Introduction

The goal of this reviewer is to provide a knowledge foundation for Math Subject Tests. This can be used to review for the SAT Math Subject Test, EJU Math Tests, MEXT Government Scholarship Application Math Tests, and may more. The first six sections are for Course 1 Mathematics. The following ten sections are Advanced Mathematics.

1 Numbers and Expressions

1.1 Numbers and Sets

1.1.1 Logic

Mathematical logic is an important part in analysis. Presented as follows are some relevant concepts.

Table 1: Common logic terms and definitions

conjunction	$p \wedge q$	p and q
disjunction	$p \vee q$	p or q
statement	$p \Longrightarrow q$	"p implies q"
converse	$q \longrightarrow p$	"q implies p"
equivalence	$p \iff q$	"p if and only if q"

Table 2: "Tautologies in propositional calculus"

$p \vee \neg p$	"Law of excluded middle"
$\neg(p \wedge \neg p)$	"Law of contradiction"
$\neg(\neg p) \iff p$	"Negation of negation"
$p \Longrightarrow q \iff (\neg p \vee q)$	"Sufficiency"
$q \Longrightarrow p \iff (\neg q \vee p)$	"Necessity"
$\neg(p \wedge q) \iff \neg p \vee \neg q$	De Morgan's Law
$\neg(p \vee q) \iff \neg p \wedge \neg q$	De Morgan's Law
$(p \Longrightarrow q) \iff (\neg q \Longrightarrow \neg p)$	"Law of contraposition / the contrapositive"
$[(p \Longrightarrow q) \wedge (q \Longrightarrow r)] \Longrightarrow (p \Longrightarrow r)$	"Law of Transitivity"
$p \wedge (p \Longrightarrow q) \Longrightarrow q$	"*Modus ponens* or Law of detachment"
$q \wedge (\neg p \Longrightarrow \neg q) \Longrightarrow p$	"Indirect proof"
$(p_1 \vee p_2) \wedge (p_1 \Longrightarrow q) \wedge (p_2 \Longrightarrow q) \Longrightarrow q$	"Distinction of cases"

Table 2 is explained below:

1. Either a statement or its negation is true.
2. A statement and its negation cannot be true at the same time.
3. The negation of the negation of the statement is the statement.
4. **Sufficiency**. A premise is *sufficient* for a conclusion iff the conclusion is true whenever the premise holds.
5. **Necessity**. A conclusion is *necessary* for a premise iff the premise is true whenever the conclusion holds.
6. If the conjunction of two statements is false, then at least one of them is false.
7. If the disjunction of two statements is false, then both of them are false.

8. A premise is sufficient for a conclusion if and only if the negation of the conclusion is sufficient to show that the premise is false.

9. Logic is transitive.

10. If a statement is true and it is sufficient for a conclusion then the conclusion holds.

11. If a conclusion holds and it is necessary for a premise then the premise holds.

12. If at least one premise (case) that is sufficient for a conclusion holds, then it is necessary to conclude.

1.1.2 Sets

Sets Sets, in general, are collections of well-defined objects, or even sets! Sets can have *any whole number* of elements, called cardinality. A set with only one element is called a *singleton*, such as $\{\varnothing\}$. A set with no elements is called an *empty set or null set*, $\varnothing$.

The elements of sets should be unique and well defined. Sets can be defined in different ways:

1. "Let S be the set of even integers"

2. $S = \{0, 1, 2, 3\}$—Roster notation

3. $S = \{x : x \neq 1\}$—Set-builder notation

Subsets There are also collections within collections—subsets—denoted as $M \subset N$. Here, we have a subset M contained within N. This follows that all elements of M can be found in N.

Set operations Common sets operations include *set intersection*, $\cap$, and *set union*, $\cup$. The intersection between two well defined sets is the collection of elements present in *both* sets. The union of two well-defined sets is the collection of elements that can be found in *either* set. Other set operations include the set difference $\setminus$ and the set complement $'$. The set difference $M \setminus N = \{x | x \in M \land x \notin N\}$. The complement $M' = \Omega \setminus M$, where Ω is a universal set.

1.1.3 Real Numbers

In beginning algebra, most of the operations apply to the real number system $\mathbb{R}$. This number system is then divided into two sets—the set of rational numbers $\mathbb{Q}$ and the set of irrational numbers $\mathbb{Q}'$.

Rational Numbers Rational numbers are numbers that can be expressed as $\frac{p}{q}$ where p, q are integral. This means that these numbers *can be expressed as fractions.* Examples of rational numbers are $\frac{1}{3}$, $\frac{134}{235}$, and $\frac{3}{1}$. The set of irrational numbers is the complement of $\mathbb{Q}$ in $\mathbb{R}$. These numbers, hence, *cannot be expressed as fractions.* Examples of irrational numbers are e, π, $\sqrt{2}$.

Integers $\mathbb{Q}$ is subdivided into the set of integers $\mathbb{Z}$ and the set of non-integers. Subsets of $\mathbb{Z}$ the set of whole numbers, $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$, and the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$.

More subsets can be defined, by parity, such as the even number set and the odd number sets within $\mathbb{N}$. There is also a special subset or natural numbers called the set of prime numbers. This set will be described in some detail in a later section.

1.1.4 Propositions

Properties of subsets:

1. Reflexivity – "A set is a subset of itself"
2. Transitivity – "$M \subset N \wedge N \subset P \Longrightarrow M \subset P$"
3. $\varnothing \subset M \ \forall \ M$

These properties imply that:

1. The number of subsets of any set M is $2^{N(M)}$, where $N(M)$ is the cardinality of M.
2. The number of proper subsets of M is $2^{N(M)} - 1$

De Morgan's Law The law applies to the set operations $', \cap, \cup$. The propositions are stated as follows:

1. $(M \cup N)' = M' \cap N'$
2. $(M \cap N)' = M' \cup N'$

Cartesian Product The Cartesian Product of two sets X and Y, $X \times Y = \{(x, y) | x \in X \wedge y \in Y\}$. This means that $\mathbb{R}^2$ is the two-dimensional Cartesian plane, $\mathbb{R}^3$ is the three-dimensional Cartesian plane, and so on.

1.2 Calculation of Expressions

Expressions, in mathematics, are segments of statements. These expressions often involve operations on quantities. Examples are as follows:

1. $1 + 1$—Constant expression
2. x—Monomial
3. $(3x - 2)^2$—Polynomial
4. $\frac{3x+2}{4x+3}$—Rational expression
5. $(x + y)^4$—Expression in multiple variables

1.2.1 Expansion and Factorization of Expressions

FOIL method For multiplication of binomials, $(3x + 2) \cdot (4x + 3)$, the FOIL method for expansion provides an equivalent of the expression $(3x + 2)(4x + 3) = 12x^2 + 17x + 6$. The FOIL method is an application of the *distributive* property of multiplication over addition; however this is limited to multiplying binomials.

