INTEGRAIS DE LINHA

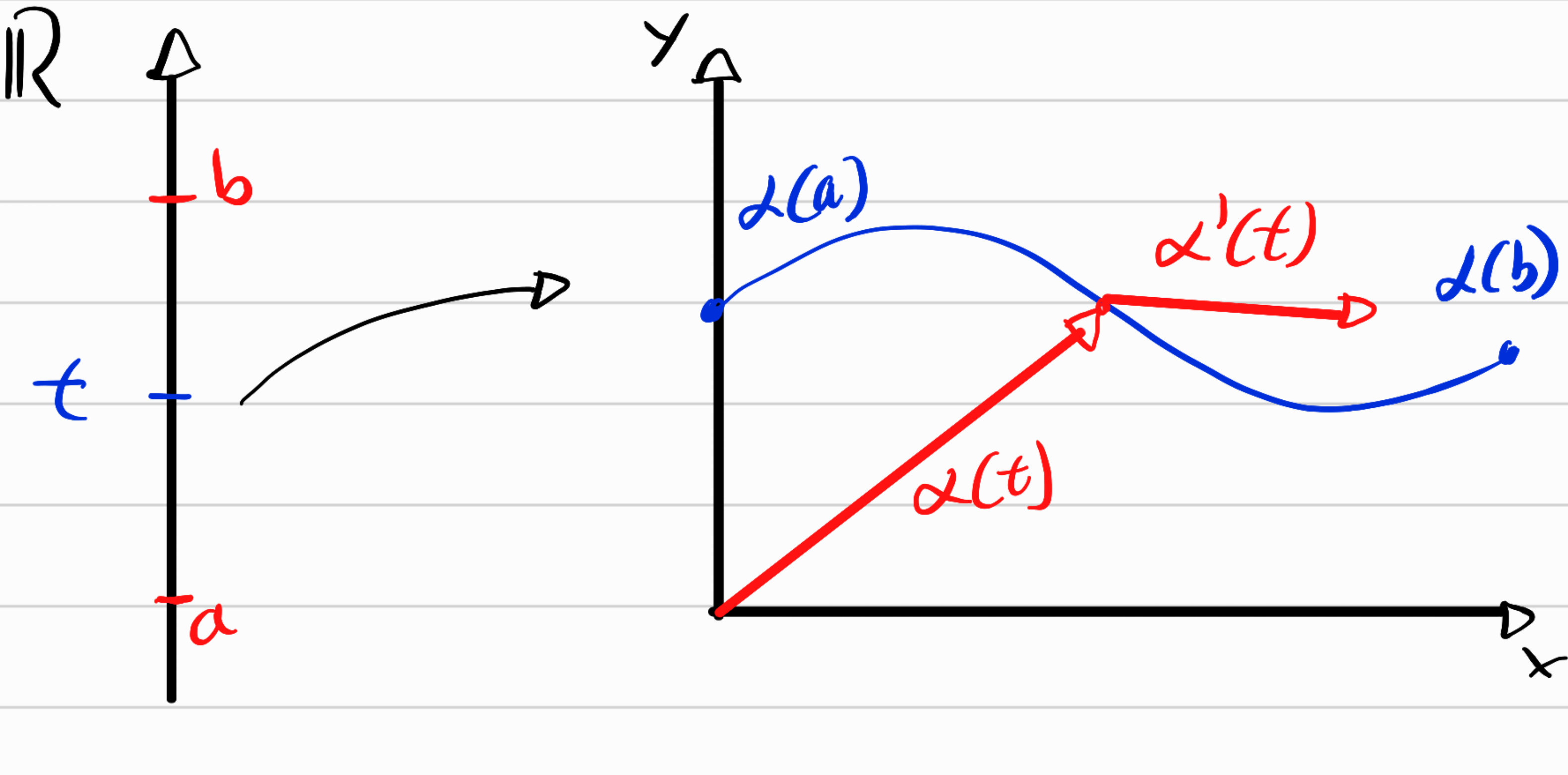
(CURVAS): $\alpha : A \subset \mathbb{R} \longrightarrow \mathbb{R}^n$

