DERIVADAS DIRECIONAIS

SEJA $f(x,y)$ UMA FUNÇÃO. A DERIVADA DIRECIONAL DE f EM (x_0, y_0) NA DIREÇÃO DE UM VETOR UNITÁRIO $\vec{u}=(a,b)$ É DADO POR:

$$D_{\vec{u}}f(x_0,y_0) = \frac{df}{d\vec{u}}(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0,y_0)}{h}$$

TEOREMA: $f(x,y)$ É DIFERENCIÁVEL EM (x_0,y_0), E $\vec{u}=(a,b)$. ENTÃO A DERIVADA DIRECIONAL DE f NA DIREÇÃO $\vec{u}$ É:

$$\frac{df}{d\vec{u}}(x_0,y_0) = f_x(x_0,y_0)\cdot a + f_y(x_0,y_0)\cdot b$$

NO PONTO (x_0,y_0)

$$= \overline{(f_x, f_y)}\cdot \underbrace{(a,b)}_{\vec{u}}$$

DEMONSTRAÇÃO DO TEOREMA:

$$g(h) = f(\underbrace{x_0+ah}_{x}, \underbrace{y_0+bh}_{y})$$

PELA REGRA DA CADEIA:

$$g'(h) = \boxed{\frac{df}{dx}} \cdot \frac{dx}{dh} + \boxed{\frac{df}{dy}} \cdot \frac{dy}{dh}$$

$$= f_x \cdot \frac{dx}{dh} + f_y \cdot \frac{dy}{dh}$$

$$= \boxed{f_x \cdot a + f_y \cdot b}$$

POR OUTRO LADO:

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + ah, y_0 + bh) - f(x_0, y_0)}{h} = D_{\vec{u}} f(x_0, y_0)$$

$$\therefore \boxed{D_{\vec{u}} f(x_0, y_0) = f_x \cdot a + f_y \cdot b} = f_x \cdot \cos(\theta) + f_y \cdot \text{sen}(\theta)$$

$D_{\hat{i}} f = f_x \cdot 1 + f_y \cdot 0 = f_x$

$\vec{u} = (\cos(\theta), \text{sen}(\theta))$

$D_{\hat{j}} f = f_x \cdot 0 + f_y \cdot 1 = f_y$

EX: $f(x,y) = x^2 - 3xy + 4y^2$

$\vec{u} = \left(\cos(\pi/6), \text{sen}(\pi/6)\right)$

$D_{\vec{u}} f(1,2) = ?$

$f_x(1,2) = 3x^2 - 3y = 3 \cdot 1^2 - 3 \cdot 2 = 3 - 6 = -3$

$f_y(1,2) = -3x + 8y = -3 \cdot 1 + 8 \cdot 2 = -3 + 16 = 13$

$\vec{u} = \left(\cos(\pi/6), \text{sen}(\pi/6)\right) = \left(\frac{\sqrt{3}}{2}, 1/2\right)$

$D_{\vec{u}} f(1,2) = f_x \cdot a + f_y \cdot b$

$= -3 \cdot \frac{\sqrt{3}}{2} + 13 \cdot \frac{1}{2}$

$= \frac{-3\sqrt{3}}{2} + \frac{13}{2}$

$$D_{\vec{u}} f(1,2) = \frac{13 - 3\sqrt{3}}{2}$$

DEFINIÇÃO: SEJA $f: A \subset \mathbb{R}^n \to \mathbb{R}$ DIFERENCIÁVEL. O VETOR:

$$\nabla f(x_1, \dots, x_n) = \left(\frac{df}{dx_1}, \frac{df}{dx_2}, \dots, \frac{df}{dx_n} \right)$$

É CHAMADO VETOR GRADIENTE DE f.

$$D_{\vec{u}} f(x_0, y_0) = \nabla f(x_0, y_0) \cdot \vec{u}$$

VETOR VEZES ESCALAR

$$= |\nabla f(x_0, y_0)| \cdot \underbrace{|\vec{u}|}_{\to 1} \cdot \cos(\theta)$$

A MÁXIMA DA DERIVADA DIRECIONAL SERÁ QUANDO $\theta = 0°$, E $\cos(0) = 1$

TEOREMA: O VALOR MÁXIMO DA DERIVADA DIRECIONAL NO PONTO (x_0, y_0) É $\|\nabla f(x_0, y_0)\|$ E OCORRE NA DIREÇÃO DO VETOR GRADIENTE.