Rainbow method For multiplying more complex polynomials, the rainbow method is a generalized version of the FOIL method. $(x^2 + 2x + 1)(x + 1) = x^3 + 2x^2 + x^2 + 2x + x + 1 = x^3 + 3x^2 + 3x + 1$. Again, this is also derived from the distributive property of multiplication over addition.

Regrouping For factoring expressions, $a^2 + 2ab + b^2$, regrouping terms with common factors is reasonable so that common terms will be factored out: $a^2 + ab + b^2 + ab = a(a + b) + b(a + b) = (a + b)^2$. This is also a consequence of the distributive property. In order to properly factor expressions, there must be ways to redistribute the terms.

Commonly used expressions Some of the commonly used expressions are listed below:

1. Square of a binomial: $(x+a)^2 = x^2 + a^2 + 2ax$
2. Square of a trinomial: $(x+a+b)^2 = x^2 + a^2 + b^2 + 2ax + 2bx + 2ab$
3. Difference of two squares: $x^2 - a^2 = (x-a)(x+a)$
4. Difference of two cubes: $x^3 - a^3 = (x-a)(x^2 + ax + a^2)$
5. Sum of two cubes: $x^3 + a^3 = (x+a)(x^2 - ax + a^2)$
6. Difference of powers of five: $x^5 - a^5 = (x-a)(x^4 + ax^3 + a^2x^2 + a^3x + a^4)$
7. Sum of powers of five: $x^5 + a^5 = (x+a)(x^4 - ax^3 + a^2x^2 - a^3x + a^4)$

1.2.2 Linear inequalities

There are two kinds of inequalities: *strict* and *non-strict* inequalities. Strict inequalities are statements of non-equality providing no solutions for equality. Non-strict inequalities, on the other hand, provide solutions for equality.

Examples of inequalities are the following. Also, note that the statements are complements ($-ve$s) of each other.

1. $x + 3 \leq 4$—Non-strict inequality
2. $x + 3 > 4$—Strict inequality

Solving linear inequalities is similar to solving linear equations.

1.2.3 Equations and inequalities with $|x|$

Absolute value function $|x|$ This function gives the distance of the value of the expression from 0. Since it returns a distance, it should be *non-negative.* Formally the absolute value is defined as:

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

Examples are shown below:

1. $|3| = 3$
2. $|-10| = 10$

There are two cases for the value of a real-valued expression: it is non-negative or negative. Thus, both cases must be considered.

Equations with $|x|$ Since the absolute value of a number is its distance from 0 in the real number system, there are two solutions to equations of absolute values of linear equations. If $|x+a| = 10 \iff x + a = 10 \vee x + a = -10$.

Inequalities with $|x|$ Inequalities containing absolute values are slightly different from equations. Since absolute values are non-negative, then the inequality $|13-x|<0$ would not make sense. What makes absolute value inequalities different is the set of cases used for solving.

Shown are the four cases and their logical equivalents:

1. $|x+3| \leq 4 \iff -4 \leq x+3 \leq 4$
2. $|x+3| < 4 \iff -4 < x+3 < 4$
3. $|x+3| \geq 4 \iff (-4 \geq x+3) \vee (4 \leq x+3)$
4. $|x+3| > 4 \iff (-4 > x+3) \vee (4 < x+3)$

Triangle inequality (Algebraic) This is a famous inequality involving absolute values. This will be restated in a later section (see section 5).

$$|x+y| \leq |x| + |y|$$

2 Quadratic Functions

2.1 Quadratic Functions and Their Graphs

Relations and Functions A relation is a mapping between two well-defined sets M and N. Operating on real-valued sets, every relation $S \subset \mathbb{R}^2$ and can be graphed in a two-dimensional Cartesian plane. There are different kinds of relations depending on the number of associations of elements of each set in another. Of such relations there are only some that are considered functions. Functions are operations on expressions that map values to another set of values. Hence, if a value or expression in the **domain** is mapped to multiple elements in a certain **codomain** set the relation is not a function.

Vertical and Horizontal line test The vertical line test is a method to determine whether a relation is a function. The validity of this test arises from the definition of a function. The horizontal line test is a method to determine whether a function is injective or non-injective.

The four kinds of relations are as follows:

One-to-one relations/injective In this kind of relation, an element in set N is associated with *at most one* element in M. Given $M = 3, 5, 6$ and $N = 6, 10, 12$. A one-to-one relation for the given sets is given as "the elements of N are twice the elements of M". Equivalently, this can be expressed as $N = \{2x | \forall x \in M\}$. Another one-to-one relation for the given sets is the relation which defines the set $S = (3, 12), (5, 10), (6, 6)$. This relation is a function since there is exactly one element mapped to each element in M in N. This function is injective since at most one element in M is mapped to every element in N.

One-to-many relations In this kind of relation, an element in set M is associated to more than one element in N. Given $M = 3, 5, 6$ and $N = 6, 10, 12$. A subset of the relation is $S = (3, 10), (3, 12), (5, 6), (5, 10)$. As shown 3 and 5 from M pair are mapped with at least one element in N. This relation cannot be a function since there are more than one values that are mapped to an element in the domain to the codomain.

Many-to-one relations In this kind of relation, more than one element in set M, domain, may be associated to *exactly one* element in N, codomain. Given $M = 3, 5, 6$ and $N = 6, 10, 12$, A relation between M and N of this kind is $S = (3, 6), (5, 6), (6, 10)$. As shown there can be more than one mapped element to the codomain from the domain. This relation is a function since there is only one element from N that is mapped to every element in M.

Many-to-many relations In this kind of relation, more than one element in M may be associated to an element in N and vice versa. This cannot be a function, by definition.

Quadratic expressions and functions Quadratic expressions are *polynomial expressions of degree* 2. Examples of quadratic expressions are x^2 and $x^2 + 6x + 9$. Quadratic functions are operations on a domain, $\mathbb{R}$ that is defined with mapping with quadratic relations. An example is $f(x) = x^2; f(3) = 9$

2.1.1 Variation in values of quadratic functions

Quadratic functions vary with respect to the square and of the value of the operand. Hence the variation varies linearly with respect to the operand—via getting the derivative (see section 15) of the function. Values of quadratic functions either increase then decrease or decrease then increase. Geometrically, the *locus* (set of points) defining the relation is a *vertical parabola*. This then means that quadratic functions have maximum or minimum values.

2.1.2 Maximum and minimum values of quadratic functions

A quadratic function, being parabolic, is either concave up or concave down. Depending on its orientation, a parabola can have either a minimum value or a maximum value. If the function is led with a negative coefficient, then it is concave down. Otherwise, it is concave up.

It can be shown by the derivative that *concave down quadratic functions have local maxima* and *concave up quadratic functions have local minima.* Not surprisingly, since the concavity is a constant function of the operand—via the second derivative test (see section 15)—*local extrema are also absolute extrema.*

For a given quadratic function $f(x) = ax^2 + bx + c$, by the first derivative test (see section 15), the local extremum can be found at $x = \frac{-b}{2a}$. The value of the extremum, by substitution is $y = \frac{4ac-b^2}{4a^2}$.

