Perturbed Kepler Orbital Element Problem in Geometric Algebra

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1 Introduction

Arakida and Fukushima[1] cast the Kustanheimo-Stiefel (KS) equations of motion in a compact and convenient quaternion notation involving the perturbation of orbit elements. Hestenes cast the KS equations into geometric algebra, including perturbed spinor equations and unperturbed orbital elements. This equation was slightly reordered by Doran and Lasenby because they defined rotors reversed from Hestene's formulation. This paper extends Hestenes derivation using Doran's and Lasenby's formulation to perturbed orbital elements in geometric algebra, using Arakida and Fukushima's paper as a guide.

2 Derivation

The equation of motion in perturbed spinor formulation is

$$U'' - \frac{1}{2}EU = \mathbf{fr}U$$

U is the spinor representing the position $\mathbf{r}$. Given basis vector $\mathbf{e}_1$, $\mathbf{r} = U\mathbf{e}_1U^\dagger$. The symbol† represents the reverse of U. The vector $\mathbf{f}$ is the perturbing force. The accent " represents the double-derivative of U with respect to s, while a single accent ' represents the single derivative $\frac{d}{ds} = r\frac{d}{dt}$.

From Arakida and Fukushima, the equation of motion is

$$\mathbf{u}'' + \omega_0^2\mathbf{u} \equiv \mathbf{F}$$

where $\mathbf{u} \equiv U$ and $\mathbf{F}$ is an effective force. The following relations apply. The negative of the Kepler orbit energy:

$$h_k = -E$$

The average angular velocity of the initial, unperturbed orbit, where h_{k0} is the negative Kepler orbit energy of the initial unperturbed orbit:

$$\omega_0^2 = \frac{1}{2}h_{k0}$$

$$= -\frac{1}{2}E_0$$

The work done by the perturbing force:

$$\begin{aligned} W &= \frac{1}{2}h_k - \omega_0^2 \\ &= \frac{1}{2}\left(h_k - h_{k0}\right) \\ &= \frac{1}{2}\left(E_0 - E\right) \end{aligned}$$

which leads to:

$$h_k = 2\left(W + \omega_0^2\right)$$

Using these relations, we can manipulate the geometric algebra formulation as follows.

$$\begin{aligned} U'' + \frac{1}{2}h_k U &= \frac{1}{2}\mathbf{fr}U \\ U'' + \left(W + \omega_0^2\right)U &= \frac{1}{2}\mathbf{fr}U \\ U'' + \omega_0^2 U &= \frac{1}{2}\mathbf{fr}U - WU \\ &= \frac{1}{2}\mathbf{fr}U - \frac{1}{2}\left(E_0 - E\right)U \\ &= \frac{1}{2}\left(\mathbf{fr} - E_0 - E\right)U \end{aligned}$$

This means that

$$\begin{aligned} \mathbf{F} &\equiv \frac{1}{2}\mathbf{fr}U - WU \\ & \quad \frac{1}{2}\left(\mathbf{fr} - E_0 - E\right)U \end{aligned}$$

which allows us to rewrite the differential equations from Arakida and Fukushima.

Some further definitions are required. The orbital elements Arakida and Fukushima use are (α, β, E, t) where

$$\begin{aligned} \alpha &\equiv \mathbf{u}\cos\omega_0 s - \frac{\mathbf{u}'}{\omega_0}\sin\omega_0 s \\ \beta &\equiv \mathbf{u}\sin\omega_0 s + \frac{\mathbf{u}'}{\omega_0}\cos\omega_0 s \end{aligned}$$

W is defined previously, and t_0 is the time at $s = 0$. The meaning of α and β in geometric algebra can be found by considering their relation to U and U' at $s = 0$, which are denoted by U_0 and U_0'.

$$\begin{aligned} \alpha|_{s=0} &= U_0 \\ \beta|_{s=0} &= \frac{U_0'}{\omega_0} \end{aligned}$$

