TUTORIAL AND EXAM QUESTIONS FOR ECON 321

Ben Davies

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Tutorial Problems

1.1. Find the first and second derivatives of the following functions.

a) $a(x) = \ln(x^2)$.
b) $b(x) = x \ln(x)$.
c) $c(x) = (\ln(x))^2$.

1.2. Let $f(x)$ be an arbitrary function with $x \in \mathbb{R}$. Find the first and second derivatives of the following functions.

a) $a(x) = f(x^2)$.
b) $b(x) = (f(x))^2$.
c) $c(x) = f(\sqrt{x})$.
d) $d(x) = \sqrt{f(x)}$.

2.1. Recall that a function $f(x)$ has a *stationary* point at x^* if $f'(x^*) = 0$. Find the stationary points of the following functions.

a) $a(x) = (x-1)^2$.
b) $b(x) = x^2/(1-x^2)$.
c) $c(x) = x(x^2-3)$.

2.2. Let $f(x)$ and $g(x)$ be arbitrary functions with $x \in \mathbb{R}$, and let

$$h(x) = \frac{f(x)}{g(x)}.$$

Apply the product rule to $f(x) = g(x)h(x)$ to show that

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

2.3. Sketch each of the following sets as a shaded region in $\mathbb{R}^2$. State which of these sets are convex.

a) $A = \{x \in \mathbb{R}^2 : x_1 + 2x_2 \leq 1\}$.
b) $B = \{x \in \mathbb{R}^2 : x_1^2 \leq x_2\}$.
c) $C = \{x \in \mathbb{R}^2 : x_1^2 \geq x_2\}$.
d) $D = \{x \in \mathbb{R}^2 : x_1 x_2 \leq 1\}$.
e) $E = \{x \in \mathbb{R}^2 : x_1 x_2 \geq 1\}$.

3.1. a) Show that the unit disk

$$X = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

is a convex set.

b) Show that the intersection of two convex sets is also a convex set.

4.1. Suppose that a consumer has utility function

$$u(x) = \ln(x_1 x_2),$$

where $x_i \geq 1$ is the quantity demanded for each good $i \in \{1, 2\}$.

a) Use Jensen's inequality to show that $u(x)$ is concave.

b) Show that $u(x)$ has convex contours.

c) Find an expression for the consumer's marginal rate of substitution as a function of x_1 and x_2.

d) Show that the consumer optimally spends the same amount of wealth on each good.

4.2. Suppose that a consumer has utility function

$$u(x) = x_1^\alpha x_2^\beta,$$

where $x_i > 0$ is the quantity demanded for each good $i \in \{1, 2\}$, and α and β are positive constants.

a) Find an expression for the consumer's marginal rate of substitution as a function of x_1 and x_2.

b) Show that the optimal expenditure ratio

$$\frac{p_1 x_1^*}{p_2 x_2^*} = \frac{\alpha}{\beta}. \tag{1}$$

c) Use (1) and the budget constraint

$$p_1 x_1^* + p_2 x_2^* = w$$

to show that

$$\frac{\partial x_1^*}{\partial \alpha} = \frac{p_2 x_2^*}{(\alpha + \beta) p_1}.$$

5.1. Suppose that market demand for a good is given by the linear function

$$x(p) = b - ap,$$

where $p > 0$ is the per-unit price of the good and $a, b > 0$ are constants. Assume that the market is a monopoly and that the firm produces each unit at a constant cost $c > 0$.

a) Write an expression for the firm's profit function $\pi(p)$ in terms of the price p, and the parameters a, b and c.

b) Show that the profit function $\pi(p)$ is concave.

c) Derive the profit-maximising price p^*.

d) Find the firm's maximal level of profit $\pi(p^*)$.

5.2. Suppose that a tax is imposed on a perfectly competitive market. The demand curve is defined by

$$p = a - bx,$$

where $p > 0$ is the market price, x is the quantity demanded and $a, b > 0$ are constants. The supply curve is defined by

$$p = c + dx + t,$$

where $t \geq 0$ is the amount of the tax and $c, d > 0$ are constants.

a) Find the equilibrium quantity x^* and price p^*.

b) Write an expression for the amount of tax revenue $r(t)$ collected in terms of the choice variable t, and the parameters a, b, c and d.

c) Solve the constrained maximisation problem

$$\max_t r(t) \text{ subject to } t \geq 0 \tag{2}$$

for the revenue-maximising amount of tax t^*.

d) The solution to (2) is strictly positive if and only if $a > c$. Is this condition reasonable?

5.3. Let X be a convex set and consider the constrained maximisation problem

$$\max_x f(x) \text{ subject to } x \in X,$$

where $f(x)$ is increasing and strictly concave. Use Jensen's inequality to show that the optimal solution to this problem is unique.

6.1. Consider the constrained maximisation problem

$$\max_x f(x) = x(x^2 - 3) \text{ s.t. } x \leq 1. \tag{3}$$

a) Write down the Lagrangian for this problem.

b) Derive the first-order and complementary slackness conditions for an optimal solution to (3).

c) Find the optimal solution x^* to (3).