2.1.3 Determining quadratic functions

Connect the dots Since functions are operations on some real number x it would be possible to determine functions from points that define it. As a general rule, to determine a polynomial function of degree n, at least $n + 1$ points would be needed. Since quadratic functions are degree 2 polynomials, there should be about three points. Given three points, it would then be possible to construct systems of linear equations in $\mathbb{R}^3$. It would be similar to solving the following matrix:

$$\begin{bmatrix} x_1^2 & x_1 & 1 \\ x_2^2 & x_2 & 1 \\ x_3^2 & x_3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix}$$

Given the roots of the function This is a special case of "connect the dots". A quadratic function has either two real roots or no real roots. This applies to the first case. The function can be determined from roots $x_1 and x_2$ as $f(x) = (x - x_1)(x - x_2)$

Given the y-intercept and vertex This is another special case of "connect the dots". Here c would be the value of the y-intercept, $h = \frac{-b}{2a}$, and finally $k = \frac{4ac-b^2}{4a^2}$.

Given vertex (h, k) and latus rectum length $4p$ From the standard form of a vertical parabola, $(x-h)^2 = 4p(y-k)$, it would be easy to determine the function through mapping of the given values.

Given the sum and product of roots This also applies if related values can be derived. Let the sum and product of roots be s and p, respectively. The quadratic function, by Vieta's theorem is given by $f(x) = x^2 - sx + p$.

2.2 Quadratic Equations and Inequalities

As a note, *equations and inequalities are not functions; they are statements.* Functions are *operations* on given operands. Examples of such statements are:

1. $x^2 + 3x + 7 = 5$—a quadratic equation in x
2. $a^4 + 2a^2 + 1 = 0$—a quadratic equation in a^2
3. $x^2 + 3x + 7 \geq 5$—a quadratic inequality in x
4. $a^4 + 2a^2 + 1 < 0$—a quadratic inequality in a^2

2.2.1 Solutions of quadratic equations

Recall that polynomials that are *factorable* may be reduced to their factors. Polynomials that cannot be factored are classified as *prime* polynomials.

Factoring If the quadratic expressions not prime, the roots (solutions) can be determined from equating the factors to 0.

Completing the square Perfect squares, similar to absolute values, are non-negative. Then statements such as $ax^2 = -1 \;\forall\; a > 0$ do not make sense in the real number system. Again, similar to absolute value expressions, cases will be considered in solving statements. First, transform the statement into one containing a *perfect square trinomial* in one side. In solving $(x-a)^2 = b$, there are two cases $x - a = \pm\sqrt{b}$. The solutions follow.

Quadratic formula As a shortcut to completing the square, the quadratic formula was defined from the process.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The Discriminant Half of the time, quadratic equations do not have real roots. This happens when the quadratic function $f(x) = 0$, when graphed, does not touch the x-axis. This can be checked by calculating the discriminant $D = b^2 - 4ac$. If $D > 0$, the function has *two real roots.* If $D = 0$, the function has *one real root.* This also means that the expression is a *perfect square trinomial.* Lastly, if $D < 0$, the function has *no real roots.*

2.2.2 Quadratic equations and graphs of quadratic functions

Quadratic equations are statements with a quadratic expression. The quadratic expression can be treated as a function of an unknown, and the "other-hand-side" as the function value, i.e $f(a) = 3a^2 = 27$. In solving the quadratic equation $f(a) - c = 0$, we find the zeros of a quadratic function $g(a) = f(a) - c$. The intersections of the graph of g and the x-axis are the roots of the quadratic equation $f(a) - c = 0$.

2.2.3 Quadratic inequalities and graphs of quadratic functions

Quadratic functions, such as $f(x)$, divide $\mathbb{R}^2$ into two regions of inequality. Regions above the graphs of quadratic functions are regions $S = \{(x, y)|y > f(x)\ \forall\ x \in R\}$. On the other hand, the regions below the functions are regions $T = \{(x, y)|y < f(x)\ \forall\ x \in R\}$.

Similarly from the previous subsection, we can determine the boundaries where the inequality is satisfied and when they are not.

If g is led by a positive coefficient This means that the graph of g is concave up. If $g(a) > 0 \iff f(a) > c$ must be satisfied then the solution set is:

$$S = \{x|x < x_1 \vee x > x_2\}$$

. Else if $g(a) < 0 \iff f(a) < c$ must be satisfied then the solution set is:

$$S = \{x|x_1 < x < x_2\}$$

. In both cases x_1 and x_2 are the zeros of $g(a)$ and the roots of $f(a) - c = 0$.

If g is led by a negative coefficient This means that the graph of g is concave down. The whole inequality can be negated for simplicity. Note that the *inequality symbol must be flipped as well.*

3 Probability

3.1 Number of possible outcomes

Events Events can be broken down into sets of events either *dependent* or *independent* of each other.

Independent events These are events wherein the outcome of an event does not rely on another. An example of such events are "picking an ace" and "picking a red card" from a standard deck.

Dependent events These are events wherein the outcome of an event relies on the outcome of a first event. An example of such events are first "picking a black card", and throwing it away then "picking a 'Queen'".

Mutually inclusive events These are events that happen simultaneously. Events should be *necessary* consequences of a prior set of events. Examples of events are "getting a 6 at the top face of a die" and "getting a 1 at the bottom face of a die".

Mutually exclusive events These are events that cannot happen simultaneously. Events should *not* be *necessary* consequences of a prior set of events. Examples of events are "being a *Slytherin*" and "being a *Ravenclaw*".

3.1.1 Principles of counting

Cardinality refers to the number of elements in a set. Such sets—also called the *outcome space*—can be the sets of possible outcomes for independent events or mutually exclusive events. In order to determine **probabilities** of related events occuring, there are certain rules we must follow.

Addition rule This rule states that the cardinality of the **union** of the outcome space of *mutually exclusive* events occuring is the sum of the cardinalities of both sets. This is logical since the events cannot happen at the same time. An example would be finding the number of students from either *Slytherin* and *Ravenclaw*. By the addition rule, the total number of students in either house is the sum of the number of students in each house. In cases where the events are not mutually exclusive, a **Venn diagram** will be useful.

Multiplication rule This rule states that the cardinality of the **intersection** of the outcome space of combinations of *independent events* is the product of the cardinalities of the outcome spaces of each event. An example would be finding the number of cards in a standard deck. By the multiplication rule, the total number of possible cards in a standard deck is the product of the number of suits and the number of ranks.

3.1.2 Permutations and combinations

There are some cases wherein the numbers of possible arrangements and combinations of a set of elements is desired. There may be cases with *indistinguishable* objects. Cases described in this section involve mostly distinguishable objects. Formulae for cases involving indistinguishable objects may be derived.

Permutations Permutations are arrangements of elements in a set. Here, the order of elements is concerned. Hence $\{1, 2, 3\}$ is different from $\{3, 1, 2\}$. It should be noted that choosing numbers sequentially in a set are independent events.

