TUTORIAL AND EXAM QUESTIONS FOR ECON 321

SUGGESTED SOLUTIONS

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Tutorial Problems

1.1. Find the first and second derivatives of the following functions.

a) $a(x) = \ln(x^2)$.
By the chain rule, the first derivative

$$\begin{aligned} a'(x) &= \frac{2x}{x^2} \\ &= \frac{2}{x} \end{aligned}$$

and the second derivative

$$a''(x) = -\frac{2}{x^2}.$$

b) $b(x) = x\ln(x)$.
By the product rule, the first derivative

$$\begin{aligned} b'(x) &= \ln(x) + \frac{x}{x} \\ &= \ln(x) + 1 \end{aligned}$$

and the second derivative

$$b''(x) = \frac{1}{x}.$$

c) $c(x) = (\ln(x))^2$.
By the chain rule, the first derivative

$$c'(x) = \frac{2\ln(x)}{x}.$$

By the quotient rule, the second derivative

$$\begin{aligned} c''(x) &= \frac{(2/x)x - 2\ln(x)}{x^2} \\ &= \frac{2 - 2\ln(x)}{x^2}. \end{aligned}$$

1.2. Let $f(x)$ be an arbitrary function with $x \in \mathbb{R}$. Find the first and second derivatives of the following functions.

a) $a(x) = f(x^2)$.
By the chain rule, the first derivative

$$a'(x) = 2xf'(x^2).$$

By the product and chain rules, the second derivative

$$a''(x) = 2f'(x^2) + 4x^2 f''(x^2).$$

b) $b(x) = (f(x))^2$.
By the chain rule, the first derivative

$$b'(x) = 2f'(x)f(x).$$

By the product rule, the second derivative

$$b''(x) = 2f''(x)f(x) + 2(f'(x))^2.$$

c) $c(x) = f(\sqrt{x})$.
By the chain rule, the first derivative

$$c'(x) = \frac{f'(\sqrt{x})}{2\sqrt{x}}.$$

By the quotient rule, the second derivative

$$\begin{aligned} c''(x) &= \frac{(f''(\sqrt{x})/2\sqrt{x})2\sqrt{x} - f'(\sqrt{x})/\sqrt{x}}{4x} \\ &= \frac{f''(\sqrt{x})\sqrt{x} - f'(\sqrt{x})}{4x^{3/2}}. \end{aligned}$$

d) $d(x) = \sqrt{f(x)}$.
By the chain rule, the first derivative

$$d'(x) = \frac{f'(x)}{2\sqrt{f(x)}}.$$

By the quotient rule, the second derivative

$$\begin{aligned} d''(x) &= \frac{2f''(x)\sqrt{f(x)} - 2f'(x)(f'(x)/2\sqrt{f(x)})}{4f(x)} \\ &= \frac{2f(x)f''(x) - (f'(x))^2}{4(f(x))^{3/2}}. \end{aligned}$$

2.1. Recall that a function $f(x)$ has a *stationary* point at x^* if $f'(x^*) = 0$. Find the stationary points of the following functions.

a) $a(x) = (x-1)^2$.
Differentiating $a(x)$ with respect to x gives

$$a'(x) = 2(x-1).$$

The equation $a'(x^*) = 0$ has a unique solution $x^* = 1$. Hence $a(x)$ has a stationary point at $x^* = 1$.

b) $b(x) = x^2/(1-x^2)$.
Differentiating $b(x)$ with respect to x gives

$$b'(x) = \frac{2x}{(1-x^2)^2}.$$

The equation $b'(x^*) = 0$ has unique solution $x^* = 0$. Hence $b(x)$ has a stationary point at $x^* = 0$.

c) $c(x) = x(x^2 - 3)$.
Differeniating $c(x)$ with respect to x gives

$$\begin{aligned} c'(x) &= x^2 - 3 + x(2x) \\ &= 3x^2 - 3. \end{aligned}$$

The equation $c'(x^*) = 0$ has a two solutions $x^* = \pm 1$. Hence $c(x)$ has stationary points at $x^* = \pm 1$.

2.2. Let $f(x)$ and $g(x)$ be arbitrary functions with $x \in \mathbb{R}$, and let

$$h(x) = \frac{f(x)}{g(x)}.$$

Apply the product rule to $f(x) = g(x)h(x)$ to show that

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

Applying the product rule to $f(x) = g(x)h(x)$ gives

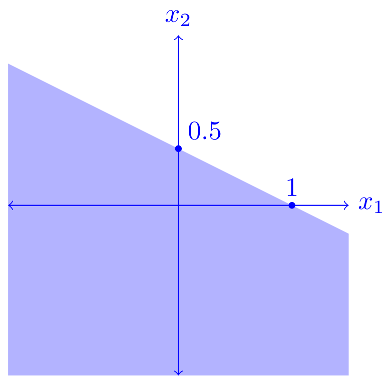
$$f'(x) = g'(x)h(x) + g(x)h'(x),$$

which we can rearrange to obtain

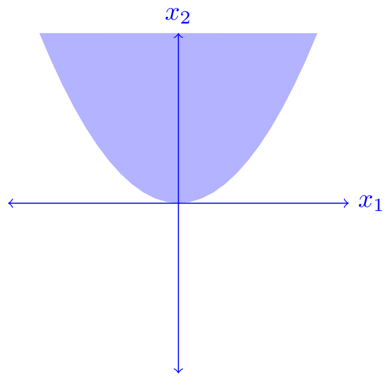
$$\begin{aligned} h'(x) &= \frac{f'(x) - g'(x)h(x)}{g(x)} \\ &= \frac{f'(x) - g'(x)f(x)/g(x)}{g(x)} \times \frac{g(x)}{g(x)} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}. \end{aligned}$$

2.3. Sketch each of the following sets as a shaded region in $\mathbb{R}^2$. State which of these sets are convex.

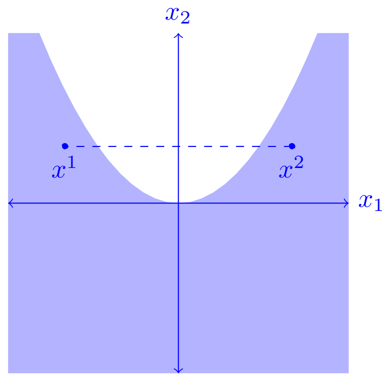
a) $A = \{x \in \mathbb{R}^2 : x_1 + 2x_2 \leq 1\}$.
The set A is convex and is shown below.



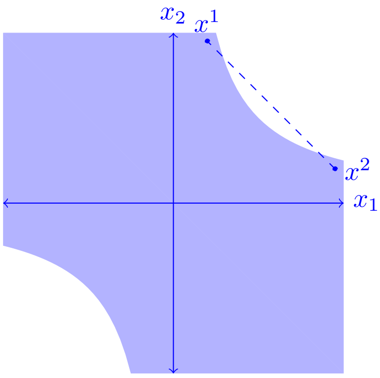
b) $B = \{x \in \mathbb{R}^2 : x_1^2 \leq x_2\}$.
The set B is convex and is shown below.