$t \longmapsto \alpha(t) = (x_1(t), x_2(t), \dots, x_n(t))$

$\alpha'(t) = \left(x_1'(t), \dots, x_n'(t)\right)$ → VETOR VELOCIDADE

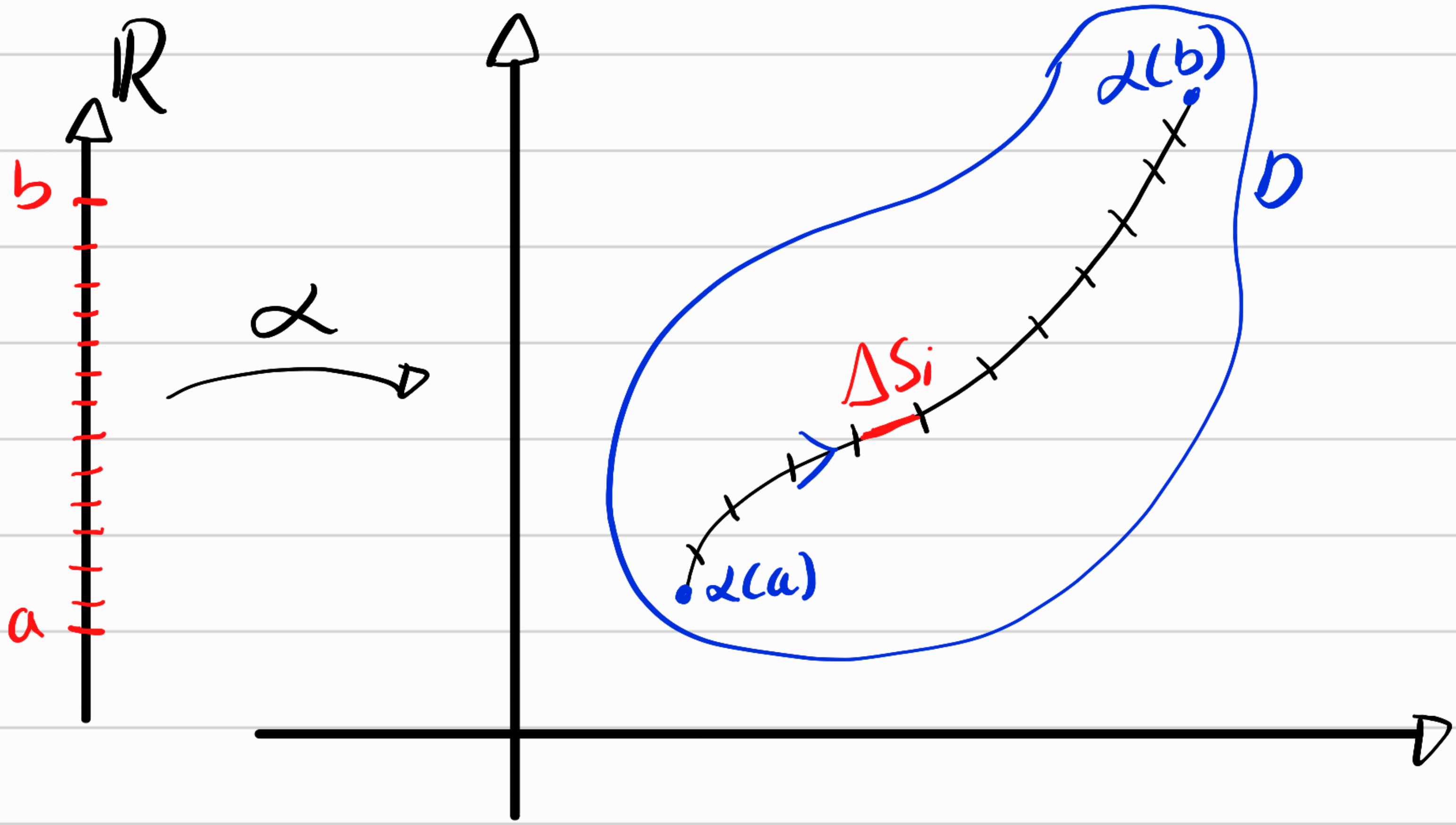
$$\boxed{T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}}$$: TANGENTE UNITÁRIO

$$L = \int_a^b \underbrace{|\alpha'(t)|\, dt}_{ds} \qquad \frac{ds}{dt} = |\alpha'(t)|$$