In a set of N distinguishable objects, there should be N ways of choosing a first element, $N - 1$ ways of choosing a second element, and so on until a last element. By the multiplication rule, the total number of possible arrangements (**permutations**) is:

$$\prod_{n=1}^{N} n = n!$$

3.2 Probability and its fundamental properties

Probabilities In some cases we are concerned of the chances of events occuring. These are called **probabilities**. A probability is a ratio of the cardinalities of a space where a set of conditions are satisfied and the outcome space of an event. Another definition of probability would be then that it is the ratio of the cardinality of the set difference of the set of outcomes that *satisfy* conditions and the outcome space and the cardinality of the outcome space.

Since the outcome space contains *all possible events*, then there must be *no* outcome that satisfies none of the possible conditions. Consider the *Sorting Ceremony*, there are only *four* possible outcomes (*Ravenclaw, Gryffindor, Slytherin,* and *Hufflepuff*). Since the house *Rowlingger* does not exist—it is not part of the outcome space—the probability of it occuring is 0. This is a “worst case” for a probability–**the lowest value of a probability is** 0.

There may be some conditions that are satisfied for a *subset* of the outcome space and are false for the complement. Consider, again, the *Sorting Ceremony*, the probability of being sorted to the House *Hufflepuff*, is non-zero, since it is in the outcome space, but it is less than, or approaching, 1. Depending on the amount of **bias** in assignment of the *Sorting Hat*, it is possible that the probability of being sorted to House *Gryffindor* may be approaching 1 but not equal to since there are other possible outcomes—**probabilities can range anywhere between** 0 **and** 1.

In some cases, there is only *one outcome* for an event. Consider, again, the *Sorting Ceremony* where the *Sorting Hat* is defective and is only able to classify everything into *Ravenclaw*, the probability of being sorted into the House is 1 since it is the only outcome possible. This is another "worst case" for a probability—**the highest value for a probability** is 1.

3.3 Independent trials and probability

Trials Empirical evidence of the outcomes of events involve **trials**. A trial is a ***controlled* event**. This means that multiple trials are held in the same enviromnent. Trials should be designed such that they *cannot affect* the outcome of another trial—they should be *independent.* The independence of trials is the basis behind most statistical measures. It makes numerical modeling easier to process and understand.

Numerical modeling of empirical events (statistical modeling) relies heavily on probabilities. Probabilities provide a means to measure, in general, the likelihood of an outcome to an event. There are two ways of prediction *a priori*, *prior* to trials, and *a posteriori*, *after* trials.

A priori This approach of determining probabilities relies on *assumptions.* A usual assumption made using this approach is that bias is absent—all outcomes are equally probable. An example would be tossing a *fair*, two-sided coin, with *distinguishable* sides. The probability of a toss resulting to a "heads" would be the same as the probability of the *same* toss resulting to a "tails". Since the elements of the outcome space have an equal chance of happening the probability is 1/2. From this probability, other related probabilities can be determined such as the probability of getting three consecutive "heads".

Since trials are also events, it would be possible to determine the probability of outcomes satisfying conditions among combinations of independent trials using the *multiplication rule.*

Mathematically, let A be the outcome of a first event, and B be the outcome of a second event, and given that the events are *independent*, the probability of both happening is:

$$P(A \cap B) = P(A) \cdot P(B)$$

A posteriori This approach of determining probabilities relies on *results.* The probability, then, of an event is defined as the ratio of the number of successes to the total number of trials:

$$P(A) = N(A)/N$$

3.4 Conditional probability

The concept of conditional probability is based on *partial dependence* of events. Events that are considered in this subsection are *not necessarily independent* of each other. While the principles observed are similar to those with independent events, analysis would be significantly different. These cases happen most of the time in real life examples.

In some conditions, the probabilities of outcomes may differ, this suggests **dependence**. To handle this, the probability of outcomes A and B of two *dependent events* happening is:

$$P(A \cap B) = P(B|A) \cdot P(A)$$

where $P(B|A)$ is the probability of B happening given that A happened.

4 Properties of Integers

Euclidean division An integer N can be expressed as a sum of the product of any two integers and an integral remainder:

$$N = P(Q) + R$$

Examples are shown below:

$$35 = 5(7) + 0$$
$$123 = 6(20) + 3$$

4.1 Divisors and Multiples

Divisors A divisor of $n \in \mathbb{N}$ is a number $k \in \mathbb{N}$ such that $\frac{n}{k} \in \mathbb{N}$, i.e. the quotient between the dividend and divisor is a natural number. Divisors can also be called the *factors* of a natural number.

Proper divisors A proper divisor of a number is a divisor that is *not equal* to the number.

Greatest Common Factor/Divisor The greatest common factor or divisor (GCD) of a set of natural numbers is the greatest natural number, not necessarily an element of the set, that divides all of the numbers in the set. Formally:

$$GCD(S \subset \mathbb{N}) = x \in \mathbb{N} \; s.t. \; x \mid a \; \forall \; a \in S$$

Multiples A multiple of $n \in \mathbb{N}$ is a number $M \in \mathbb{N}$ such that $\exists \; k \in \mathbb{N} : M = kn$ i.e. there exists a natural number k where M is a product of k and n.

Least Common Multiple The least common multiple (LCM) of a set of natural numbers is the least natural number, not necessarily an element of the set, such that every element in the set divides the number. Formally:

$$LCM(S \subset \mathbb{N}) = x \in \mathbb{N} \; s.t. \; a \mid x \; \forall \; a \in S$$

These definitions imply that:

1. A natural number n is a divisor *and* a multiple but *not* a proper divisor of itself.
2. If $k \mid n$ then k is a divisor of n and n is a divisor of k.

4.2 Euclidean Algorithm

The Euclidean algorithm is a fast *recursive* method to calculate the GCF of a two numbers. Reduction algorithms can be applied to a list of numbers based on the algorithm. The algorithm assumes that the GCF of two numbers is the same as the GCF of one of the numbers and the difference of the numbers. Formally, with $x > y$:

$$GCD(x, y) = GCD(x\%y, y) = GCD(x\%y, x)$$

4.3 Applications of the properties of integers

Fundamental theorem of arithmetic The theorem states that natural numbers other than 1 are either *prime numbers* or can be uniquely represented as *products* of prime numbers.

Number of divisors of a number From the fundamental theorem of arithmetic and counting techniques (see section 3), it is possible to determine the number of factors of an integer. As examples:

$$100 = 2^2 \cdot 5^2$$
$$45 = 3^2 \cdot 5^1$$

The first number (100) has $(2+1)(2+1) = 9$ factors. The second number (45) has $(2+1)(1+1) = 6$ factors.

Number of "trailing zeros" in a factorial In finding the number of trailing zeros in a factorial m, it is necessary to find the largest power of 10 that can divide the factorial expression. Since $10 = 2 \cdot 5$ we find $min(f(2),\ f(5))$ where $f(n \in P)$ gives the exponent of the prime number n in the prime factorization of the factorial. The value of $f(n)$ can be obtained from continuous division of n from the operand m.