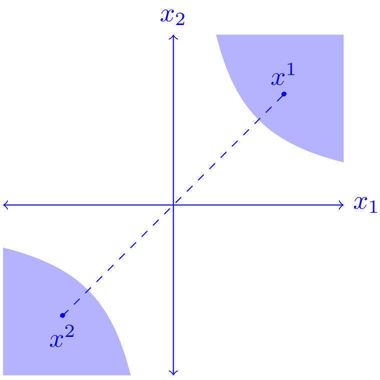
c) $C = \{x \in \mathbb{R}^2 : x_1^2 \geq x_2\}$.
The set C is shown below. It is *not* convex because the line segment connecting the points $x^1, x^2 \in C$ contains points that are not in C.



d) $D = \{x \in \mathbb{R}^2 : x_1 x_2 \leq 1\}$.
The set D is shown below. It is *not* convex because the line segment connecting the points $x^1, x^2 \in D$ contains points that are not in D.



e) $E = \{x \in \mathbb{R}^2 : x_1 x_2 \geq 1\}$.
The set E is shown below. It is *not* convex because the line segment connecting the points $x^1, x^2 \in E$ contains points that are not in E.



3.1. a) Show that the unit disk

$$X = \{x \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$

is a convex set.
First notice that the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(t^2) = 2$$

is positive for all $t \in \mathbb{R}$. Hence the square function is convex and therefore

$$(\lambda t_1 + (1-\lambda)t_2)^2 \leq \lambda t_1^2 + (1-\lambda)t_2^2$$

for all $t_1, t_2 \in \mathbb{R}$ and whenever $0 \leq \lambda \leq 1$ by Jensen's inequality. Now, let $x^1, x^2 \in X$ be arbitrary and define

$$x^3 = \lambda x^1 + (1-\lambda)x^2$$

for some $\lambda \in [0,1]$. Then

$$\begin{aligned}
(x_1^3)^2 + (x_2^3)^2 &= (\lambda x_1^1 + (1-\lambda)x_1^2)^2 + (\lambda x_2^1 + (1-\lambda)x_2^2)^2 \\
&\leq \lambda(x_1^1)^2 + (1-\lambda)(x_1^2)^2 + \lambda(x_2^1)^2 + (1-\lambda)(x_2^2)^2 \\
&< \lambda + (1-\lambda) \\
&= 1
\end{aligned}$$

so that $x^3 \in X$. It follows that X is a convex set.

b) Show that the intersection of two convex sets is also a convex set.

Let X_1 and X_2 be convex sets with intersection

$$X_3 = X_1 \cap X_2.$$

Let also $x^1, x^2 \in X_3$ be arbitrary and define

$$x^3 = \lambda x^1 + (1-\lambda)x^2$$

where $0 \leq \lambda \leq 1$. Then $x^3 \in X_1$ because $x^1, x^2 \in X_1$ and X_1 is a convex set. Similarly, we have $x^3 \in X_2$ because $x^1, x^2 \in X_1$ and X_2 is a convex set. Hence $x^3 \in X_3$ and therefore X_3 is a convex set. It follows that the intersection of two convex sets is also a convex set.

4.1. Suppose that a consumer has utility function

$$u(x) = \ln(x_1 x_2),$$

where $x_i \geq 1$ is the quantity demanded for each good $i \in \{1, 2\}$.

a) Use Jensen's inequality to show that $u(x)$ is concave.

First notice that the second derivative

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln(t) = -\frac{1}{t^2}$$

is negative whenever $t \geq 1$. Hence the natural logarithm is concave and therefore

$$\ln(\lambda t_1 + (1-\lambda)t_2) \geq \lambda \ln(t_1) + (1-\lambda)\ln(t_2)$$

for all $\lambda \in [0,1]$ and $t_1, t_2 \geq 1$ by Jensen's inequality. Now, define the set

$$X = \{x \in \mathbb{R}^2 : x_1 \geq 1 \text{ and } x_2 \geq 1\}$$

and let $x^1, x^2 \in X$ be arbitrary. Then for all $\lambda \in [0,1]$ we have

$$\begin{aligned}
u(\lambda x^1 + (1-\lambda)x^2) &= \ln((\lambda x_1^1 + (1-\lambda)x_1^2)(\lambda x_2^1 + (1-\lambda)x_2^2)) \\
&= \ln(\lambda x_1^1 + (1-\lambda)x_1^2) \\
&\quad + \ln(\lambda x_2^1 + (1-\lambda)x_2^2) \\
&\geq \lambda \ln(x_1^1) + (1-\lambda)\ln(x_1^2) \\
&\quad + \lambda \ln(x_2^1) + (1-\lambda)\ln(x_2^2) \\
&= \lambda \ln(x_1^1 x_2^1) + (1-\lambda)\ln(x_2^1 x_2^2) \\
&= \lambda u(x^1) + (1-\lambda)u(x^2).
\end{aligned}$$

It follows that $u(x)$ is concave by Jensen's inequality.

b) Show that $u(x)$ has convex contours.

Let $c \geq 0$ be arbitrary and consider the contour defined by $u(x) = c$. We can write

$$\begin{aligned}
\exp(c) &= \exp(\ln(x_1 x_2)) \\
&= x_1 x_2
\end{aligned}$$

because the exponential and natural logarithm are inverse functions. Dividing both sides by x_1 gives

$$x_2 = f(x_1),$$

where the function

$$f(x_1) = \frac{c}{x_1}.$$

Then the second derivative

$$f''(x_1) = \frac{2c}{x_1^3}$$

is positive because $x_1 \geq 1$ and $c \geq 0$. It follows that $u(x)$ has convex contours.

c) Find an expression for the consumer's marginal rate of substitution as a function of x_1 and x_2.

By the implicit function theorem, we have

$$\begin{aligned} \left.\frac{\mathrm{d}x_2}{\mathrm{d}x_2}\right|_{\mathrm{d}u=0} &= -\frac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} \\ &= -\frac{1/x_1}{1/x_2} \\ &= -\frac{x_2}{x_1} \end{aligned}$$

so that the consumer's marginal rate of substitution

$$\mathrm{MRS}(x) = -\frac{x_2}{x_1}.$$

d) Show that the consumer optimally spends the same amount of wealth on each good.

The optimal demand x_i^* for each good $i \in \{1, 2\}$ satisfies the tangency condition

$$\begin{aligned} -\frac{p_1}{p_2} &= \mathrm{MRS}(x^*) \\ &= -\frac{x_2^*}{x_1^*}, \end{aligned}$$

which implies that $p_1 x_1^* = p_2 x_2^*$. Hence the consumer optimally spends the same amount of wealth on each good.