EX: $f(x,y) = \text{sen}(x) + e^{xy}$

$$\frac{df}{d\vec{u}}(0,1) = ? \qquad \vec{u} = (2,5)$$

$\vec{u} = (a,b)$ $\quad \|\vec{u}\| = \sqrt{a^2+b^2}$

NORMALIZAR O VETOR $\Rightarrow$ $\dfrac{\vec{u}}{\|\vec{u}\|} = \left(\dfrac{a}{\sqrt{a^2+b^2}}, \dfrac{b}{\sqrt{a^2+b^2}}\right)$

$$\|\vec{u}\| = \sqrt{2^2+5^2} = \sqrt{4+25} = \sqrt{29}$$

$$\frac{\vec{u}}{\|\vec{u}\|} = \frac{(2,5)}{\sqrt{29}} = \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right)$$

$$\frac{df}{dx}(0,1) = \cos(x) + e^{xy} \cdot y = \cos(0) + e^{0\cdot 1} \cdot 1 = 1 + 1 \cdot 1 = 2$$

$$\frac{df}{dy} = x \cdot e^{xy} = 0 \cdot e^{0 \cdot 1} = 0 \cdot e^{0} = 0 \cdot 1 = 0$$

$$D_{\vec{u}} f(0,1) = \nabla f(0,1) \cdot \frac{\vec{u}}{\|\vec{u}\|}$$

$$\nabla f(0,1) = \left(\frac{df}{dx}, \frac{df}{dy} \right) = (2,0)$$

$$D_{\vec{u}} f(0,1) = (2,0) \cdot \left(\frac{2}{\sqrt{29}}, \frac{5}{\sqrt{29}} \right)$$

$$= \boxed{\frac{4}{\sqrt{29}}}$$

Ex: $f(x,y) = x^2 \cdot y^3 - 4y \quad , \quad \vec{u} = (2,5)$

$$\frac{df}{d\vec{u}} = D_{\vec{u}} f(2,-1) = ?$$

$$f_x(2,-1) = 2x \cdot y^3 = 2 \cdot 2 \cdot (-1)^3 = 4 \cdot (-1) = -4$$

$$f_y(2,-1) = 3y^2 \cdot x^2 - 4 = 3 \cdot (-1)^2 \cdot 2^2 - 4 = 3 \cdot 1 \cdot 4 - 4 = 8$$

$$\nabla f(2,-1) = (-4,8)$$

$$\frac{df}{d\vec{u}} = (-4,8)\cdot\left(\frac{2}{\sqrt{29}},\frac{5}{\sqrt{29}}\right) = -\frac{8}{\sqrt{29}} + \frac{40}{\sqrt{29}} = \boxed{\frac{32}{\sqrt{29}}}$$

Ex: $f(x,y,z) = x\cdot sen(yz)$

a) $\nabla f = ?$

b) $\frac{df}{d\vec{u}}(1,3,0) = ?$ $\quad \vec{u} = (1,2,-1)$

$$\|\vec{u}\| = \sqrt{1^2+2^2+(-1)^2} = \sqrt{1+4+1} = \sqrt{6}$$

a) $\nabla f(x,y,z) = (f_x, f_y, f_z)$

$$= (sen(y,z),\ xz\cdot sen(yz),\ xy\cdot sen(yz))$$

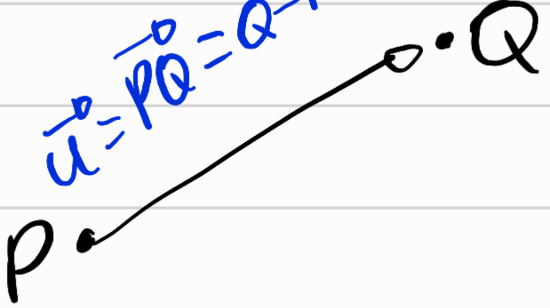
$$\nabla f(1,3,0) = (0,0,3) \qquad \frac{\vec{u}}{\|\vec{u}\|} = \left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$$

b) $$\frac{df}{d\vec{u}}(1,3,0) = \nabla f(1,3,0)\cdot\frac{\vec{u}}{\|\vec{u}\|} = (0,0,3)\cdot\left(\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},-\frac{1}{\sqrt{6}}\right)$$

$$= -\frac{3}{\sqrt{6}} = -\frac{3}{\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = -\frac{3\sqrt{6}}{6} = -\frac{\sqrt{6}}{3}$$

Ex: $f(x,y) = x \cdot e^{y}$

$$P(2,0) \to Q(1/2, 2)$$



$$\vec{u} = \vec{PQ} = Q - P = (1/2 - 2, 2 - 0) = (-3/2, 2)$$

$$\|\vec{u}\| = \sqrt{\frac{9}{4} + 4} = \sqrt{\frac{9+16}{4}}$$

$$= \sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{4}$$

TAXA DE VARIAÇÃO DA FUNÇÃO NO PONTO $f(2,0)$ NA DIREÇÃO DE Q.

$$\nabla f(2,0) = ?$$