Modulo Congruence

Chinese Remainder Theorem

5 Properties of Figures

5.1 Plane Figures

Planes A plane is an infinite set of lines that is determined by two non-parallel lines. It is a *flat surface* where other figures (surfaces) can be drawn.

Polygons Polygons are plane figures that are made of finite non-intersecting line segments surrounding a closed space. The minimum number of sides a polygon can have is 3, called a *triangle*.

Convex polygons A subset of polygons wherein *all diagonals* (line segments connecting two non-adjacent sides of a polygon) are contained *within* the polygon. Non-convex polygons are polygons that are not convex in nature.

Regular polygons These are convex polygons having equal side lengths. As a consequence, the measures of the internal angles are all equal.

Circles Circles are plane figures composed of an infinite amount of points equidistant to a fixed point called a *center*. More formally, a circle is a locus of points equidistant to a fixed center.

5.1.1 Properties of triangles

Triangles can classified into three groups based on the lengths of side measures.

They can also be classified according to their angle measures.

scalene	no equal sides
isosceles	two equal sides
equilateral	three equal sides

acute	all angles are *acute* $< 90°$
right	an angle is *right* $90°$
obtuse	an angle is *obtuse* $> 90°$

Triangle inequality (Geometric) The triangle inequality states that the measure of the longest side of the triangle should be less than, or equal to (degenerate case), the sum of the lengths of the shorter sides.

A theorem on sides and angles The *longest side* of a triangle is the side opposite the *largest angle*. The *shortest side* of a triangle is the side opposite the *smallest angle*. As will be seen in a later section (see section 9), this is a consequence of the "law of sines".

Hinge theorem and its converse Hinge theorem states that if two sides of a triangle are congruent to another and the included angle in the first triangle is larger than the second triangle, then the side opposite the included angle in the first triangle is longer than the side opposite the included angle in the second triangle. The converse of this theorem is also true. As will be seen in a later section (see section 9), this is a consequence of the "law of cosines".

Congruent triangles Congruent triangles are triangles of the same size and shape. This means that all *corresponding sides* and *corresponding angles* are congruent.

SSS Postulate of Congruence This postulate says that if all corresponding sides of triangles are congruent, then the triangles are congruent.

SAS Postulate of Congruence This postulate says that if two corresponding sides of triangles are congruent and the corresponding included angles are congruent, then the triangles are congruent.

ASA Postulate of Congruence This postulate says that if two corresponding angles of triangles are congruent and the included sides are congruent then, the triangles are congruent.

AAS Theorem of Congruence This theorem says that if two corresponding angles of triangles are congruent and any corresponding side is congruent between triangles, then they are congruent.

Corresponding Parts of Congruent Triangles are Congruent Often abbreviated to **CPCTC**, this is a useful fact when it comes to proving identities relying on figure congruence.

Similar triangles Similar triangles are triangles of the same shape. This means that all of the *corresponding angles* are *congruent* and all of the *corresponding sides share a common ratio*

SAS Theorem of Similarity This theorem says that if two of the corresponding sides share a ratio and the included angles are congruent then the triangles are similar.

SSS Theorem of Similarity This theorem says that if all corresponding sides share a ratio then the triangles are similar.

AA Theorem of Similarity This theorem says that if any two corresponding angles of triangles are congruent then the triangles are similar.

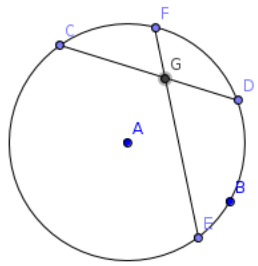
5.1.2 Properties of circles

Parts of a circle A circle is an infinite collection of points equidistant to a fixed point called a *center*. By connecting some points we can define some useful properties of circles and lines.

Radii, diameters, and circumferences The radius of a circle is the line segment connecting the center and a point on the circle. The diameter of a circle is the line segment whose endpoints lie on the circle and *passes through the center*. The circumference of a circle is its *perimeter*—the distance around the circle. The constant π is the ratio between the circumference and the diameter.

Secants, tangents, and chords Secants are *lines* that intersect a circle at *two points*. Tangents are the "limited versions" of secants. They are *lines* that intersect a circle at *one point*. Tangents are *perpendicular to the radii* of the circles they intersect. Chords are *line segments* whose endpoints lie on the circle. The diameter of a circle is the longest chord.

"Power of a Point" theorems This set of theorems state the relationships between lengths of segments in figures containing circles.



Two chord case The two chord case of the "power of a point" theorems states that measurements in the figure above follow the relation below. It can be proven using properties of similar triangles.

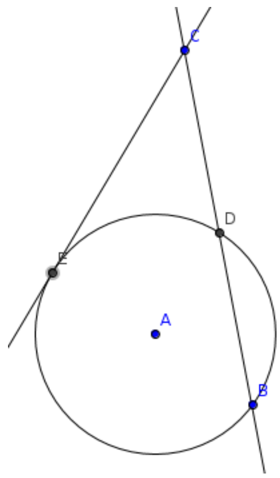
$$FG \cdot GE = CG \cdot GD$$

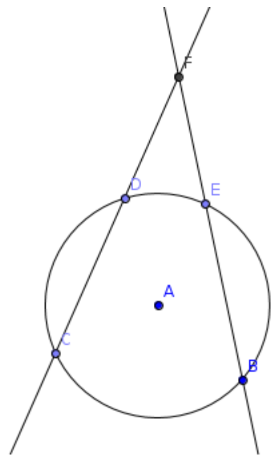
Secant-tangent case The secant-tangent case of the "power of a point" theorems states that measurements in the figure above follow the relation below. It can also be proven using properties of similar triangles.

$$AB^2 = CD \cdot CE$$

Two secant case The two secant case of the "power of a point" theorems states that the measurements in the figure above follow the relation below. It can also be proven using properties of similar triangles.

$$FC \cdot FD = FE \cdot FB$$





5.2 Solid/Space Figures

5.2.1 Lines and Planes

5.2.2 Polyhedrons

Euler's theorem

6 Miscellaneous Expressions

6.1 Expressions and proofs

6.1.1 Division of polynomials, rational expressions, and binomial theorem identities

Euclidean division of polynomials The *Euclidean division*, or simply division, of polynomials is the counterpart of the Euclidean division of integers to polynomials. A polynomial $P(x)$ can be expressed as:

$$P(x) = Q(x) \cdot D(x) + R(x)$$

where $D(x)$ is the divisor, $Q(x)$ is the *quotient*, and $R(x)$ is the remainder upon division. As examples:

1. $x^2 + 3x + 2 = (x+1)(x+2) + 0$
2. $x^2 = (x-1)(x+1) + 1$

This division of polynomials can be carried out using *long divison* and *synthetic division*. Application of these techniques are **not** covered in this reviewer.