4.2. Suppose that a consumer has utility function

$$u(x) = x_1^\alpha x_2^\beta,$$

where $x_i > 0$ is the quantity demanded for each good $i \in \{1, 2\}$, and α and β are positive constants.

a) Find an expression for the consumer's marginal rate of substitution as a function of x_1 and x_2.
The partial derivatives

$$\frac{\partial u(x)}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta}$$

and

$$\frac{\partial u(x)}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1}.$$

By the implicit function theorem, we have

$$\begin{aligned}\left.\frac{\mathrm{d}x_2}{\mathrm{d}x_1}\right|_{\mathrm{d}u=0} &= -\frac{\partial u(x)/\partial x_1}{\partial u(x)/\partial x_2} \\ &= -\frac{\alpha x_1^{\alpha-1} x_2^{\beta}}{\beta x_1^{\alpha} x_2^{\beta-1}} \\ &= -\frac{\alpha x_2}{\beta x_1}\end{aligned}$$

so that the consumer's marginal rate of substitution

$$\mathrm{MRS}(x) = -\frac{\alpha x_2}{\beta x_1}.$$

b) Show that the optimal expenditure ratio

$$\frac{p_1 x_1^*}{p_2 x_2^*} = \frac{\alpha}{\beta}. \tag{1}$$

The optimal demand x_i^* for each good $i \in \{1, 2\}$ satisfies the tangency condition

$$\begin{aligned}-\frac{p_1}{p_2} &= \mathrm{MRS}(x^*) \\ &= -\frac{\alpha x_2^*}{\beta x_1^*}\end{aligned}$$

which can be rewritten as

$$\frac{p_1 x_1^*}{p_2 x_2^*} = \frac{\alpha}{\beta}.$$

c) Use (1) and the budget constraint

$$p_1 x_1^* + p_2 x_2^* = w$$

to show that

$$\frac{\partial x_1^*}{\partial \alpha} = \frac{p_2 x_2^*}{(\alpha + \beta) p_1}.$$

First write the tangency condition as

$$\beta p_1 x_1^* = \alpha p_2 x_2^*.$$

Differentiating both sides with respect to α gives

$$\beta p_1 \frac{\partial x_1^*}{\partial \alpha} = p_2 x_2^* + \alpha p_2 \frac{\partial x_2^*}{\partial \alpha} \tag{2}$$

by the product rule. We can also differentiate the budget constraint with respect to α to obtain

$$p_1 \frac{\partial x_1^*}{\partial \alpha} + p_2 \frac{\partial x_2^*}{\partial \alpha} = 0$$

so that

$$\frac{\partial x_2^*}{\partial \alpha} = -\frac{p_1}{p_2} \frac{\partial x_1^*}{\partial \alpha}.$$

Substituting this latter expression into (2) gives

$$\beta p_1 \frac{\partial x_1^*}{\partial \alpha} = p_2 x_2^* - \alpha p_1 \frac{\partial x_1^*}{\partial \alpha},$$

which we can rearrange to obtain

$$\frac{\partial x_1^*}{\partial \alpha} = \frac{p_2 x_2^*}{(\alpha + \beta) p_1}.$$

5.1. Suppose that market demand for a good is given by the linear function

$$x(p) = b - ap,$$

where $p > 0$ is the per-unit price of the good and $a, b > 0$ are constants. Assume that the market is a monopoly and that the firm produces each unit at a constant cost $c > 0$.

a) Write an expression for the firm's profit function $\pi(p)$ in terms of the price p, and the parameters a, b and c.
The firm's profit function is given by

$$\begin{aligned} \pi(p) &= px(p) - cx(p) \\ &= p(b - ap) - c(b - ap) \\ &= -ap^2 + (b + ac)p - bc, \end{aligned}$$

where a, b and c are treated as parameters.

b) Show that the profit function $\pi(p)$ is concave.
The second derivative

$$\pi''(p) = -2a$$

is negative because $a > 0$. It follows that $\pi(p)$ is concave.

c) Derive the profit-maximising price p^*.
The profit-maximising price p^* satisfies the first-order condition

$$\begin{aligned} 0 &= \pi'(p^*) \\ &= -2ap^* + b + ac \end{aligned}$$

so that

$$p^* = \frac{b + ac}{2a}.$$

The second-order condition is satisfied because $\pi(p)$ is concave.

d) Find the firm's maximal level of profit $\pi(p^*)$.
The firm's maximal level of profit

$$\begin{aligned}
\pi(p^*) &= -a(p^*)^2 + (b+ac)p^* - bc \\
&= -a\left(\frac{b+ac}{2a}\right)^2 + (b+ac)\left(\frac{b+ac}{2a}\right) - bc \\
&= \frac{-(b+ac)^2 + 2(b+ac)^2 - 4abc}{4a} \\
&= \frac{(b-ac)^2}{4a}.
\end{aligned}$$

5.2. Suppose that a tax is imposed on a perfectly competitive market. The demand curve is defined by

$$p = a - bx,$$

where $p > 0$ is the market price, x is the quantity demanded and $a, b > 0$ are constants. The supply curve is defined by

$$p = c + dx + t,$$

where $t \geq 0$ is the amount of the tax and $c, d > 0$ are constants.

a) Find the equilibrium quantity x^* and price p^*.
The equilibrium price quantity x^* satisfies

$$a - bx^* = c + dx^* + t,$$

which has unique solution

$$x^* = \frac{a-c-t}{b+d}.$$

Hence the equilibrium price

$$\begin{aligned}
p^* &= a - bx^* \\
&= a - b\left(\frac{a-c-t}{b+d}\right) \\
&= \frac{ad + b(c+t)}{b+d}.
\end{aligned}$$

b) Write an expression for the amount of tax revenue $r(t)$ collected in terms of the choice variable t, and the parameters a, b, c and d.
We have

$$\begin{aligned}
r(t) &= tx^* \\
&= \frac{(a-c)t - t^2}{b+d}.
\end{aligned}$$

c) Solve the constrained maximisation problem

$$\max_t r(t) \text{ subject to } t \geq 0 \tag{3}$$

for the revenue-maximising amount of tax t^*.

Assume that $t^* > 0$. The second derivative

$$r''(t) = -\frac{2}{b+d}$$

is strictly negative and therefore $r(t)$ is concave. Hence the revenue-maximising tax t^* satisfies the first-order condition

$$\begin{aligned} 0 &= r'(t^*) \\ &= \frac{(a-c) - 2t^*}{b+d} \end{aligned}$$

so that

$$t^* = \frac{a-c}{2}.$$

Now $r(0) = 0$ so t^* is optimal if and only if $a - c$.

d) The solution to (3) is strictly positive if and only if $a > c$. Is this condition reasonable?

Yes; the condition $a > c$ is a necessary condition for x^* to be positive.

5.3. Let X be a convex set and consider the constrained maximisation problem

$$\max_x f(x) \text{ subject to } x \in X,$$

where $f(x)$ is increasing and strictly concave. Use Jensen's inequality to show that the optimal solution to this problem is unique.