6.2. Consider the constrained maximisation problem

$$\max_x f(x) = \ln(x_1 x_2) \text{ s.t. } x_1 + x_2 \leq 1 \text{ and } x_1 + 2x_2 \geq 1. \tag{4}$$

a) Write down the feasible set for this problem. Is this set convex?

b) Write down the Lagrangian for this problem.

c) Derive the first-order and complementary slackness conditions for an optimal solution to (4).

d) Find the optimal solution to (4).

6.3. Consider the constrained *minimisation* problem

$$\min_x f(x) = (x_1 - 1)^2 + (x_2 - 2)^2 \text{ s.t. } x_1 + x_2 \leq 1. \tag{5}$$

a) Write down the Lagrangian for this problem.

b) Derive the first-order and complementary slackness conditions for an optimal solution to (5).

c) Find the optimal solution x^* to (5).

d) Interpret the Lagrange multiplier for the contraint $x_1 + x_2 \leq 1$.

7.1. Suppose that a consumer solves

$$\max_x u(x) = \sqrt{x_1 x_2} \text{ subject to } x_1 + 2x_2 \leq w,$$

where $w > 0$ is the consumer's disposable income and $x_i \geq 0$ is quantity demanded of each good $i \in \{1, 2\}$.

a) Write down the Lagrangian for the consumer's problem. Ignore any nonnegativity constraints.

b) Derive the first-order and complementary slackness conditions for an optimal solution to the consumer's problem.

c) Find the optimal solution to the consumer's problem. Interpret the value of the Lagrange multiplier for the budget constraint.

d) Show that the consumer's indirect utility function is linear in w.

e) Suppose that the budget constraint changes to $x_1 + 3x_2 \leq w$. Without any calculations, explain what will happen to x_1^*.

7.2. Consider the constrained minimisation problem

$$\min_x v_1 x^2 + v_2(1-x)^2 + 2cx(1-x) \text{ subject to } r_1 x + r_2(1-x) \geq R,$$

where v_1, v_2, c, r_1, r_2 and R are positive constants, and $v_1 + v_2 > 2c$.

a) Write down the Lagrangian for this problem.

b) Show that the Lagrangian from part a) is maximised by

$$x^* = \frac{\delta(r_1 - r_2)}{2(v_1 + v_2 - 2c)} + \frac{v_2 - c}{v_1 + v_2 - 2c},$$

where δ is the Lagrange multiplier for the inequality constraint.

c) Interpret the Lagrange multiplier δ from part b).

d) Show that if $r_1 = r_2$ then $x^* > 0$ if and only if $v_2 > c$.

9.1. A consumer solves

$$\max_x u(x) \text{ subject to } px \leq w,$$

where x denotes the demand for a single good with price $p > 0$ and $w > 0$ denotes the consumer's wealth. Assume that the utility function $u(x)$ is strictly increasing and concave in x.

a) Show that the optimal solution x^* satisfies

$$u'(x^*) = \delta p,$$

where $\delta \geq 0$ is the Lagrange multiplier for the budget constraint.

b) Show that x^* is (i) increasing in w and (ii) decreasing in p.

c) Let $v(p, w) = u(x^*)$ denote the consumer's indirect utility function. Show that $v(p, w)$ is (i) increasing in w and (ii) decreasing in p.

9.2. A consumer with initial wealth w_0 and utility function $u(w) = \ln(w)$ is exposed to a risk that will decrease his wealth by an amount $x \in (0, w_0)$ with probability p or increase his wealth by x with probability $(1 - p)$.

a) Write down an expression for the consumer's expected utility.

b) Suppose that ϕ satisfies the indifference condition

$$\mathrm{E}[u(w)] = u(w_0 - \phi).$$

Use Jensen's inequality to show that $\phi > (2p - 1)x$.

c) Let $p = 0.5$. Show graphically that $\phi < x$.

Hint: sketch the consumer's utility function $u(w) = \ln(w)$ in $\mathbb{R}^2$ and label the points $(w_0 - x, u(w_0 - x))$, $(w_0, u(w_0))$, etc., and compare the positions of $w_0 - x$ and $w_0 - \phi$.

10.1. Suppose that a consumer with income y suffers a loss of size $L < y$ with probability $p \in (0, 1)$. The consumer can buy $c \in [0, L]$ units of insurance coverage at the per-unit price π and solves

$$\max_c \mathrm{E}[u(w)] = pu(y - \pi c - L + c) + (1 - p)u(y - \pi c),$$

where the utility function $u(w)$ is strictly increasing and concave in w.

a) Derive the first-order condition for the optimal level of coverage c^*. Show that the second-order condition holds.

b) Show that c^* is increasing in p.

c) Show that c^* is increasing in y if and only if the Arrow-Pratt measure of absolute risk aversion

$$A(w) = -\frac{u''(w)}{u'(w)}$$

is increasing in w.

d) Let π_L be the premium rate at which $c^* = L$ and π_0 the premium rate at which $c^* = 0$. Show that $\pi_L < \pi_0$.

e) Now assume that $u(w) = \ln(w)$. Find an expression for c^* in terms of the parameters y, π, p and L.

