(X'X)^{-1}]$
11. $E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = E[(X'X)^{-1}X'uu'X(X'X)^{-1}]$
12. $E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = (X'X)^{-1}X'E(uu')X(X'X)^{-1}$
13. $E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = (X'X)^{-1}X'\sigma^2 IX(X'X)^{-1}$
14. $E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = \sigma^2(X'X)^{-1}X'X(X'X)^{-1}$
15. $E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = \sigma^2 I(X'X)^{-1}$
16. $Var(\hat{\beta}) = E[(\hat{\beta} - \beta)(\hat{\beta} - \beta)'] = \sigma^2(X'X)^{-1}$