Assume that $x^1, x^2 \in X$ are distinct optimal solutions. Then

$$\lambda x^1 + (1-\lambda)x^2 \in X$$

for all $\lambda \in [0, 1]$ because X is a convex set. But $f(x)$ is strictly concave and therefore

$$\begin{aligned} f(\lambda x^1 + (1-\lambda)x^2) &> \lambda f(x^1) + (1-\lambda)f(x^2) \\ &\geq \min\{f(x^1), f(x^2)\} \end{aligned}$$

by Jensen's inequality. This contradicts our assumption that x^1 and x^2 are optimal solutions. Hence the problem must have a unique solution.

6.1. Consider the constrained maximisation problem

$$\max_x f(x) = x(x^2 - 3) \text{ s.t. } x \leq 1. \tag{4}$$

a) Write down the Lagrangian for this problem.

The Lagrangian is given by

$$\mathcal{L}(x, \delta) = x(x^2 - 3) + \delta(1 - x),$$

where δ is a Lagrange multiplier.

b) Derive the first-order and complementary slackness conditions for an optimal solution to (4).

The first-order conditions for a maximum is

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x} \\ &= 3(x^2 - 1) - \delta \end{aligned}$$

and the complementary slackness condition is

$$\delta(1 - x^*) = 0.$$

c) Find the optimal solution x^* to (4).

If $\delta > 0$ then $x^* = 1$ and if $\delta = 0$ then the first-order condition has roots $x = \pm 1$. Now $f(1) = -2$ and $f(-1) = 2$, and so the optimal solution $x^* = -1$.

6.2. Consider the constrained maximisation problem

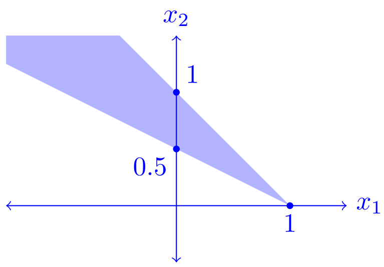
$$\max_x f(x) = \ln(x_1 x_2) \text{ s.t. } x_1 + x_2 \leq 1 \text{ and } x_1 + 2x_2 \geq 1. \tag{5}$$

a) Write down the feasible set for this problem. Is this set convex?

The feasible set is given by

$$X = \{x \in \mathbb{R}^2 : x_1 + x_2 \leq 1 \text{ and } x_1 + 2x_2 \geq 1\}.$$

This set is drawn below and is clearly convex.



b) Write down the Lagrangian for this problem.

The Lagrangian is given by

$$\mathcal{L}(x, \delta) = \ln(x_1 x_2) + \delta_1(1 - x_1 - x_2) + \delta_2(x_1 + 2x_2 - 1),$$

where δ_1 and δ_2 are Lagrange multipliers.

c) Derive the first-order and complementary slackness conditions for an optimal solution to (5).

The first-order conditions for a maximum are

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x_1} \\ &= \frac{1}{x_1} - \delta_1 + \delta_2 \\ 0 &= \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x_2} \\ &= \frac{1}{x_2} - \delta_1 + 2\delta_2 \end{aligned}$$

and the complementary slackness conditions are

$$\delta_1(1 - x_1^* - x_2^*) = 0$$
$$\delta_2(x_1^* + 2x_2^* - 1) = 0.$$

d) Find the optimal solution to (5).
Suppose that $\delta_1, \delta_2 > 0$. Then the complementary slackness conditions can be written as the linear system

$$\begin{aligned} x_1^* + x_2^* &= 1 \qquad (6) \\ x_1^* + 2x_2^* &= 1 \end{aligned}$$

which has unique solution $x^* = (1, 0)$. But the objective function is unbounded below as $x^* \to 0$, so this can't be a solution. Hence at most one constraint can bind.
Suppose that $\delta_1 > 0 = \delta_2$. Then the first-order conditions imply that

$$x_1^* = x_2^*.$$

Condition (6) then gives $x_1^* = 1/2$ and $x_2^* = 1/2$; a feasible solution. Hence the optimal solution $x^* = (1/2, 1/2)$.

6.3. Consider the constrained *minimisation* problem

$$\min_x f(x) = (x_1 - 1)^2 + (x_2 - 2)^2 \text{ s.t. } x_1 + x_2 \leq 1. \qquad (7)$$

a) Write down the Lagrangian for this problem.
The Lagrangian is given by

$$\mathcal{L}(x, \delta) = -(x_1 - 1)^2 - (x_2 - 2)^2 + \delta(1 - x_1 - x_2),$$

where δ is a Lagrange multiplier.

b) Derive the first-order and complementary slackness conditions for an optimal solution to (7).
The first-order conditions for a minimum are

$$\begin{aligned} \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x_1} &= -2x_1^* + 2 - \delta \\ &= 0 \qquad (8) \\ \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x_2} &= -2x_2^* + 4 - \delta \\ &= 0 \qquad (9) \end{aligned}$$

and the complementary slackness condition is

$$\delta(1 - x_1^* - x_2^*) = 0.$$

c) Find the optimal solution x^* to (7).
Substituting (8) into (9) so as to eliminate δ gives

$$2 - 2x_1^* = 4 - 2x_2^*$$

so that

$$x_2^* = 1 + x_1^*.$$

Hence

$$\begin{aligned} 1 &\geq x_1^* + x_2^* \\ &= 1 + 2x_1^* \end{aligned}$$

and therefore

$$x_1^* \leq 0.$$

If $x_1^* < 0$ then $x_2^* < 1$ and the complementary slackness condition implies that $\delta = 0$. But then (8) gives $x_1^* = 1 \not< 0$, a contradiction. Hence we must have $x_1^* = 0$. This gives $x_2^* = 1$ and $\delta = 2$, which is feasible. It follows that the optimal solution $x^* = (0, 1)$.

d) Interpret the Lagrange multiplier for the contraint $x_1 + x_2 \leq 1$.

We know that $\delta = 2$ from part c). The constraint is binding on the optimal solution because $\delta > 0$. If we relaxed this constraint then we would expect the optimal objective function value to *decrease*.

7.1. Suppose that a consumer solves

$$\max_x u(x) = \sqrt{x_1 x_2} \text{ subject to } x_1 + 2x_2 \leq w,$$

where $w > 0$ is the consumer's disposable income and $x_i \geq 0$ is quantity demanded of each good $i \in \{1, 2\}$.

a) Write down the Lagrangian for the consumer's problem. Ignore any nonnegativity constraints.

The Lagrangian is given by

$$\mathcal{L}(x, \delta) = \sqrt{x_1 x_2} + \delta(w - x_1 - 2x_2),$$

where δ is a Lagrange multiplier.

b) Derive the first-order and complementary slackness conditions for an optimal solution to the consumer's problem.