Long Division As the name suggests, this is the method of division that shows all of the parts of the polynomials being divided. This is useful for finding shorter techniques for dividing polynomials. The technique of synthetic division is derived from the application of long division.

Synthetic Division This division technique is derived from the long division of polynomials. This makes use only of the coefficients of the terms of the dividend and the divisor. This is normally used when dividing polynomials of a single term to a linear polynomial of the same term, *eg.* $3x^3 - 4x + 5$ divided by $x + 2$. Extensions to quadratic forms have been made and further generalizations can also be made.

Rational Expressions Rational expressions are expressions involving *ratios* of polynomials. There are two kinds of rational expressions—*proper* and *improper* rational expressions. A **proper** rational expression is a ratio between a polynomial of a higher degree, or the same, to another polynomial. An **improper** rational expression is a ratio between a polynomial of a lower degree to another polynomial. Examples of rational expressions are:

1. $\frac{x^2+x+1}{x^3+4x+2}$—a proper rational expression
2. $\frac{x^3+4x+2}{x^2+x+1}$—an improper rational expression

Since improper rational expressions is a division operation on two polynomials, it is necessary that they can be expressed in the form $Q(x) \cdot D(x) + R(x)$ by Euclidean division. From expression:

$$\begin{aligned} P(x) &= Q(x) \cdot D(x) + R(x) \\ \frac{P(x)}{D(x)} &= Q(x) + \frac{R(x)}{D(x)} \end{aligned}$$

Here, the LHS is the *improper* rational expression, $Q(x)$ is the polynomial quotient, and the last term is a *proper* rational fraction, the remainder.

6.1.2 Proofs of equations and inequalities

This subsection will contain important equations and inequalities used for proving. Important equations for proving algebraic equalities are shown in section 1.

QM-AM-GM-HM inequalities This family of inequalities is famous for applications in proving related inequalities. This is a relation between the *quadratic mean* (QM), *arithmetic mean* (AM), *geometric mean* (GM), and *harmonic mean* (HM) of *positive real numbers.* Equality is achieved when *all* numbers in the set are *equal* The relation is given as follows:

$$\begin{aligned} QM \geq AM & \qquad \geq GM \geq HM \\ \sqrt{\frac{1}{N} \cdot \sum_{n=1}^{N} a_n^2} \geq \frac{1}{N} \cdot \sum_{n=1}^{N} a_n & \qquad \geq \Big(\prod_{n=1}^{N} a_n \Big)^{\frac{1}{N}} \geq \Big(\frac{1}{N} \cdot \sum_{n=1}^{N} \frac{1}{a_n} \Big)^{-1} \end{aligned}$$

Tips Use these inequalities when the expressions appear *cyclic*. This means that terms have repeating "patterns".

Cauchy-Schwarz Inequality This inequality is also used in proving some related inequalities. These state that the product of a sum of squares is grater than or equal to the sum of the square of the sum of the products of the terms. Equality holds when $\frac{a_n}{b_n}$ $\forall\ n \leq N$. Symbollically,

$$(a_1^2 + a_2^2 + a_3^2 + \ldots + a_n^2)(b_1^2 + b_2^2 + b_3^2 + \ldots + b_n^2) \geq (a_1b_1 + a_2b_2 + a_3b_3 + \ldots + a_nb_n)^2$$

Tips Use this inequality when comparing inequalities involving *both squares and their products.* Applications may not be obvious.

6.2 Equations of higher degree

Equations of degree 3 and above do not follow quadratic, and other simple, relations and are most of the time harder to solve. Completing the cube (for degree 3 polynomials) and so forth for higher degree polynomials are very tedious to do.

6.2.1 Complex numbers and solutions of quadratic equations

The set of complex numbers $\mathbb{C}$ contains the set of real numbers $\mathbb{R}$ and the set of non-real numbers $\mathbb{R}'$. Complex numbers have imaginary components and can be expressed in the form $a + bi$ where $a, b \in \mathbb{R}$ and i is the imaginary unit ($i = \sqrt{-1}$).

As stated in section 2, quadratic expressions with $D < 0$ have no real roots but they have *two non-real, complex roots.* Given that $\sqrt{-1} = i$, it is then possible to calculate the complex roots of quadratic functions.

6.2.2 Factor theorem

This theorem states that if $P(a) = 0$, then $x - a$ is a factor of $P(x)$ *ie. it leaves a remainder of* 0 *upon division.* This theorem can be shown as a consequence of Euclidean division.

6.2.3 Properties of higher degree polynomials and methods of solving them

Rational root theorem This theorem states that given a polynomial $P(x) = a_nx^n + a_{n-1}x^n - 1 + a_{n-2}x^{n-2} + \ldots + a_0$, the set S:

$$S = \{\frac{b_0}{b_n}|b_0 \mid a_0; b_n \mid a_n\}$$

contains all possible *rational roots* of the equation.

7 Figures and Equations

7.1 Lines and Circles

7.1.1 Coordinates of a Point

Considering $\mathbb{R}^2$, there are two parts to the coordiantes of a point. The first part is the x-coordinate known as the *abscissa* of the point. The second part is the y-coordinate known as the *ordinate* of the point. Using the abscissa and ordinate of a point, it is possible to plot them onto $\mathbb{R}^2$.

The Four Quadrants The Cartesian plane is divided into 4 regions as quadrants.

7.1.2 Equations of Lines

A *line* is a collection of points that extend in both directions. Lines can be represented in $\mathbb{R}^2$ as relations between x and y. Examples of these relations are:

1. $L_1 = \{(x, y) \mid y = x\}$
2. $L_2 = \{(x, y) \mid y = -x\}$
3. $L_3 = \{(x, y) \mid y = 0\}$
4. $L_4 = \{(x, y) \mid x = 0\}$

Items 1 and 2 above show a relation that exists in both x and y. When all points following this relation (Item 1) are plotted, a line passing through the origin at $45°$ will be seen. When all points following the second relation (Item 2) are plotted, a line passing through the origin at $-45°$ will be seen. Items 3 and 4 show relations that depend only on one of x and y. Item 3 presents a horizontal line passing through the origin—also called the x-axis. Item 4 presents a vertical line passing through the origin—also called the y-axis.

Intercepts Intercepts are the values at which the line crosses an axis. The x and y intercepts—a and b—of Items 1 and 2 above are 0 and 0, respectively.

Slope The slope of a line is a measure of "steepness". Its value is calculated as the *first derivative* of the equation as a function or "rise over run".

Forms of Equations of Lines Equations of lines can be expressed in different forms such as the *general form* and the *standard forms.*

General Form This form is easily distinguishable since *one side of the equation is* 0. Examples are $x \pm y = 0$, $x = 0$, $y = 0$.

Standard Forms There are many different standard forms of equations of lines. These forms show the different properties of the lines they define. Listed below are some of them and the properties they give.