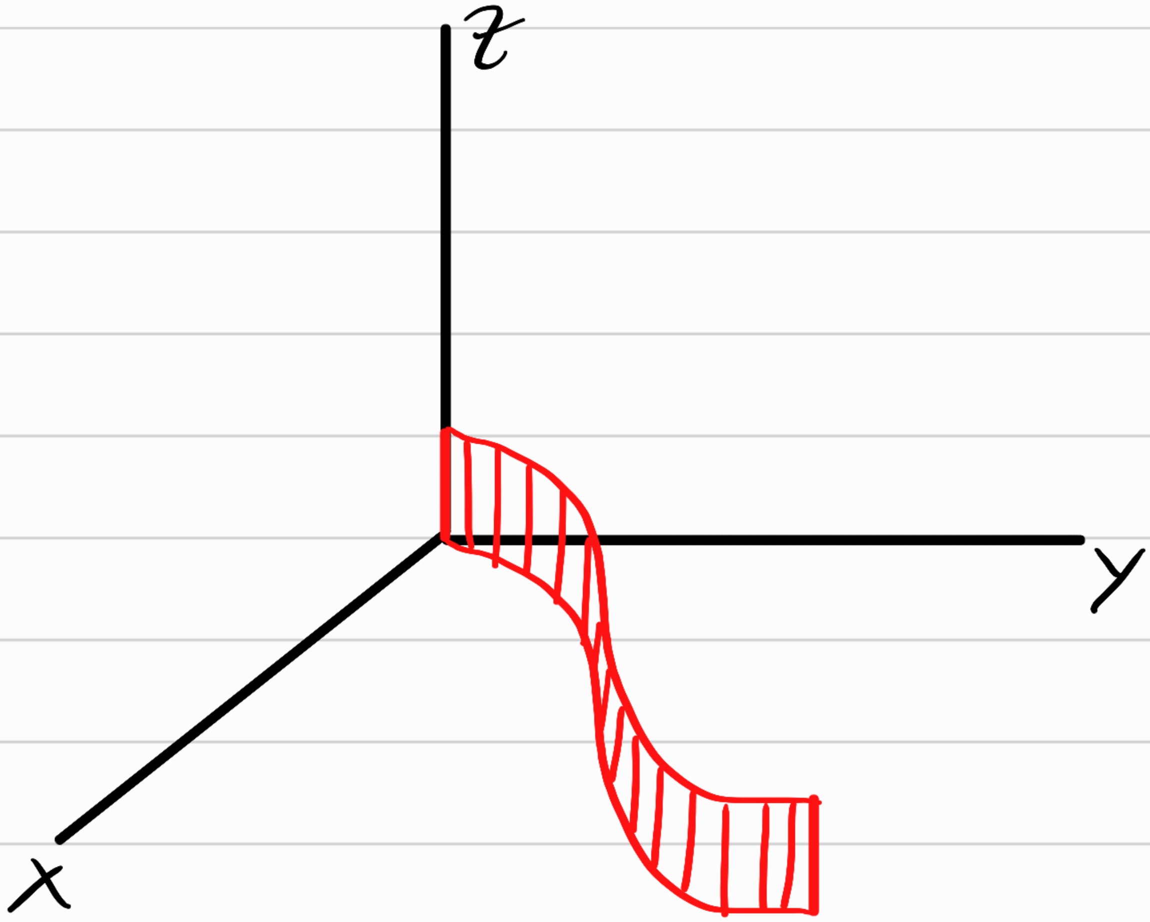
$f: D \subset \mathbb{R}^2 \to \mathbb{R}$: CONTÍNUA

$\alpha: A \subset \mathbb{R} \to \mathbb{R}^2$: SUAVE, $\operatorname{Im}\alpha \subset D$