The first-order conditions for a maximum are

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x_1} \\ &= \frac{1}{2}\sqrt{\frac{x_2^*}{x_1^*}} - \delta \qquad (10) \\ 0 &= \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x_2} \\ &= \frac{1}{2}\sqrt{\frac{x_1^*}{x_2^*}} - 2\delta \qquad (11) \end{aligned}$$

and the complementary slackness condition is

$$\delta(w - x_1^* - 2x_2^*) = 0.$$

c) Find the optimal solution to the consumer's problem. Interpret the value of the Lagrange multiplier for the budget constraint.
Substituting (10) into (11) so as to eliminate δ gives

$$\sqrt{\frac{x_2^*}{x_1^*}} = \frac{1}{2}\sqrt{\frac{x_1^*}{x_2^*}},$$

which implies that $x_1^* = 2x_2^*$. The Lagrange multipler

$$\begin{aligned} \delta &= \frac{1}{2}\sqrt{\frac{x_2^*}{2x_2^*}} \\ &= \frac{1}{2\sqrt{2}} \end{aligned}$$

is strictly positive and so the budget constaint is binding. Hence

$$\begin{aligned} w &= x_1^* + 2x_2^* \\ &= 2x_1^* \end{aligned}$$

so that $x^* = (w/2, w/4)$. The Lagrange multiplier value implies that an increase in wealth of \$1 would generate an additional $1/2\sqrt{2}$ units of utility.

d) Show that the consumer's indirect utility function is linear in w.
The consumer's objective function is given by

$$\begin{aligned} v(w) &= u(x^*) \\ &= \sqrt{\frac{w^2}{8}} \\ &= \frac{w}{2\sqrt{2}} \end{aligned}$$

The second derivative

$$v''(w) = 0$$

is nonnegative and therefore $v(w)$ is linear.

e) Suppose that the budget constraint changes to $x_1 + 3x_2 \leq w$. Without any calculations, explain what will happen to x_1^*.
For Cobb-Douglas utility functions with $\alpha = \beta$, the consumer spends the same amount of wealth on each good. But the price of good 1 hasn't changed and so the optimal demand for good 1 will remain at $x_1^* = w/2$.

7.2. Consider the constrained minimisation problem

$$\min_x v_1 x^2 + v_2(1-x)^2 + 2cx(1-x) \text{ subject to } r_1 x + r_2(1-x) \geq R,$$

where v_1, v_2, c, r_1, r_2 and R are positive constants, and $v_1 + v_2 > 2c$.

a) Write down the Lagrangian for this problem.
The Lagrangian is given by

$$\mathcal{L}(x, \delta) = -v_1 x^2 - v_2(1-x)^2 - 2cx(1-x) + \delta(r_1 x + r_2(1-x) - R),$$

where $\delta \geq 0$ is a Lagrange multiplier.

b) Show that the Lagrangian from part a) is maximised by

$$x^* = \frac{\delta(r_1 - r_2)}{2(v_1 + v_2 - 2c)} + \frac{v_2 - c}{v_1 + v_2 - 2c}, \tag{12}$$

where δ is the Lagrange multiplier for the inequality constraint.
The maximiser x^* satisfies the first-order condition

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x} \\ &= -2v_1 x^* + 2v_2(1 - x^*) - 2c(1 - 2x^*) + \delta(r_1 - r_2) \\ &= 2v_2 - 2c + \delta(r_1 - r_2) - 2x^*(v_1 + v_2 - 2c), \end{aligned}$$

which can be rearranged for (12). Now

$$\frac{\partial^2 \mathcal{L}(x^*, \delta)}{\partial x^2} = -2(v_1 + v_2 - 2c)$$

is strictly negative and therefore x^* is indeed a maximum.

c) Interpret the Lagrange multiplier δ from part b).
We have

$$\frac{\partial \mathcal{L}(x^*, \delta)}{\partial R} = -\delta.$$

So if R increases by one unit then the maximal value of the objective function *decreases* by about δ units.

d) Show that if $r_1 = r_2$ then $x^* > 0$ if and only if $v_2 > c$.
If $r_1 = r_2$ then the maximiser

$$x^* = \frac{v_2 - c}{v_1 + v_2 - 2c}.$$

Now $v_1 + v_2 - 2c > 0$ and therefore $\operatorname{sign}(x^*) = \operatorname{sign}(v_2 - c)$.

9.1. A consumer solves

$$\max_x u(x) \text{ subject to } px \leq w, \tag{13}$$

where x denotes the demand for a single good with price $p > 0$ and $w > 0$ denotes the consumer's wealth. Assume that the utility function $u(x)$ is strictly increasing and concave in x.

a) Show that the optimal solution x^* satisfies

$$u'(x^*) = \delta p, \tag{14}$$

where $\delta \geq 0$ is the Lagrange multiplier for the budget constraint.
The Lagrangian for (13) is given by

$$\mathcal{L}(x, \delta) = u(x) + \delta(w - px).$$

The optimal demand x^* satisfies the first-order condition

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}(x^*, \delta)}{\partial x} \\ &= u'(x^*) - \delta p, \end{aligned}$$

which can be rearranged for (14).

b) Show that x^* is (i) increasing in w and (ii) decreasing in p.

From (14) we have

$$\delta = \frac{u'(x^*)}{p},$$

which is strictly positive since $u(x)$ is strictly increasing. Hence the budget constraint is binding and so

$$px^* = w.$$

Differentiating with respect to w gives

$$p\frac{\partial x^*}{\partial w} = 1$$

so that

$$\frac{\partial x^*}{\partial w} = \frac{1}{p}$$

is strictly positive, while differentiating with respect to p gives

$$x^* + p\frac{\partial x^*}{\partial p} = 0$$

so that

$$\frac{\partial x^*}{\partial p} = -\frac{x^*}{p}$$

is strictly negative because $x^* = w/p > 0$. Hence x^* is increasing in w and decreasing in p.

c) Let $v(p, w) = u(x^*)$ denote the consumer's indirect utility function. Show that $v(p, w)$ is (i) increasing in w and (ii) decreasing in p.

Differentiating $v(p, w)$ with respect to w gives

$$\begin{aligned}\frac{\partial v(p,w)}{\partial w} &= u'(x^*)\frac{\partial x^*}{\partial w}\\ &= \frac{u'(x^*)}{p}\\ &= \delta,\end{aligned}$$

which is strictly positive. Hence $v(p, w)$ is increasing in w. Similarly, differentiating $v(p, w)$ with respect to p gives

$$\begin{aligned}\frac{\partial v(p,w)}{\partial p} &= u'(x^*)\frac{\partial x^*}{\partial p}\\ &= -\frac{u'(x^*)x^*}{p}\end{aligned}$$

which is strictly negative. Hence $v(p, w)$ is decreasing in p.

9.2. A consumer with initial wealth w_0 and utility function $u(w) = \ln(w)$ is exposed to a risk that will decrease his wealth by an amount $x \in (0, w_0)$ with probability p or increase his wealth by x with probability $(1 - p)$.

a) Write down an expression for the consumer's expected utility.

