22

$$\int_1^2 \int_1^3 \frac{1}{1+x+y}\,dx\,dy = \int_1^2 \ln(1+x+y)\Big|_1^3 \,dy$$

$$= \int_1^2 \ln(4+y) - \ln(2+y)\,dy$$

$$= (4+y)\ln(4+y) - (4+y) - \big((2+y)\ln(2+y) - (2+y)\big)\Big|_1^2$$

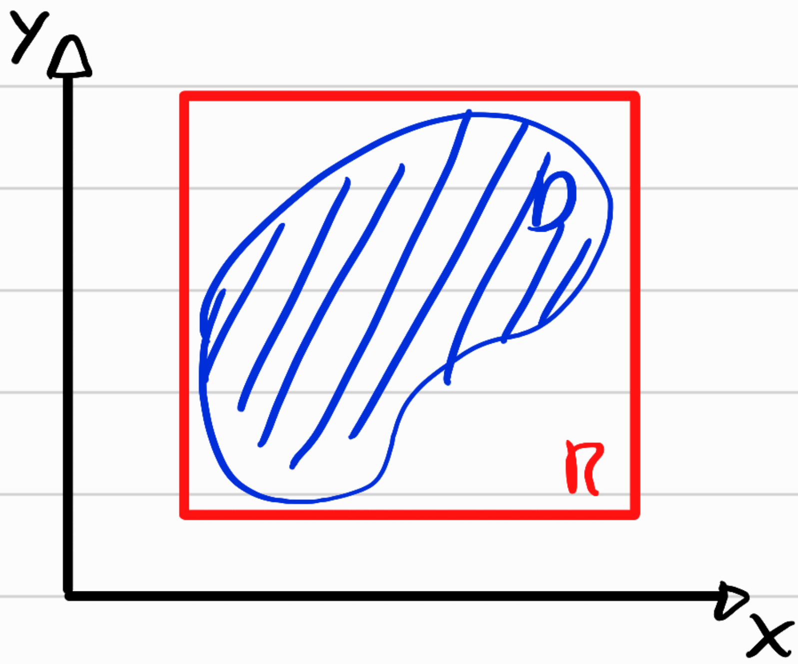
$$= 6\ln(6) - 6 - (4\ln(4) - 4) - [5\ln(5) - 5 - (3\ln(3) - 3))$$

$$= (-6+4+5-3) + 6\ln(6) - 4\ln(4) - 5\ln(5) + 3\ln(3)$$

$$= 6[\ln(2) + \ln(3)] - 8\ln(2) - 5\ln(5) + 3\ln(3)$$

$$= \boxed{-2\ln(2) + 9\ln(3) - 5\ln(5)}$$

INTEGRAL DUPLA SOBRE REGIÕES QUAISQUER



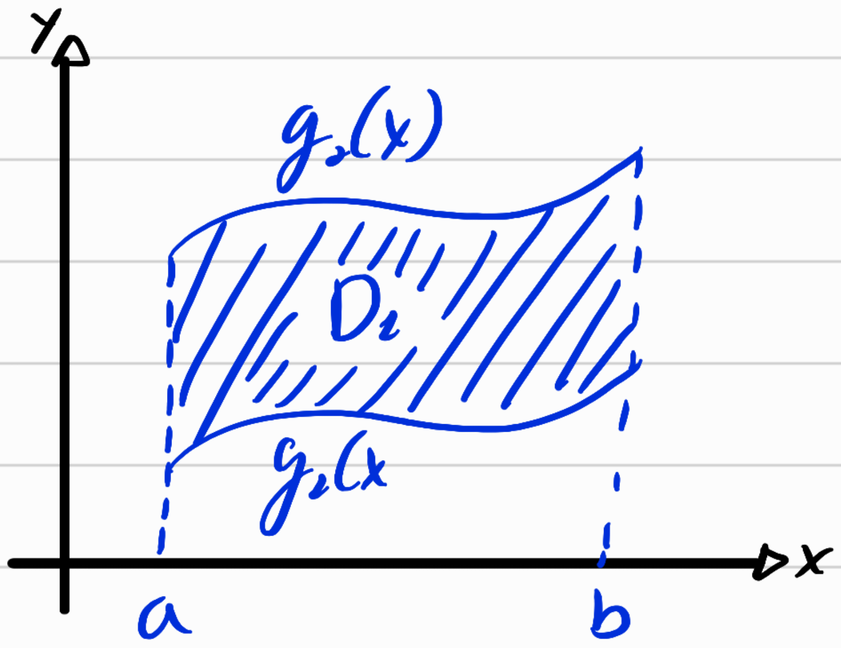
SEJA $f: A \subset \mathbb{R}^2 \to \mathbb{R}$ CONTÍNUA, $D \subset A$

$$\iint_D f(x,y)\, dA$$

$$F(x,y) = \begin{cases} f(x,y) & , (x,y) \in D \\ 0 & , (x,y) \notin D \end{cases}$$