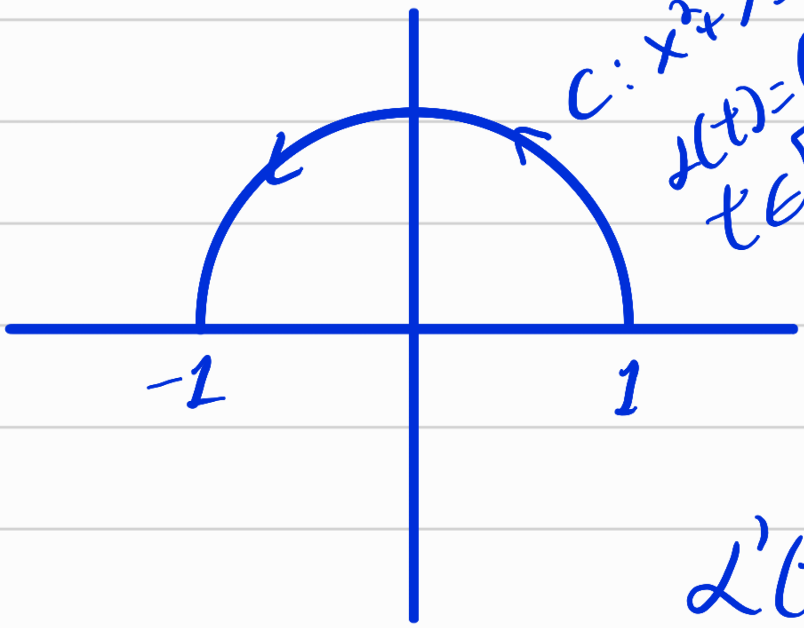


$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*) \cdot \Delta S_i = \int_C f(x,y)\, ds =$$

$$= \int_a^b f(x(t), y(t)) \cdot |\alpha'(t)|\, dt$$



Ex: $\int_C (2+x^2y)\,ds$, C: círculo unitário com $y \geq 0$



$C: x^2+y^2=1$

$\alpha(t)=(\cos(t),\operatorname{sen}(t))$

$t\in[0,\pi]$

$\alpha'(t)=(-\operatorname{sen}(t),\cos(t))$

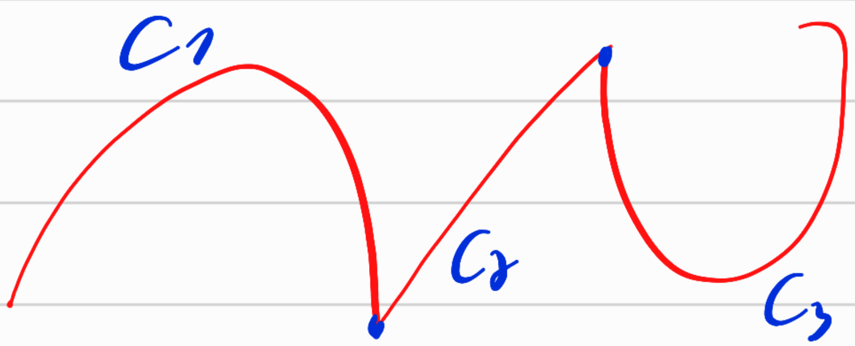
$|\alpha'(t)|=1$

$$\int_C (2+x^2y)\,ds = \int_0^{\pi} (2+\cos^2(t)\cdot \operatorname{sen}(t))\cdot 1\cdot dt =$$

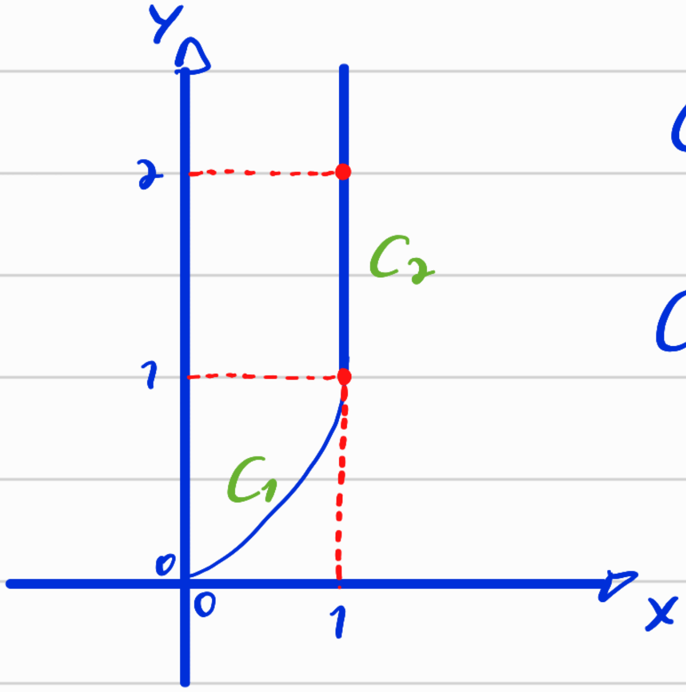
$$= 2t - \frac{\cos^3(t)}{3}\Bigg|_0^{\pi} = 2\pi + \frac{1}{3} - \left(0 - \frac{1}{3}\right) = \frac{6\pi+2}{3} = \frac{2(3\pi+2)}{3}$$