The consumer has expected utility

$$\mathrm{E}[u(w)] = p \ln(w_0 - x) + (1 - p) \ln(w_0 + x).$$

b) Suppose that ϕ satisfies the indifference condition

$$\mathrm{E}[u(w)] = u(w_0 - \phi). \tag{15}$$

Use Jensen's inequality to show that $\phi > (2p - 1)x$.

Recall that $\ln(t)$ is strictly concave in t. Hence, by Jensen's inequality, we have

$$\begin{aligned} \mathrm{E}[u(w)] &= p \ln(w_0 - x) + (1 - p) \ln(w_0 + x) \\ &< \ln(p(w_0 - x) + (1 - p)(w_0 + x)) \\ &= \ln(w_0 + (1 - 2p)x) \end{aligned}$$

so that

$$\ln(w_0 - \phi) < \ln(w_0 + (1 - 2p)x).$$

But $\ln(t)$ is strictly increasing in t and therefore

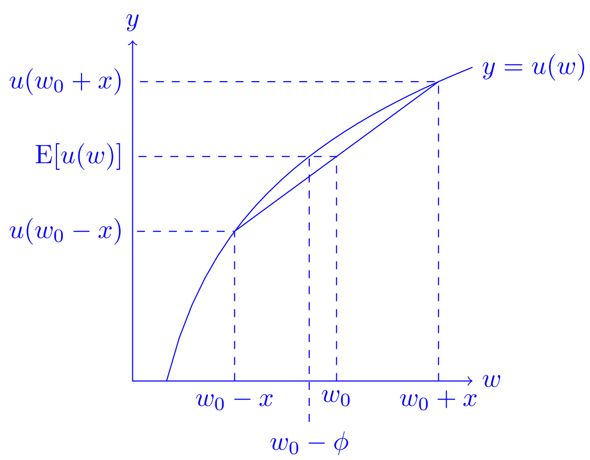
$$w_0 - \phi < w_0 + (1 - 2p)x$$

from which it follows that $\phi > (2p - 1)x$.

c) Let $p = 0.5$. Show graphically that $\phi < x$.

Hint: sketch the consumer's utility function $u(w) = \ln(w)$ in $\mathbb{R}^2$ and label the points $(w_0 - x, u(w_0 - x))$, $(w_0, u(w_0))$, etc., and compare the positions of $w_0 - x$ and $w_0 - \phi$.

The consumer's prospects can be shown graphically as follows.



The consumer's expected utility $\mathrm{E}[u(w)]$ lies at the midpoint of the chord between the points $(w_0 - x, u(w_0 - x))$ and $(w_0 + x, u(w_0 + x))$ because $p = 0.5$. The level of wealth $w_0 - \phi$ satisfying (15) is shown. Clearly $w_0 - x < w_0 - \phi$ and therefore $\phi < x$ as required.

10.1. Suppose that a consumer with income y suffers a loss of size $L < y$ with probability $p \in (0, 1)$. The consumer can buy $c \in [0, L]$ units of insurance coverage at the per-unit price π and solves

$$\max_c \mathrm{E}[u(w)] = pu(y - \pi c - L + c) + (1 - p)u(y - \pi c),$$

where the utility function $u(w)$ is strictly increasing and concave in w.

a) Derive the first-order condition for the optimal level of coverage c^*. Show that the second-order condition holds.

The optimal level of coverage c^* satisfies the first-order condition

$$\begin{aligned} 0 &= \left.\frac{\partial \mathrm{E}[u(w)]}{\partial c}\right|_{c=c^*} \\ &= (1 - \pi)pu'(w - \pi c^* - L + c^*) - \pi(1 - p)u'(w - \pi c^*). \end{aligned} \tag{16}$$

The second derivative

$$\frac{\partial^2 \mathrm{E}[u(w)]}{\partial c^2} = (1 - \pi)^2 pu''(y - \pi c - L + c) + \pi^2(1 - p)u''(y - \pi c)$$

is strictly negative since $u(w)$ is strictly concave and so the second-order condition holds.

b) Show that c^* is increasing in p.

Let $w_1^* \equiv y - \pi c^* - L + c^*$ and $w_2^* \equiv y - \pi c^*$ so that

$$0 = (1 - \pi)pu'(w_1^*) - \pi(1 - p)u'(w_2^*) \tag{17}$$

from (16). Differentiating with respect to p gives

$$\begin{aligned} 0 = (1 - \pi)u'(w_1^*) + (1 - \pi)^2 p \frac{\partial c^*}{\partial p} u''(w_1^*) \\ + \pi u'(w_2^*) + \pi^2(1 - p)\frac{\partial c^*}{\partial p} u''(w_2^*) \end{aligned}$$

so that

$$\frac{\partial c^*}{\partial p} = -\frac{(1 - \pi)u'(w_1^*) + \pi u'(w_2^*)}{(1 - \pi)^2 pu''(w_1^*) + \pi^2(1 - p)u''(w_2^*)},$$

and therefore c^* is increasing in p since $u(w)$ is strictly increasing and concave in w.

c) Show that c^* is increasing in y if and only if the Arrow-Pratt measure of absolute risk aversion

$$A(w) = -\frac{u''(w)}{u'(w)}$$

is increasing in w.

Differentiating (17) with respect to y gives

$$0 = (1-\pi)pu''(w_1^*)\left(1 + (1 - \pi)\frac{\partial c^*}{\partial y}\right) - \pi(1-p)u''(w_2^*)\left(1 - \pi\frac{\partial c^*}{\partial y}\right)$$

so that
$$\frac{\partial c^*}{\partial y} = \frac{\pi(1-p)u''(w_2^*) - (1-\pi)pu''(w_1^*)}{(1-\pi)^2 pu''(w_1^*) + \pi^2(1-p)u''(w_2^*)}$$
and therefore
$$\text{sign}\left(\frac{\partial c^*}{\partial y}\right) = \text{sign}((1-\pi)pu''(w_1^*) - \pi(1-p)u''(w_2^*))$$
from the second-order condition. But we know from (17) that
$$(1-\pi)p = \frac{\pi(1-p)u'(w_2^*)}{u'(w_1^*)}$$
and so
$$\begin{aligned}\text{sign}\left(\frac{\partial c^*}{\partial y}\right) &= \text{sign}\left(\frac{\pi(1-p)u'(w_2^*)}{u'(w_1^*)}u''(w_1^*) - \pi(1-p)u''(w_2^*)\right)\\ &= \text{sign}\left(\frac{u''(w_1^*)}{u'(w_1^*)} - \frac{u''(w_2^*)}{u'(w_2^*)}\right)\end{aligned}$$
since $\pi(1-p)u'(w_2^*) > 0$. Hence
$$\text{sign}\left(\frac{\partial c^*}{\partial y}\right) = \text{sign}(A(w_2^*) - A(w_1^*)).$$
But $w_2^* \geq w_1^*$. It follows that c^* is increasing in w if and only if $A(w)$ is increasing in w.