10.2. Suppose that a consumer has initial wealth w_0 and utility function

$$u(w) = 1 - \exp(-w).$$

Suppose also that the consumer suffers a loss of size l with probability p or no loss with probability $(1 - p)$, where $p \in (0, 1)$.

a) Write an expression for the consumer's expected utility $\mathrm{E}[u(w)]$ in terms of the parameters w_0, l and p.

b) Suppose that ϕ that satisfies the indifference condition

$$\mathrm{E}[u(w)] = u(w_0 - \phi).$$

Find an expression for ϕ in terms of the parameters w_0, l and p.

c) Explain why ϕ from part b) is positive and independent of w_0.

11.1. An investor with initial wealth w_0 is considering putting money into an investment that earns the interest rate r_1 with probability p and r_2 with probability $(1-p)$, where $r_1 < 0 < r_2$. The investor has final wealth

$$\tilde{w} = w_0 + \tilde{r}x,$$

where $\tilde{r}$ denotes the random interest rate and x his allocation to the risky investment. The investor solves

$$\max_x f(x) = \mathrm{E}[u(\tilde{w})],$$

where his utility function $u(w)$ is strictly increasing and strictly concave in w, and E is the expectation operator.

a) Write down an expression for $\mathrm{E}[u(\tilde{w})]$ in terms of the choice variable x, and the parameters w_0, r_1, r_2 and p.

b) Find the first-order conditions for the optimal choice of investment x^* Show that x^* satisfies the second-order condition for a maximum.

c) Show that $x^* > 0$ if and only if the expected interest rate $\mathrm{E}[\tilde{r}] > 0$.

d) Show that the optimal choice of investment x^* is increasing in w_0 if and only if

$$A(w_0 + r_1 x^*) > A(w_0 + r_2 x^*),$$

where $A(w) \equiv -u''(w)/u'(w)$ is the Arrow-Pratt measure of absolute risk aversion.

11.2. Consider an investor with initial wealth w_0 and utility function

$$u(w) = \frac{w^{1-\gamma} - 1}{1-\gamma},$$

where $\gamma > 0$.[1] Show that the investor's degree of absolute risk aversion is decreasing in his initial wealth.

11.3. Consider an investor with initial wealth w_0 and utility function

$$u(w) = aw - bw^2,$$

where $a, b > 0$. Show that the investor's degree of absolute risk aversion is increasing in his initial wealth.

[1] One can use L'Hôpital's rule to show that $u(w) \to \ln(w)$ as $\gamma \to 1$.

Exam Questions

E.1. Suppose that a consumer has utility function $u(x) = a\ln(x_1) + b\ln(x_2)$ and solves

$$\max_x u(x) \text{ subject to } p_1x_1 + p_2x_2 \leq w, \tag{6}$$

where x_i and $p_i > 0$ respectively denote the quantity demanded and price of good $i \in \{1,2\}$, $w > 0$ denotes his wealth, and a and b are positive constants. Assume that the optimal demands $x_1^* > 0$ and $x_2^* > 0$.

a) Write down the Lagrangian for the consumer's problem. Derive the first-order and complementary slackness conditions for the optimal demands x_1^* and x_2^*.

b) Show that the optimal expenditure ratio

$$\frac{p_1x_1^*}{p_2x_2^*} = \frac{a}{b}.$$

c) Show that the consumer's budget constraint is binding.

d) Show that x_1^* is decreasing in p_1.

e) Use Jensen's inequality to show that $u(x)$ is concave in x.

f) Suppose that a second consumer has utility function $u(x) = x_1^a x_2^b$ and solves (6) with the same parameters. Explain why both consumers will have the same demands for goods 1 and 2.

E.2. Suppose that a consumer receives income y at each date $t \in \{0,1\}$. At date 1, the consumer incurs a loss of size $L > 0$ with probability $p > 0$. At date 0, he buys an insurance contract that provides c units of coverage at date 1 if the loss occurs. The consumer pays π per contracted unit of coverage and solves

$$\max_c v(c) = u(y - \pi c) + \beta[pu(y - L + c) + (1-p)u(y)],$$

where $u(w)$ is strictly increasing and strictly concave in w, and $\beta \in (0,1)$ is his intertemporal discount factor.

a) Derive the first-order condition for the optimal level of coverage c^*. Show that the second-order condition holds.

b) Show that c^* is increasing in p.

c) Show that c^* is increasing in y if and only if

$$A(y - L + c) < A(y - \pi c),$$

where $A(w) = -u''(w)/u'(w)$ is the Arrow-Pratt measure of absolute risk aversion.

d) Let π_L denote the per-unit price of coverage at which $c^* = L$. Show that π_L is increasing in β.

e) Assume that $u(w) = \ln(w)$. Find an expression for c^* in terms of the parameters y, π, β, p and L.