SEJA $C = C_1 \cup C_2 \cup \cdots \cup C_n$ EM QUE C_i É SUAVE PARA TODO $i = 1, \ldots, n$, C É DITA SUAVE POR PARTES, E ENTÃO:

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\cdot ds + \cdots + \int_{C_n} f(x,y)\,ds$$



EX: $\int_C 2x\,ds$ EM QUE C É A PARÁBOLA $y=x^2$ DE $(0,0)$ À $(1,1)$ E O SEGMENTO DE RETA QUE LIGA $(1,1)$ À $(1,2)$



$C_1: \alpha_1(t)=(t,t^2),\ t\in[0,1]$

$C_2: \alpha_2(t)=(1,t),\ t\in[1,2]$

$$\alpha_1'(t)=(1,2t) \Rightarrow |\alpha_2'(t)|=\sqrt{1^2+(2t)^2}=\sqrt{1+4t^2}$$

$$\alpha_2'(t)=(0,1) \Rightarrow |\alpha_2'(t)|=1$$

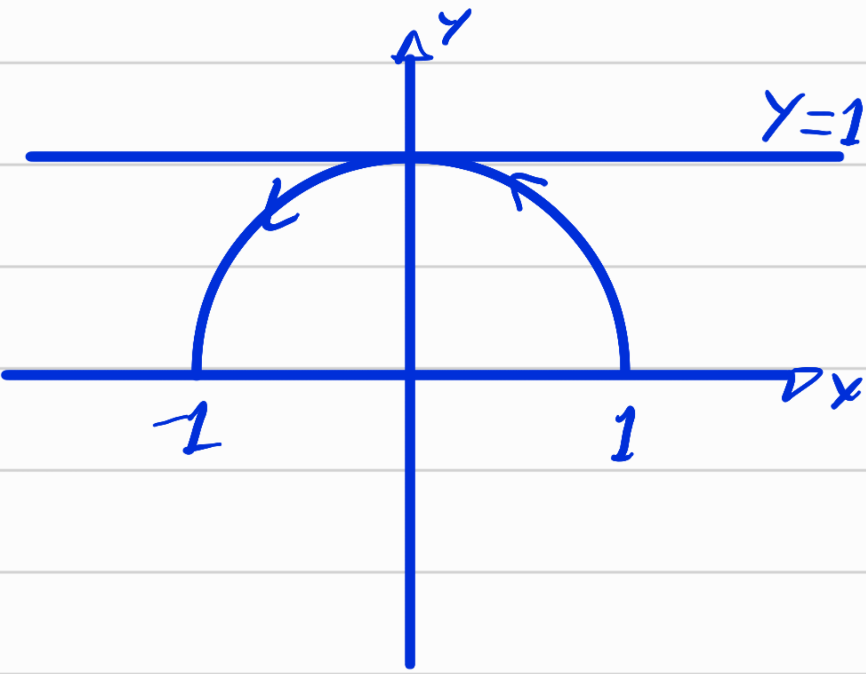
$$\int_C 2x\,ds=\int_{C_1} 2x\,ds+\int_{C_2} 2x\,ds=$$

$$=\int_0^2 2t\cdot\sqrt{1+4t^2}\,dt+\int_0^2 2\cdot 1\cdot 1\cdot dt=$$

$$= \frac{1}{4}\left(\frac{2(1+4t^2)^{3/2}}{3}\right)\Bigg|_0^1 + 2t\Bigg|_0^2 =$$

$$= \frac{1}{\cancel{4}} \cdot \frac{\cancel{2}}{3}\left(5\sqrt{5}-1\right)+(4-2) = \frac{1}{6}\cdot\left(5\sqrt{5}-1\right)+2 = \frac{5\sqrt{5}+11}{6}$$