d) Let π_L be the premium rate at which $c^* = L$ and π_0 the premium rate at which $c^* = 0$. Show that $\pi_L < \pi_0$.
From (16) we have
$$0 = (1-\pi_L)pu'(w - \pi_L L) - \pi_L(1-p)u'(w - \pi_L L)$$
so that
$$(1-\pi_L)p = \pi_L(1-p)$$
and therefore $\pi_L = p$. Similarly
$$0 = (1-\pi_0)pu'(w-L) - \pi_0(1-p)u'(w)$$
so that
$$\pi_0 = \frac{pu'(w-L)}{pu'(w-L) + (1-p)u'(w)}.$$
But $u(w)$ is concave and so $u'(w-L) > u'(w)$ since $L > 0$. Hence
$$pu'(w-L) + (1-p)u'(w) < u'(w-L)$$
and therefore $\pi_0 > p = \pi_L$.

e) Now assume that $u(w) = \ln(w)$. Find an expression for c^* in terms of the parameters y, π, p and L.
If $u(w) = \ln(w)$ then $u'(w) = 1/w$. Hence from (16) we have
$$\frac{\pi(1-p)}{w - \pi c^*} = \frac{(1-\pi)p}{w - \pi c^* - L + c^*}$$

so that

$$\pi(1-p)(w-\pi c^* - L + c^*) = (1-\pi)p(w-\pi c^*)$$

and therefore

$$c^* = \frac{(1-\pi)pw - \pi(1-p)(w-L)}{\pi(1-\pi)}.$$

10.2. Suppose that a consumer has initial wealth w_0 and utility function

$$u(w) = 1 - \exp(-w).$$

Suppose also that the consumer suffers a loss of size l with probability p or no loss with probability $(1-p)$, where $p \in (0,1)$.

a) Write an expression for the consumer's expected utility $\mathrm{E}[u(w)]$ in terms of the parameters w_0, l and p.
The consumer has expected utility

$$\begin{aligned}\mathrm{E}[u(w)] &= pu(w_0 - l) + (1-p)u(w_0)\\ &= p(1-\exp(-w_0+l)) + (1-p)(1-\exp(-w_0))\\ &= 1 - p\exp(-w_0)\exp(l) - (1-p)\exp(-w_0). \end{aligned} \tag{18}$$

b) Suppose that ϕ that satisfies the indifference condition

$$\mathrm{E}[u(w)] = u(w_0 - \phi). \tag{19}$$

Find an expression for ϕ in terms of the parameters w_0, l and p.
First write

$$\begin{aligned}u(w_0-\phi) &= 1-\exp(-w_0+\phi)\\ &= 1-\exp(-w_0)\exp(\phi).\end{aligned}$$

It follows from (18) and (19) that

$$\exp(\phi) = p\exp(l) + (1-p),$$

and therefore

$$\phi = \ln(p\exp(l) + (1-p)).$$

c) Explain why ϕ from part b) is positive and independent of w_0.
The quantity ϕ represents the maximum premium that the consumer would be willing to pay to remove their risk exposure. Notice that the Arrow-Pratt measure of absolute risk aversion

$$\begin{aligned}A(w_0) &= -\frac{u''(w_0)}{u'(w_0)}\\ &= -\frac{-\exp(-w_0)}{\exp(-w_0)}\\ &= 1\end{aligned}$$

is a positive constant. Hence (i) the consumer is risk averse and is willing to pay a positive premium to remove the risk, and (ii) the consumer's aversion to absolute risks is independent of wealth. The latter explains why ϕ does not change when w_0 changes: the consumer's aversion to the absolute risk does not change.

11.1. An investor with initial wealth w_0 is considering putting money into an investment that earns the interest rate r_1 with probability p and r_2 with probability $(1-p)$, where $r_1 < 0 < r_2$. The investor has final wealth

$$\tilde{w} = w_0 + \tilde{r}x,$$

where $\tilde{r}$ denotes the random interest rate and x his allocation to the risky investment. The investor solves

$$\max_x f(x) = \mathrm{E}[u(\tilde{w})],$$

where his utility function $u(w)$ is strictly increasing and strictly concave in w, and E is the expectation operator.

a) Write down an expression for $\mathrm{E}[u(\tilde{w})]$ in terms of the choice variable x, and the parameters w_0, r_1, r_2 and p.

The investor has expected utility

$$\mathrm{E}[u(\tilde{w})] = pu(w_0 + r_1 x) + (1-p)u(w_0 + r_2 x).$$

b) Find the first-order conditions for the optimal choice of investment x^* Show that x^* satisfies the second-order condition for a maximum.

The optimal level of investment x^* satisfies the first-order condition

$$\begin{aligned} 0 &= f'(x) \\ &= r_1 p u'(w_0 + r_1 x^*) + r_2(1-p)u'(w_0 + r_2 x^*). \end{aligned} \tag{20}$$

The second derivative

$$f''(x^*) = r_1^2 p u''(w_0 + r_1 x^*) + r_2^2 (1-p) u''(w_0 + r_2 x^*)$$

is negative because $u(w)$ is strictly concave and so x^* satisfies the second-order condition for a maximum.

c) Show that $x^* > 0$ if and only if the expected interest rate $\mathrm{E}[\tilde{r}] > 0$.

First rewrite the first-order condition as

$$-\frac{r_2(1-p)}{r_1 p} = \frac{u'(w_0 + r_1 x^*)}{u'(w_0 + r_2 x^*)}.$$

Now $r_1 < 0 < r_2$ and so $x^* > 0$ if and only if $w_0 + r_1 x^* < w_0 + r_2 x^*$. This occurs precisely when

$$1 < \frac{u'(w_0 + r_1 x^*)}{u'(w_0 + r_2 x^*)}$$

so that

$$r_1 p > -r_2(1-p)$$

and therefore $0 < r_1 p + r_2(1-p) = \mathrm{E}[\tilde{r}]$.

d) Show that the optimal choice of investment x^* is increasing in w_0 if and only if

$$A(w_0 + r_1 x^*) > A(w_0 + r_2 x^*),$$

where $A(w) \equiv -u''(w)/u'(w)$ is the Arrow-Pratt measure of absolute risk aversion.