Table 3: Standard forms and properties of lines

Slope-intercept form	$y = mx + b$	Slope and y-intercept of the line
Point-slope form	$(y - y_1) = m(x - x_1)$	Slope and a point (x_1, y_1) on the line
Two point form	$(y - y_{1/2}) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)\left(x - x_{1/2}\right)$	Two points on the line
Two-intercept form	$\frac{x}{a} + \frac{y}{b} = 1$	*Non-zero* intercepts of the line

7.1.3 Equations of Circles

Distance formula This formula is based on the Pythagorean theorem and it is used to get distances between points.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

As said in section 5, a circle is a collection of points *equidistant* to a fixed point called the *center.* Using the distance formula, we can find a relation wherein the distance of points to a "center" is *constant*—a circle.

$$(x-h)^2 + (y-k)^2 = r^2$$

Given above is the standard form of the equation of a circle. Like lines, a general form ($RHS = 0$) for the equation can also be expressed.

7.1.4 Relative Positions of a Circle and a Line

Lines can be drawn alongside a circle and relative positions can be determined. Two of the relative positions were already stated in section 5—tangent and secant. In this section, we define a third relative position called the **external line**. As the name suggests, an external line is a line that does not intersect the circle.

To determine the relative position of a line with respect to a circle, the following system should be solved.

$$\begin{cases} (x-h)^2 + (y-k)^2 &= r^2 \\ y - mx &= b \end{cases}$$

Solving the system means finding points that satisfy *both* equations. This then means that such points can be found on both the line and the circle. If there are *two* solutions, then the line is a *secant* to the circle. If there is *only one* solution, then the line is *tangent* to the circle. Lastly, if there are no solutions, then the line is an *external line* to the circle.

7.2 Locus and Region

7.2.1 Locus defined by an equality

A **locus** is a set of points that satisfies a given equality. Loci (relations) are visually represented in $\mathbb{R}^2$. As examples related to this section, a circle is a locus equdistant to a center, a line is a locus of constant variation.

7.2.2 Region defined by an equality

Loci, whether closed or not closed, divide $\mathbb{R}^2$ into two areas or **regions** of inequality. Regions above the locus satisfy a standard $\geq$ relation while regions below the locus satisfy a standard $\leq$ relation.

8 Exponential and Logarithmic Functions

8.1 Exponential functions

Exponential functions are mappings that involve exponentiations of expressions with independent variables with respect to constants. Such expressions are *transcendental* in nature; hence, they do not follow simple algebraic relations and operations as will be pointed out later. These functions are injective, *the domain and follow a one-to-one relation* Examples of these expressions are the following:

1. $f(x) = 2^x$
2. $g(k) = 4^{k+2}$
3. $h(x) = e^{3x^2+2}$

8.1.1 Expansion of exponents

Exponentiation can be defined as the recursive multiplication of *base* expressions due to multiplicities of an *exponent*. The expression x^3 is equivalent to $x \cdot x \cdot x$. Here, x is multiplied thrice to itself.

Multiple expressions can be made from these properties of exponents. The properties can be described as the laws of exponents as shown by the table below.

Table 4: Laws of Exponents

$x^0 = 1; \forall\ x \neq 0$	$x^1 = x$
$x^a \cdot x^b = x^{a+b}$	$\frac{x^a}{x^b} = x^{a-b}$
$x^-a = \frac{1}{x^a}$	$x^{\frac{1}{a}} = \sqrt[a]{x}$
$(x^a)^b = x^{ab}$	$a^x \cdot b^x = (ab)^x$

8.1.2 Exponential functions and their graphs

Injectivity As noted above, the exponential functions are *injective*. This means that there will only be *one* value in the range for every value in the domain. Generally, singular exponential functions are injective. Combinations or superpositions (from addition or subraction) can be *unpredictable*, but their behaviors can be shown by calculating the derivatives of the said functions.

Span Singular exponential expressions span half of $\mathbb{R}^2$. They have horizontal asymptotes, and depending on the sign of the coefficient is the direction of the graph. The horizontal asymptotes can be calculated by finding the limits at infinity ($-\infty$ or ∞). This means that the range of simple exponential functions are restricted to half of $\mathbb{R}$, while the domain spans $\mathbb{R}$ in general.

Concavity It can be noted that the derivatives of exponential functions are also exponential. The term exponential *growth* refers to the nature of the rate of change (derivative) is very similar to the expression of the represented function:

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx} - e^x = -e^x$$

If the coefficient is positive, as in $3^x = f(x)$, the graph is convex (concave up) and increasing towards $-\infty$ (shown by calculating the limits at∞). If the coefficient is negative, as in $-5^{x+2} = g(x)$, the graph is concave (concave down) and decreasing towards $-\infty$.

8.2 Logarithmic functions

Logarithmic functions are mappings that involve logarithmic expressions, which are defined as inverses of exponential expressions. The logarithm can be defined as:

$$\log_a b = c$$

where

$$a^c = b$$

Examples of logarithmic functions are:

1. $f(x) = \log_3 x$
2. $g(k) = \log_e(x + 2) = \ln(x + 2)$

Table 5: Laws on Logarithms

[h!] $\log_a 1 = 0 \forall\ a \neq 0$	$\log_a a = 1$
$\log_a x + \log_a y = \log_a(xy)$	$\log_a x - log_a y = log_a(\frac{x}{y})$
$-\log_a x = \log_a \frac{1}{x}$	$c \log_a b = \log_a(b^c)$
$\frac{log_a x}{log_a y} = \log_y x$	

3. $h(q) = \log_1 0x = logx$

Similar to exponential functions, they are also transcendental and follow the same rules as their inverses. The laws of exponents can be translated to the logarithmic counterparts as follows

8.2.1 Properties of logarithms

8.2.2 Logarithmic functions and their graphs

Logarithmic functions are, generally, the inverses of exponential functions. This means that the graphs of simple logarithmic functions are *reflections* of their respective inverses with respect to $f(x) = x$.

Injectivity Simple logarithmic functions are also injective. However, their superpositions are not linear (unpredictable). Derivatives are useful tools for the general description for function behavior.

Span They also span half of $\mathbb{R}^2$. Since they are diagonal reflections, the horizontal asymptotes of corresponding exponential functions will be numerically be equivalent to the vertical asymptotes. Similarly, the domain is restricted to half of $\mathbb{R}$, while the range spans $\mathbb{R}$.

Concavity The second derivative of a logarithmic expression is:

$$\frac{d^2}{dx^2} \log_a(x) = -\frac{1}{x^2 \cdot \ln a}$$

which is always *negative* $\forall\ x > 1$ but *positive* $\forall 0 < x < 1$.

Hence, the graph is *concave* $\forall\ x > 1$ ang *convex* $\forall\ 0 < x < 1$.