EX: UM ARAME TEM FORMA DE SEMICÍRCULO $x^2+y^2=1$ COM $y \geq 0$. SUA DENSIDADE É PROPORCIONAL À DISTÂNCIA À RETA $y=1$, EM CADA PONTO DO FIO. CALCULE O CENTRO DE MASSA DO ARAME



$$\rho(x,y) = K(1-y)$$

$$m = \int_C \rho(x,y)\,ds = \int_C K\cdot(1-y)\,ds = \int_0^{\pi} K\cdot(1-\operatorname{sen}(t))\,1\cdot dt$$

$$= K\cdot\left(t+\cos(t)\right)\Big|_0^{\pi} = K\cdot\left(\pi+(-1)-(0+1)\right) = K(\pi-2)$$

$$\bar{X} = \frac{1}{m} \cdot \int x \cdot \rho(x,y)\, ds = \frac{1}{K(\pi-2)} K \cdot \int_0^{\pi} \cos(t) \cdot (1-\text{sen}(t))\, dt = 0$$

$$\bar{Y} = \frac{1}{m} \cdot \int y \cdot \rho(x,y)\, ds = \frac{1}{K(\pi-2)} \cdot K \int \text{sen}(t)(1-\text{sen}(t))\, dt$$

$$\bar{Y} = \frac{1}{\pi-2} \left(-\cos(t) - \frac{1t}{2} + \frac{1}{4} \cdot \text{sen}(2t) \right) \Bigg|_0^{\pi} = 0{,}38$$

EX: $\int_C y \cdot \text{sen}(z)\, ds$ EM QUE C É A HÉLICE CIRCULAR

$$\alpha(t) = (\cos(t), \text{sen}(t), t), \quad t \in [0, 2\pi]$$

$$\alpha'(t) = (-\text{sen}(t), \cos(t), 1) \Rightarrow |\alpha'(t)| = \sqrt{2}$$

$$\int_C y\,\text{sen}(z)\, ds = \int_0^{2\pi} \text{sen}(t) \cdot \text{sen}(t) \cdot \sqrt{2}\, dt =$$

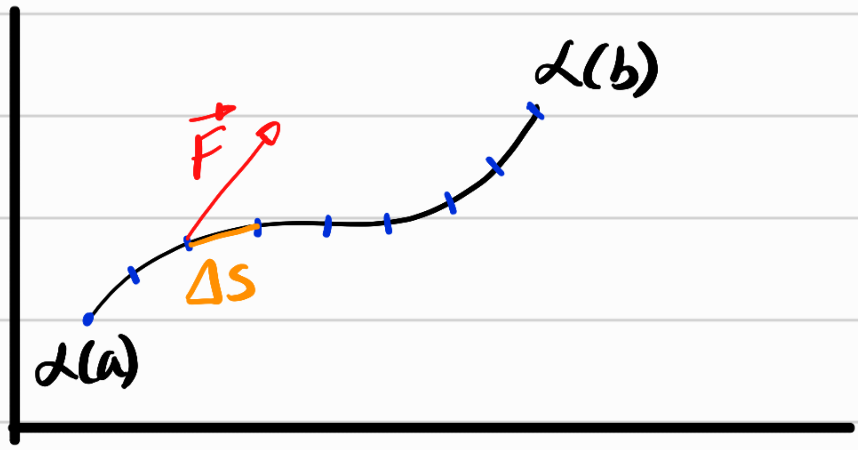
$$= \sqrt{2} \cdot \left(\frac{t}{2} - \frac{\text{sen}(2t)}{4} \right) \Bigg|_0^{2\pi} = \frac{\sqrt{2}}{2} \left(2\pi - \frac{1}{2} \cdot 0 - (0-0) \right) = \sqrt{2}\pi$$

