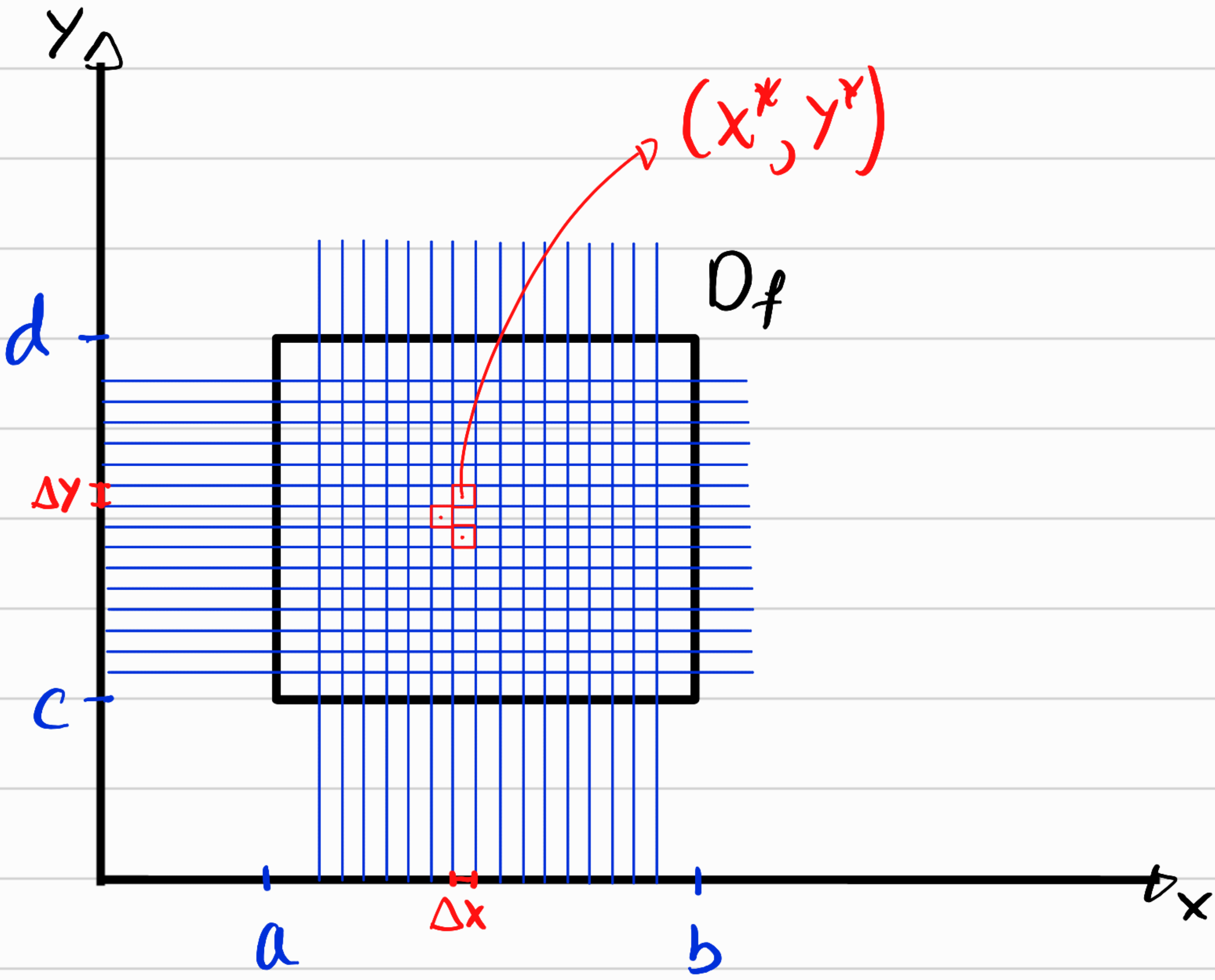
INTEGRAL DUPLA

$f(x,y)$, $D_f = [a,b] \times [c,d] = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b$
$e\ c \leq y \leq d\}$



$\text{ÁREA } \boxed{\cdot} = \Delta x_i \cdot \Delta y_i$

VOLUME ABAIXO DO GRÁFICO (ENTRE O GRÁFICO E O PLANO XY) DA FUNÇÃO

$$\underbrace{f:[a,b]\times[c,d]}_{R} \longrightarrow \mathbb{R} \text{ É:}$$

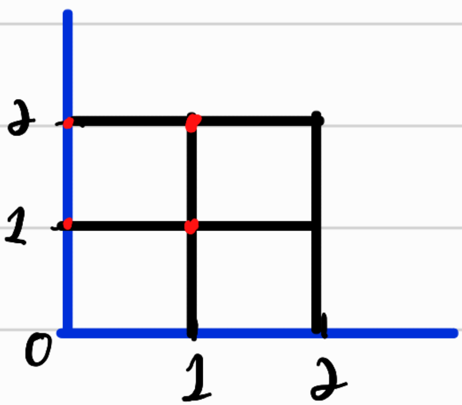
$$V = \lim_{n,m\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{m} f(x^*, y^*)\,\Delta x_i\,\Delta y_i$$

A INTEGRAL DUPLA DE f SOBRE O RETÂNGULO R

$$\int_a^b\int_c^d f(x,y)\,dy\,dx = V$$

SE ESSE LIMITE EXISTIR.