9 Trigonometric Functions

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10 Sequences of Numbers

10.1 Sequences and their sums

Sequences A sequence is a properly defined collection of symbols or numbers. Real sequences $\{a_n\}$ where $a_i \in \mathbb{R}$ can be defined in terms of the progression of the terms in them. Examples of real sequences are:

1. $\{a_n\} = \{23, 24, 25, 26, 27, 28\}; n = 6$

2. $\{g_n\} = \{-2, 4, -8, 16, -32, 64\}; n = 6$
3. $\{t_n\} = \{1, 1, 2, 3, 5, 8, 13, 21, 34\}; n = 9$
4. $\{f_n\} = \{1, 3, 6, 10, 15, 21, 28\}; n = 7$

Series A series is defined as the sum of terms of a given sequence.

Convergence and divergence of infinite series An infinite series is said to be *converging* if the limit to infinity of a sum can be defined. Otherwise, it is said to be divergent.

10.1.1 Arithemetic progressions and geometric progressions

Arithmetic sequences Arithmetic sequences are sequences of real numbers wherein $\exists\ d \in \mathbb{R} a_n = a_{n-1} + d$. This means that consecutive elements of the sequence are separated by a *common difference* d. The first example given above is an arithmetic sequence with $d = 1$ and $n = 6$. Arithmetic sequences can only be [strictly] *increasing* or *decreasing* depending on the sign of d. If $d > 0$, the sequence is increasing, else if $d < 0$, the sequence is decreasing.

Arithemetic series Generally the series sums are expressed as the following:

$$S_a(n) = \frac{a_1 + a_n}{2} \cdot n = \frac{2a_1 + d(n-1)}{2} \cdot n$$

where n is the number of terms, a_1 is the first term, a_n is the last term, and d is the common difference.

Divergence of series Generally, the series can be expressed in terms of the first term. Given the equation above, it can be seen that the sum linearly increases as the number of terms increases, *the limit cannot be defined*; hence, all *infinite arithmetic series are divergent.*

Arithmetic mean The arithmetic mean is the numerical *average* of the elements of a sequence. It, $mean(\{a_n\})$ can be expressed as:

$$mean(\{a_n\}) = \frac{S_a(n)}{n} = \frac{a_1 + a_n}{2} = \frac{2a_1 + d(n-1)}{2}$$

where the first part of the statement is the definition of the mean, the second and third parts came from the definition of the arithmetic series. From the above statements, it can be said that the arithmetic mean, more informally "the midle term", of an arithmetic sequence is equal to the arithmetic mean of the first and last terms.

Geometric sequences Geometric sequences are sequences of real numbers wherein $\exists d \in \mathbb{R} g_n = r \cdot g_{n-1}$. This means that consecutive elements of the sequence are separated by a *common ratio* r. The second example given above is an arithmetic sequence with $r = -2$ and $n = 6$. If $absr > 1$, the absolute value of every succeeding g_n is increasing, else if $absr < 1$, the absolute value of every succeeding g_n is decreasing.

Finite geometric series The finite geometric series can be expressed as follows:

$$S(n) = a_0 \cdot \frac{1 - r^n}{1 - r}$$

where a_0 is the first term of the series, r is the common ratio, and n is the number of terms.

Infinite geometric series The series sum can expressed as follows:

$$S_g(n) = a_0 \cdot \frac{1}{1-r} = \lim_{n\to\infty} a_0 \cdot \frac{1-r^n}{1-r}$$

where the variable definitions still hold. Take note that the limit can only be evaluated when $absr < 1$ (*convergent*). Otherwise, the limit cannot be evaluated, DNE, when $absr > 1$ (*divergent*).

Geometric mean The geometric mean of a sequence is defined as:

$$gmean(\{g_n\}) = \sqrt[n]{S_g(n)}$$

Similar with the arithmetic mean, the geometric mean, also informally "the middle term" can also be expressed as the geometric mean of the first and last terms, or any two terms of the same "distance" from the middle term.

10.1.2 Various sequences

There are also many ways to define sequences. Presented in this subsection are common sequences and series.

Fibonacci sequence One of the more popular sequences known is the Fibonacci sequence. It is a basic sequence defined recursively from the sum of two previous terms. The third example at the beginning of this section is an example of a Fibonacci sequence starting from $a_0 = 1, a_1 = 1$ and $n = 9$.

Triangular number sequence The triangular number sequence $\{t_n\}$, is a sequence of numbers wherein each term $\{t_k\}, k \in \{1, 2, 3, \ldots, n\}$ is the sum of the arithmetic series from 1 to n. The fourth example at the beginning of this section is an example of a triangular number sequence of length 7.

Telescoping series A telescoping series is a series wherein the partial sums of the expression yield *fewer* terms due to cancellation of *equivalent* addends. An example of such a series is:

$$\sum_{n=1}^{N} \frac{1}{n^2+n} = 1 - \frac{1}{n}$$

10.2 Recurrence formulae and mathematical induction

Sequences can also be defined, like the Fibonacci sequence, recursively through **recursive formulae**. Recursion is defined as an operation in repetition with respect to previously calculated elements.

Recurrence formulae Recurrence formula are often useful when in need to define successive terms in a sequence without the need of a general expression. The recursive formula for the Fibonacci sequence is $f_n = f_{n-1} + f_{n-2}$. The recursive formula for an arithmetic sequence is $a_n = a_{n-1} + d$. The recursive formula for a geometric sequence is $g_n = r \cdot g_{n-1}$. The recursive formula for a triangular number series is $t_n = t_{n-1} + n$. Usually, simple sequences usually have simpler recursive formulae compared to their general counterparts. As a consequence, they are *simpler* to evaluate *in succession*.

Principle of mathematical induction The principle of mathematical induction is a method of proof that relies on assumptions on the recursive properties of expressions that can apply to all integers. In this paragraph, we will attempt to prove $\forall\, n \in \mathbb{N}$ that $f(n) = n^3 - n$ is divisible by 3 using mathematical induction. There are four basic steps in proving:

Proof on a trivial cases Show that the statement works on easy cases. For $n = 1$, $f(1) = 0 \equiv 0(\text{mod}3)$, and $n = 2$, $f(2) = 6 \equiv 0(\text{mod}3)$. This step ensures that there are cases wherein the statement holds.

Assumption on $k \in \mathbb{N}$ Assume that the statement holds for some $n \in \mathbb{N}$. For some $n = k$, $f(k) = k^3 - k \equiv 0(mod3)$. Adding $3k^2 + 3k$ to both sides gives:

$$\begin{aligned} k^3 + 3k^2 + 3k - k &\equiv 3k^2 + 3k(mod3) \equiv 0(mod3) \\ k^3 + 3k^2 + 3k + 1 - k - 1 &\equiv 0(mod3) \\ (k+1)^3 - (k+1) = f(k+1) &\equiv 0(mod3) \end{aligned}$$

Conclusion Hence, we have shown that if the statement holds for $n = k \in \mathbb{N}$, then it must hold for $n = k + 1$. Since $\mathbb{N}$ is closed under addition, the statement must hold $\forall\ n \in \mathbb{N}$.

11 Vectors

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12 Complex Plane

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13 Curves on a Plane

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14 Limits

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15 Differential Calculus

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16 Integral Calculus

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