$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA$$

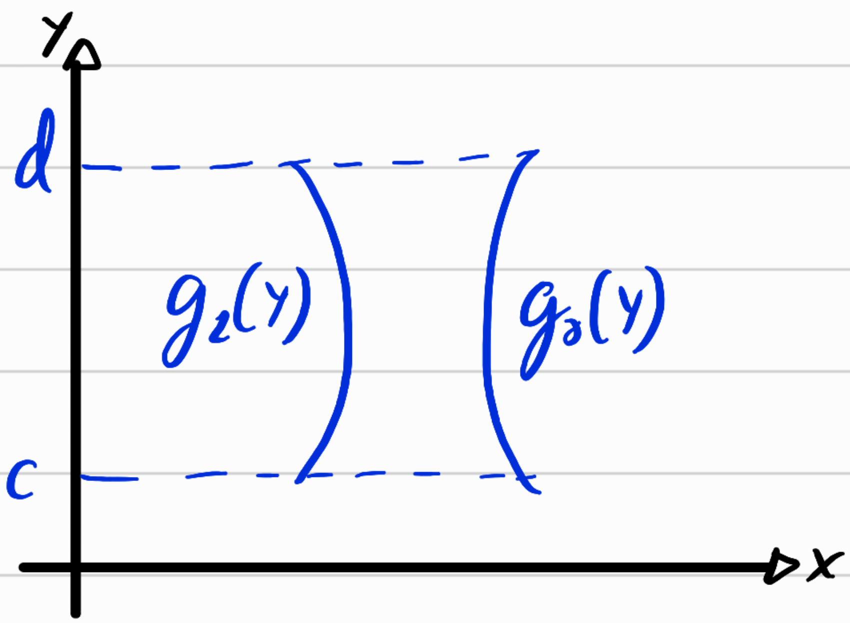
DICA 1: REGIÃO DO TIPO 1



$$D_1 = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

$$\iint_{D_1} f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

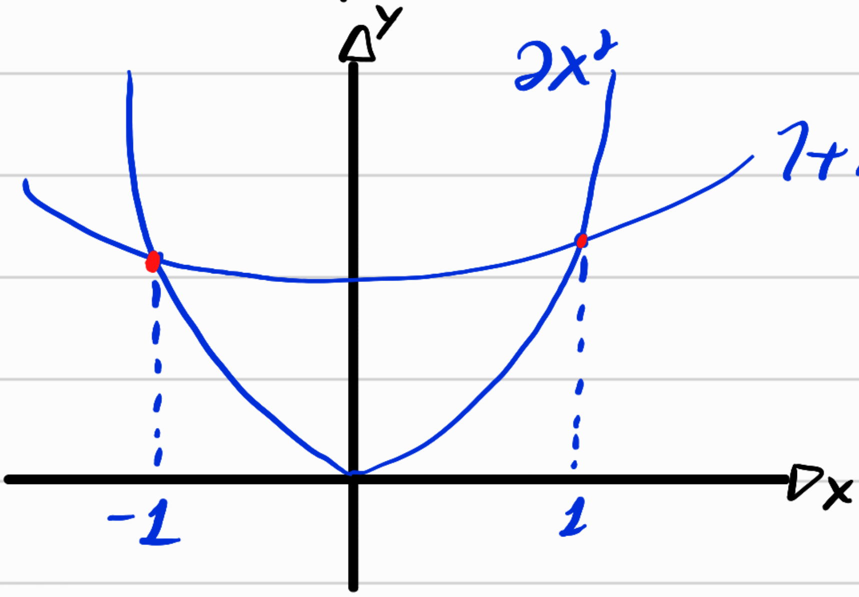
DICA 2: REGIÃO DO TIPO 2



$$\iint_{D_2} f(x,y)\,dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)\,dx\,dy$$

$$D_2 = \{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d,\ g_2(y) \leq x \leq g_1(y)\}$$

Ex: $\iint_D x+2y\,dA$ D É REGIÃO LIMITADA PELAS PARÁBOLAS $y=2x^2$ E $y=1+x^2$



ONDE OS PONTOS SÃO IGUAIS?