EX: $z = 16 - x^2 - 2y^2$ $\quad R=[0,2]\times[0,2]$



$$f(0,1)\cdot\Delta A_1 + f(0,2)\Delta A_2$$
$$+ f(1,2)\Delta A_3 + f(1,1)\Delta A_4$$

$$f(0,1)\cdot\Delta A_1+f(0,2)\Delta A_2+f(1,2)\Delta A_3+f(1,1)\Delta A_4$$

$$f(0,1)=16-0^2-2\cdot 1^2=16-2=14$$

$$f(0,2)=16-0^2-2\cdot 2^2=16-8=8$$

$$f(1,2)=16-1^2-2\cdot 2^2=16-9=7$$

$$f(1,1)=16-1^2-2\cdot 1^2=16-3=13$$

$$\Delta A_1=\Delta A_2=\Delta A_3=\Delta A_4=1$$

$$=14\cdot 1+8\cdot 1+7\cdot 1+13\cdot 1=42$$

Teorema de Fubini

Se $f: A\subset\mathbb{R}^2\to\mathbb{R}$ é contínua em $R=[a,b]\times[c,d]$, então:

$$\iint_R f(x,y)\,\underbrace{dxy}_{dA}=\int_a^b \underbrace{\boxed{\int_c^d f(x,y)\,dy}}_{A(x)}dx=\int_c^d \underbrace{\boxed{\int_a^b f(x,y)\,dx}}_{A(y)}dy$$

Ex:

$$\int_0^3\int_1^2 x^2 y \, dy \, dx$$

$$\int_0^3 x^2 \int_1^2 y \, dy \, dx = \int_0^3 x^2 \cdot \left[\frac{y^2}{2}\right]_1^2 dy$$

$$\left[\frac{2^2}{2} - \frac{1^2}{2}\right] = \left[\frac{4}{2} - \frac{1}{2}\right] = \left[\frac{4-1}{2}\right] = \frac{3}{2}$$

$$\int_0^3 \frac{3x^2}{2} \, dy = \frac{\cancel{3}}{2}\left[\frac{x^3}{\cancel{3}}\right]_0^3 = \frac{1}{2}\left[3^3 - 0^3\right] = \frac{1}{2} \cdot 27$$

$$\boxed{V = \frac{27}{2}}$$

$$\int_1^2\int_0^3 x^2 y \, dx \, dy$$

$$V=\int_1^2 y\cdot\left[\frac{x^3}{3}\right]_0^3 dy=\int_1^2 y\cdot\frac{27}{3}dy$$

$$V=\frac{27}{3}\cdot\int_1^2 y\,dy=\frac{27}{3}\cdot\left[\frac{y^2}{2}\right]_1^2=\frac{27}{3}\cdot\left[\frac{2^2}{2}-\frac{1^2}{2}\right]$$

$$V=\frac{27}{3}\cdot\left[\frac{4}{2}-\frac{1}{2}\right]=\cancel{\frac{27}{3}\cdot\frac{3}{2}}=\boxed{\frac{27}{2}}$$

EX:

$$\int_0^2\int_0^2 16-x^2-2y^2\,dx\,dy$$

$$\int_0^2\int_0^2 2(8-x^2-y^2)\,dx\,dy$$

$$\int_0^2 2\cdot\left[\int_0^2 8\,dx - \int_0^2 x^2\,dx - \int_0^2 y^2\,dx\right] dy$$

$$\int_0^2 2\cdot\left[8\cdot[x]_0^2 - \left[\frac{x^3}{3}\right]_0^2 - y^2\cdot[x]_0^2\right] dy$$

$$\int_0^2 2\cdot\left[8\cdot 2 - \frac{2^3}{3} - y^2\cdot 2\right] dy$$

$$\int_0^2 2\cdot\left[16 - \frac{8}{3} - 2y^2\right] dy$$

$$\int_0^2 32y - \frac{8}{3}y - 2\frac{y^3}{3}\,dy$$

$$32\cdot 2 - \frac{8}{3}\cdot 2 - \frac{2\cdot 2^3}{3} = 64 - \frac{16}{3} - \frac{16}{3}$$

$$64 - \frac{32}{2} = \frac{128-32}{2} = \frac{96}{2} = \boxed{48}$$

$\underline{EX:}$ $\displaystyle\int_0^{\pi}\int_1^{2} y\cdot sen(xy)\,dx\,dy$

$$-\int_0^{\pi} \cancel{y}\cdot\left[\frac{cos(xy)}{\cancel{y}}\right]_1^2 dy$$

$$-\int_0^{\pi} cos(2y)-cos(y)\,dy$$

$$-\left[\frac{sen(2y)}{2}-sen(y)\right]_0^{\pi}$$

$$\left[-\cancel{\frac{sen(2\cdot\pi)}{2}-sen(\pi)}^{\,0}\right]-\left[\cancel{\frac{sen(2\cdot 0)}{2}-sen(0)}^{\,0}\right]$$

$$=\boxed{0}$$

CASO PARTICULAR

$$\int_a^b \int_c^d f(x) \cdot g(y)\, dy\, dx$$

$$\int_a^b f(x) \cdot \boxed{\int_c^d g(y)\, dy}\, dx$$

CONSTANTE P/ dx

$$\int_c^d g(y)\, dy \cdot \int_a^b f(x)\, dx$$

Ex:

$$\int_0^{\pi/2} \int_0^{\pi/2} \cos(x)\, \operatorname{sen}(y)\, dx\, dy$$

$$\int_0^{\pi/2} \cos(x)\, dx \cdot \int_0^{\pi/2} \operatorname{sen}(y)\, dy$$

$$\left[\frac{\operatorname{sen}(x)}{x}\right]_0^{\pi/2} \cdot \left[-\frac{\cos(y)}{y}\right]_0^{\pi/2}$$

$$\left[\frac{\operatorname{sen}(0)}{0} - \frac{\operatorname{sen}(\pi/2)}{\pi/2}\right] \cdot \left[-\frac{\cos(0)}{0} + \frac{\cos(\pi/2)}{\pi/2}\right]$$

$$= (1-0)\cdot(-0+1) = 1$$