The position spinor can then be evaluated according to Hestenes as:

$$U = U_0 \cos \omega_0 s + \frac{U_0'}{\omega_0} \sin \omega_0 s$$

Therefore:

$$U = \alpha \cos \omega_0 s + \beta \sin \omega_0 s$$

We can prove this relation is correct by plugging in the expressions for α and β:

$$\begin{aligned} U &= \left(U \cos \omega_0 s - \frac{U'}{\omega_0} \sin \omega_0 s \right) \cos \omega_0 s + \left(U \sin \omega_0 s + \frac{U'}{\omega_0} \cos \omega_0 s \right) \sin \omega_0 s \\ &= U \left(\cos^2 \omega_0 s + \sin^2 \omega_0 s \right) \\ &= U \end{aligned}$$

To find the expression for U', use the equation for α (or β):

$$\begin{aligned} \alpha &= U \cos \omega_0 s - \frac{U'}{\omega_0} \sin \omega_0 s \\ U' &= \frac{\omega_0}{\sin \omega_0 s} \left(U \cos \omega_0 s - \alpha \right) \\ &= \frac{\omega_0}{\sin \omega_0 s} \left((\alpha \cos \omega_0 s + \beta \sin \omega_0 s) \cos \omega_0 s - \alpha \right) \\ &= \frac{\omega_0}{\sin \omega_0 s} \left(\alpha \cos^2 \omega_0 s + \beta \sin \omega_0 s \cos \omega_0 s - \alpha \right) \\ &= \frac{\omega_0}{\sin \omega_0 s} \left(\alpha \left(\cos^2 \omega_0 s - 1 \right) + \beta \sin \omega_0 s \cos \omega_0 s \right) \\ &= \frac{\omega_0}{\sin \omega_0 s} \left(-\alpha \sin^2 \omega_0 s + \beta \sin \omega_0 s \cos \omega_0 s \right) \\ &= \omega_0 \left(\beta \cos \omega_0 s - \alpha \sin \omega_0 s \right) \end{aligned}$$

From Arakida and Fukishima, the expression relating time and initial time is:

$$t - t_0 = \frac{\alpha^2 + \beta^2}{2} s + \frac{\alpha^2 - \beta^2}{4\omega_0} \sin 2\omega_0 s - \frac{\alpha \cdot \beta}{2\omega_0} \cos 2\omega_0 s$$

Now, we are ready to use the equations of motion from Arakida and Fukushima, mapped into geometric algebra:

$$\begin{aligned} \alpha' &= -\frac{\mathbf{F}}{\omega_0} \sin \omega_0 s \\ \beta' &= \frac{\mathbf{F}}{\omega_0} \cos \omega_0 s \end{aligned}$$

The derivative for the orbital energy must be derived separately. From Hestenes:

$$\begin{aligned} \dot{E} &= \mathbf{v} \cdot \mathbf{f} \\ E' &= r\mathbf{v} \cdot \mathbf{f} \end{aligned}$$

This translates to W as:

$$\begin{aligned}
-\frac{E}{2} &= \frac{1}{2}h_k \\
&= \frac{1}{2}\left(h_k + h_{k0} - h_{k0}\right) \\
&= W + \omega_0 \\
-\frac{E'}{2} &= W' \\
W' &= -\frac{r}{2}\mathbf{v} \cdot \mathbf{f}
\end{aligned}$$

Finally, for time t:

$$\frac{dt}{ds} = t' = r$$

For t_0, from Arakida and Fukushima:

$$t_0' = \frac{1}{\omega_0}\mathbf{F} \cdot \left[\left(\alpha \sin \omega_0 s - \beta \cos \omega_0 s\right) s + \frac{1}{2\omega_0}\left(\alpha \cos \omega_0 s + \beta \sin \omega_0 s\right)\right]$$

References

[1] Hideyoshi Arakida and Toshio Fukushima. Long-term integration error of kustaanheimo-stiefel regularized orbital motion. ii. method of variation of parameters. *The Astronomical Journal*, 121(3):1764, 2001.