INTEGRAL DE LINHA DE CAMPO VETORIAL

$$\vec{F}: A \subset \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longmapsto \vec{F}(x,y) = (P(x,y,z), Q(x,y,z), R(x,y,z))$$

$$\alpha: I \subset \mathbb{R} \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \alpha(t) = (x(t), y(t), z(t))$$

$$W = \tau = F \cdot d$$



TRABALHO DA FORÇA $\vec{F}$

$$\lim_{n \to \infty} \sum_{i=1}^{n} W_i = \sum_{i=1}^{n} \vec{F_i} \cdot \vec{T_i} \cdot \Delta S_i = \int_C \vec{F} \cdot \vec{T} \cdot ds = \boxed{\int_C \vec{F} \cdot dr}$$

$$\int_C \vec{F}\, dr = \int_a^b \vec{F}(\alpha(t)) \cdot \frac{\alpha'(t)}{|\alpha'(t)|} \cdot |\alpha'(t)|\, dt$$

$$= \int_C \vec{F} \cdot \alpha'(t)\, dt$$

$$\alpha(t) = (x(t), y(t), z(t))$$

$$\alpha'(t) = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\vec{F}(x,y,z) = (P, Q, R)$$

$$\int_C \vec{F} \cdot \alpha'(t)\, dt = \int_C (P,Q,R) \cdot \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right) dt =$$

$$= \int_C \left(P \cdot \frac{dx}{dt} + Q \frac{dy}{dt} + R \cdot \frac{dz}{dt}\right) dt$$

$$= \int_C \frac{1}{dt} \cdot (P dx + Q dy + R dz)\, dt$$

$$\boxed{= \int_C P\,dx + Q\,dy + R\,dz}$$

EX: DETERMINE O TRABALHO FEITO PELO CAMPO $\vec{F}(x,y,z) = (x^2, -xy)$ AO SE MOVER UMA PARTÍCULA AO LONGO DE UM QUARTO DE CÍRCULO $\alpha(t) = (\cos(t), \operatorname{sen}(t))$, $t \in [0, \pi/2]$:

$$W = \int_C \vec{F} \cdot dr = \int_0^{\pi/2} (\cos^2(t), -\cos(t) \cdot \operatorname{sen}(t)) \cdot (-\operatorname{sen}(t), \cos(t))\,dt$$

$$= \int_0^{\pi/2} (-\cos^2(t)\operatorname{sen}(t) - \cos^2(t)\operatorname{sen}(t)\,dt = -2\int_0^{\pi/2} \cos^2(t)\operatorname{sen}(t)\,dt =$$

$$= -2 \cdot \frac{\cos^3(t)}{3}\Big|_0^{\pi/2} = -2/3$$

EX: CALCULE $\int_C y\,dx + z\,dy + x\,dz$ EM QUE C É O SEGMENTO DE $(2,0,0)$ Á $(3,4,5)$ SEGUIDO DO SEGMENTO QUE LIGA $(3,4,5)$ À $(3,4,0)$.

$$r: P = A + t \cdot \vec{v} = A + t \cdot \overrightarrow{AB} = A + t \cdot (B - A) = A + tB - tA$$

$$P = tB + A - tA = tB + (1 - t) \cdot A$$

$$r: \quad (x,y,z) = t(3,4,5) + (1+t)(2,0,0)$$

$$(x,y,z) = (3t + 2(1-t), 4t + 0, 5t + 0)$$

$$(x,y,z) = (2+t, 4t, 5t)$$

$$C_1 : \alpha_1(t) = (2+t, 4t, 5t), \quad t \in [0,1]$$

$$C_2 : \alpha_2(t) = (3, 4, t), \quad t \in [0,5]$$

$$\int_C y\,\underline{dx} + z\,\underline{dy} + x\,\underline{dz} = \int_{C_1} y\,dx + z\,dy + x\,dz + \int_{C_2} y\,dx + z\,dy + x\,dz$$

$$= \int_0^1 (4t \cdot \underline{1} + 5t \cdot \underline{4} + (2+t) \cdot \underline{5}\, dt + \int_0^5 (4 \cdot 0 + t \cdot 0 + 3 \cdot 1)\, dt$$

$$= 29\frac{t^2}{2} + 10t\Big|_0^1 + 3t\Big|_0^5 = 39 - 0 + 15 - 0 = \boxed{54}$$