Arbitrary Frege Arithmetic

Abstractionist theories in philosophy of mathematics are systems composed by a logical theory augmented with an abstraction principle (AP), of form: $\forall X \forall Y(@X = @Y) \leftrightarrow E(X, Y)$– that introduces, namely rules and implicitly defines, a term-forming operator @ by means of an equivalence relation E. As is well-known, the seminal abstractionist program, Frege's Logicism, failed: Russell's Paradox proved its inconsistency and, *a fortiori*, its non-logicality. In the last century, both the issue of consistency and the issue of logicality have been resumed in the abstractionist debate (cf. [13], [7], [1], [4], [3]). More precisely, on the one side, different revisions of Frege's original system have been proposed in order to avoid Russell's Paradox and to obtain a consistent system that is strong enough to derive (at least, a relevant portion of) Peano Arithmetic. On the other side, given a semantical definition of logicality as permutation invariance, some abstraction principles have been proved to be logical ([1], [4]).

Nevertheless, many concerns are still open. Particularly, regarding the preliminary condition of consistency, the ways out of Russell's Paradox proposed so far do not precisely mirror a corresponding explanation of the origin of the contradiction and often imply a weakening of the hoped strength of the theory (cf. [11], [14], [6]); regarding the issue of logicality, an undesired dilemma overshadows the abovementioned results: precisely in case of logical (i.e. permutation invariant) abstraction principles, their implicit *definienda* turn out to be non logical ([1]) – so preventing a full achievement of the Logicist goal.

My preliminary aim consists in arguing that these – apparently unrelated –problems have a common source in some unquestioned assumptions of Frege's project (inherited also by the following abstractionist programs). I argue that such assumptions are part of what we can call the Traditional view of abstraction, that includes the choice of classical logic as the base theory, with the related semantical consequence of full referentiality of the vocabulary, and the choice of a so-called Canonical interpretation function for all the (both primitive and defined) expressions of the language.

In the rest of the talk, I show that by renouncing one or both of these problematic assumptions we can recover consistency and/or logicality. More precisely, I propose a double revision of Frege's Logicist program: on the one side, weakening the Canonical interpretation function for the implicitly defined (abstract) expressions of the vocabulary (cf. [3]), I prove that any consistent revision of BLV turns out to be logical (i.e. permutation invariant); on the other side, I show that such an arbitrary interpretation, on a (negative) free logic background, allows us to identify a restriction of BLV, able to precisely exclude the paradoxical concepts, namely to avoid Russell's Paradox, but, at the same time, to preserve the derivational strength necessary to derive second-order Peano axioms. This means that this system – that we'll call Arbitrary Logicism, precisely renouncing to the Traditional assumptions mentioned above, is able to recover both Frege's goals of consistency and logicality.

The logical part of the language of Arbitrary Logicism, L_F, includes denumerably many first-order variables $(x, y, z, ...)$, denumerably many second-order variables $(X, Y, Z, ...)$, logical connectives $(\neg, \rightarrow)$ and a first-order existential quantifier $(\exists)$. We can also usefully define a predicative monadic constant (E!), whose extension is equal to the range of identity: E!a $=_{def}$ $\exists x(x = a)$. The only non-logical primitive symbol is the term-forming operator ϵ which applies to monadic second-order variables to produce complex singular terms $(\epsilon(X))$.

The theory involves, as its logical part, the axioms and inference rules of non-inclusive negative free logic with identity ($NF^{=}$):

NF1) $\forall v\alpha \to (E!t \to \alpha(t/v))$;

NF2) $\exists v E!v$;

NF3) $s = t \to (\alpha \to \alpha(t//s))$;

NF4) $\forall v(v = v)$;

NF5) $P\tau_1, ..., \tau_n \to E!\tau_i$ (with $1 \leq i \leq n$);

$\forall$I): $E!a...\phi(a/x) \vdash \forall x\phi$;

$\exists$E): $\phi(a/x), E!a...\psi, \exists x\phi \vdash \psi$, where a is a new individual constant which does not occur in ϕ and ψ.

Additionally, the theory involves an axiom-schema of universal instantiation for second-order variables ($\forall X\phi(X) \to \phi(Y)$), a rule of universal generalisation (GEN), a second-order comprehension axiom schema (CA: $\exists X\forall x(Xx \leftrightarrow \alpha)$) and *modus ponens* (MP).

The abstraction principle that characterizes this theory is obtained by weakening the right-to-left conditional of Basic Law V (BLV: $\forall F\forall G(\epsilon F = \epsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx))$, i.e. BLVa (arbitrarily interpreted), by means of the condition of Permutation Invariance (cf. [1], [3]).

W-BLV: $\forall F\forall G(\epsilon F = \epsilon G \leftrightarrow \forall x(Fx \leftrightarrow Gx) \wedge \epsilon(\pi(F)) = \pi(\epsilon F))$

As well known, the ϵ operator (as defined by standard BLV), also arbitrarily interpreted, is not Permutation Invariant – because, roughly speaking, by being inconsistent it is unable to define or rule any function. We can emphasize that, given an arbitrary interpretation, Permutation Invariance fails precisely for the argument that determines its inconsistency. In other words, as can be pointed out for other consistent revisions of BLV, in any case in which it is safely restricted, ϵ turns out to satisfy Permutation Invariance, namely it is such that $\pi(\epsilon) = \epsilon$, i.e. $\forall X\forall y(\epsilon X = y \leftrightarrow \epsilon(\pi(X)) = \pi(y))$. Then, the second conjunct of the right-hand side of W-BLV requires that – no matter which object y is identical to ϵF – ϵ satisfies Permutation Invariance for the considered arguments.

Accordingly, W-BLV, as a bi-conditional, turns out to be satisfied by any concept instantiating the universal quantifier. On the one side, given an arbitrary interpretation of the abstraction operator, for any concept different from Russellian concept (R), $\pi(\epsilon) = \epsilon$. On the other side, we can consider Russell's Paradox as a *reductio ad absurdum* of the alleged truth of both the sides of the bi-conditional for the concept R: the contradiction proves that ϵR – as legitimately admitted on a free logical background – does not exist, namely it is a term devoid of denotation; accordingly, it is not identical to itself (so, falsifying the left-hand side of W-BLV) and, even if R, as any other concept, is co-extensional with itself, it falsifies Permutation Invariance of the operator. Accordingly, also the right-hand side of W-BLV is false and also the instance of the bi-conditional for the concept R is verified.

Such a restricted version of W-BLV allows us to derive a corresponding restricted version of Hume's Principle. Nevertheless, the same restriction, on HP, is trivially satisfied by any instantiation, so it actually does not represent a weakening of the principle itself and allow us to derive the main arithmetical results, including Frege's Theorem.

References

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