SÃO IGUAIS QUANDO $2x^2 = 1+x^2$

$$2x^2 - x^2 = 1 \quad \begin{cases} x = \pm\sqrt{1} \\ \boxed{x = \pm 1} \end{cases}$$

$$x^2 = 1$$

X ESTÁ ENTRE CONSTANTES E, Y ESTÁ ENTRE FUNÇÕES

$$\int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y\,dy\,dx = \int_{-1}^{1} x\cdot\left[y\right]_{2x^2}^{1+x^2} + \cancel{2}\cdot\left[\frac{y^2}{\cancel{2}}\right]_{2x^2}^{1+x^2} dx =$$

$$\int_{-1}^{1} x\cdot\left(1+x^2+2x^2\right)+\left(\left(1+x^2\right)^2-\left(2x^2\right)^2\right) dx =$$

$$\int_{-1}^{1} x+x^3+2x^3+\left(1+2x^2+x^4-4x^4\right) dx =$$

$$\int_{-1}^{1} x+x^3+2x^3+\underline{1}+2x^2+x^4-4x^4\, dx =$$

$$\int_{-1}^{1} x+3x^3+2x^2-3x^4+1\, dx =$$

$$\left[\frac{x^2}{2}\right]_{-1}^{1}+3\left[\frac{x^4}{4}\right]_{-1}^{1}+2\cdot\left[\frac{x^3}{3}\right]_{-1}^{1}-3\left[\frac{x^5}{5}\right]_{-1}^{1}+\left[x\right]_{-1}^{1}=$$

$$\left(\frac{1^2}{2}-\frac{(-1)^2}{2}\right)+3\cdot\left(\frac{1^4}{4}-\frac{(-1)^4}{4}\right)+2\left(\frac{1^3}{3}-\frac{(-1)^3}{3}\right)-3\left(\frac{1^5}{5}-\frac{(-1)^5}{5}\right)$$

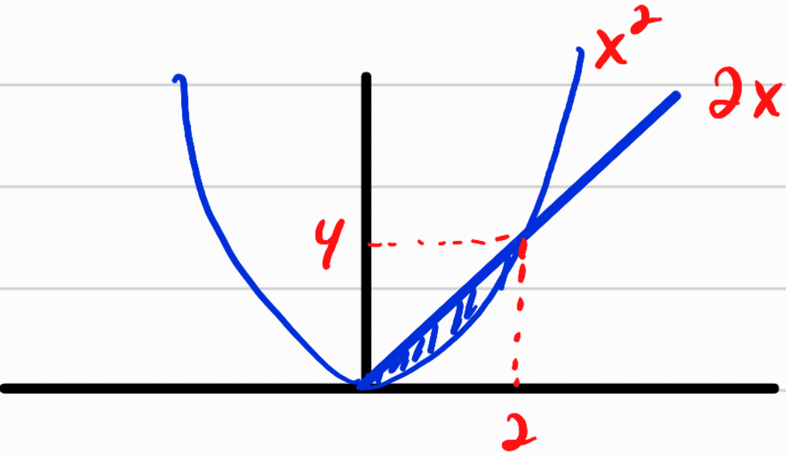
$$+1-(-1)=$$

$$\cancel{\left(\frac{1}{2}-\frac{1}{2}\right)}^{0}+3\cdot\cancel{\left(\frac{1}{4}-\frac{1}{4}\right)}^{0}+2\left(\frac{1}{3}+\frac{1}{3}\right)-3\cdot\left(\frac{1}{5}+\frac{1}{5}\right)+2$$

$$\frac{4}{3}-\frac{6}{5}+2=\frac{20-6}{15}+2=\frac{14}{15}+2$$

$$=\frac{14+30}{15}=\frac{34}{15}$$

EX: VOLUME DO SÓLIDO, ABAIXO DE $z=x^2+y^2$ E ACIMA DE $\underline{D}$, COM D LIMITADA PELA RETA $y=2x$ E PELA PARÁBOLA $y=x^2$.



$$x^2=2x \qquad \frac{x^2}{x}=2 \qquad \begin{cases} x=2 \\ y=4 \end{cases}$$

$$V=\int_0^2\int_{x^2}^{2x} x^2+y^2\,dy\,dx$$

$$V=\int_0^2 \left[x^2 \cdot y+\frac{y^3}{3}\right]_{x^2}^{2x} dx$$

$$V=\int_0^2 x^2\cdot 2x+\frac{(2x)^3}{3}-\left(x^2+\frac{(x^2)^3}{3}\right) dx$$

$$V=\int_0^2 x^2\cdot 2x+\frac{8x^3}{3}-x^2-\frac{x^6}{3}\,dx$$

$$V=\int_0^2 2x^3+\frac{8x^3}{3}-x^2-\frac{x^6}{3}\,dx$$

$$V=\int_0^2 \frac{6x^3}{3}+\frac{8x^3}{3}-x^2-\frac{x^6}{3}\,dx$$

$$V=\int_0^2 \frac{14}{3}x^3 - x^2 - \frac{x^6}{3}\,dx$$

$$V=\frac{14}{3}\cdot\left[\frac{x^4}{4}\right]_0^2-\left[\frac{x^3}{3}\right]_0^2-\left[\frac{x^7}{21}\right]$$

$$V=\frac{14}{3}\cdot 4-\frac{8}{3}-\frac{1}{3}\cdot\frac{128}{7}$$

$$V=\frac{56}{3}-\frac{8}{3}-\frac{128}{21}$$

$$V=\frac{48}{3}-\frac{128}{21}=\frac{1008-384}{63}=\frac{624}{63}=208$$