Differentiating (20) with respect to w_0 gives

$$\begin{aligned} 0 = &\left(1 + r_1 \frac{\partial x^*}{\partial w_0}\right) r_1 p u''(w_0 + r_1 x^*) \\ &+ \left(1 + r_2 \frac{\partial x^*}{\partial w_0}\right) r_2 (1-p) u''(w_0 + r_2 x^*) \end{aligned}$$

which can be rearranged for

$$\frac{\partial x^*}{\partial w_0} = -\frac{r_1 p u''(w_0 + r_1 x^*) + r_2(1-p)u''(w_0 + r_2 x^*)}{r_1^2 p u''(w_0 + r_1 x^*) + r_2^2(1-p)u''(w_0 + r_2 x^*)}.$$

But the denominator is strictly negative from the first-order condition and so

$$\text{sign}\left(\frac{\partial x^*}{\partial w_0}\right) = \text{sign}\left(r_1 p u''(w_0 + r_1 x^*) + r_2(1-p)u''(w_0 + r_2 x^*)\right).$$

Now

$$r_2(1-p) = -\frac{r_1 p u'(w_0 + r_1 x^*)}{u'(w_0 + r_2 x^*)}$$

from the first-order condition and so

$$\begin{aligned} \text{sign}\left(\frac{\partial x^*}{\partial w_0}\right) &= \text{sign}\left(r_1 p u''(w_0 + r_1 x^*) - \frac{r_1 p u'(w_0 + r_1 x^*)}{u'(w_0 + r_2 x^*)} u''(w_0 + r_2 x^*)\right) \\ &= \text{sign}\left(-\frac{u''(w_0 + r_1 x^*)}{u'(w_0 + r_1 x^*)} + \frac{u''(w_0 + r_2 x^*)}{u'(w_0 + r_2 x^*)}\right) \\ &= \text{sign}(A(w_0 + r_1 x^*) - A(w_0 + r_2 x^*)) \end{aligned}$$

since $-r_1 p u'(w_0 + r_1 x^*) > 0$. Thus x^* is increasing in w_0 if and only if $A(w_0 + r_1 x^*) > A(w_0 + r_2 x^*)$.

11.2. Consider an investor with initial wealth w_0 and utility function

$$u(w) = \frac{w^{1-\gamma} - 1}{1-\gamma},$$

where $\gamma > 0$.[1] Show that the investor's degree of absolute risk aversion is decreasing in his initial wealth.

We have

$$\begin{aligned} A(w_0) &= -\frac{u''(w_0)}{u'(w_0)} \\ &= -\frac{-\gamma w_0^{-\gamma-1}}{w_0^{-\gamma}} \\ &= \frac{\gamma}{w_0}. \end{aligned}$$

[1] One can use L'Hôpital's rule to show that $u(w) \to \ln(w)$ as $\gamma \to 1$.

Differentiating with respect to w_0 gives

$$\begin{aligned} A'(w_0) &= -\frac{\gamma}{w_0^2} \\ &< 0 \end{aligned}$$

for all $w_0 \neq 0$. Thus $u(w)$ exhibits decreasing absolute risk aversion.

11.3. Consider an investor with initial wealth w_0 and utility function

$$u(w) = aw - bw^2,$$

where $a, b > 0$. Show that the investor's degree of absolute risk aversion is increasing in his initial wealth.

We have

$$\begin{aligned} A(w_0) &= -\frac{u''(w_0)}{u'(w_0)} \\ &= -\frac{-2b}{a - 2bw_0} \\ &= \frac{1}{r - w_0}, \end{aligned}$$

where we define $r \equiv a/2b > 0$. Differentiating with respect to w_0 gives

$$\begin{aligned} A'(w_0) &= \frac{1}{(r - w_0)^2} \\ &> 0 \end{aligned}$$

for all $w_0 \neq r$. Thus $u(w)$ exhibits increasing absolute risk aversion.

Exam Questions

E.1. Suppose that a consumer has utility function $u(x) = a\ln(x_1) + b\ln(x_2)$ and solves

$$\max_x u(x) \text{ subject to } p_1x_1 + p_2x_2 \leq w, \tag{21}$$

where x_i and $p_i > 0$ respectively denote the quantity demanded and price of good $i \in \{1, 2\}$, $w > 0$ denotes his wealth, and a and b are positive constants. Assume that the optimal demands $x_1^* > 0$ and $x_2^* > 0$.

a) Write down the Lagrangian for the consumer's problem. Derive the first-order and complementary slackness conditions for the optimal demands x_1^* and x_2^*.

The Lagrangian is given by

$$\mathcal{L}(x, \delta) = a\ln(x_1) + b\ln(x_2) + \delta(w - p_1x_1 - p_2x_2),$$

where $\delta \geq 0$ is a Lagrange multiplier. The optimal demands x_1^* and x_2^* satisfy the first-order conditions

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}(x, \delta)}{\partial x_1} \\ &= \frac{a}{x_1^*} - \delta p_1 \qquad (22) \end{aligned}$$

and

$$
\begin{aligned}
0 &= \frac{\partial \mathcal{L}(x, \delta)}{\partial x_1} \\
&= \frac{b}{x_2^*} - \delta p_2, \qquad (23)
\end{aligned}
$$

and the complementary slackness condition

$$\delta(w - p_1 x_1^* - p_2 x^*) = 0. \qquad (24)$$

b) Show that the optimal expenditure ratio

$$\frac{p_1 x_1^*}{p_2 x_2^*} = \frac{a}{b}. \qquad (25)$$

The result follows immediately from substituting (22) into (23) so as to eliminate δ.

c) Show that the consumer's budget constraint is binding.
From (22) and (23), we have

$$
\begin{aligned}
\delta w &\geq \delta(p_1 x_1^* + p_2 x^*) \\
&= \delta(a + b)
\end{aligned}
$$

so that

$$\delta > \frac{a + b}{w}$$

is strictly positive. It follows from (24) that the budget constraint is binding.

d) Show that x_1^* is decreasing in p_1.
First write (25) as

$$b p_1 x_1^* = a p_2 x_2^*.$$

Differentiating both sides with respect to p_1 gives

$$b\left(x_1^* + p_1 \frac{\partial x_1^*}{\partial p_1}\right) = a p_2 \frac{\partial x_2^*}{\partial p_1}.$$

But from the budget constraint we have

$$x_1^* + p_1 \frac{\partial x_1^*}{\partial p_1} + p_2 \frac{\partial x_2^*}{\partial p_1} = 0$$

so that

$$b\left(x_1^* + p_1 \frac{\partial x_1^*}{\partial p_1}\right) = -a\left(x_1^* + p_1 \frac{\partial x_1^*}{\partial p_1}\right)$$

and therefore

$$\frac{\partial x_1^*}{\partial p_1} = -\frac{x_1^*}{p_1}.$$

But x_1^* and p_1 are strictly positive, and so x_1^* is decreasing in p_1.

e) Use Jensen's inequality to show that $u(x)$ is concave in x.
Let x^1 and x^2 be feasible bundles and let $\lambda \in [0, 1]$. Now $\ln(t)$ is strictly concave in t and so

$$\ln(\lambda t_1 + (1-\lambda)t_2) > \lambda \ln(t_1) + (1-\lambda)\ln(t_2)$$

for all $t_1 > 0$ and $t_2 > 0$ by Jensen's inequality. Hence

$$\begin{aligned} u(\lambda x^1 + (1-\lambda)x^2) &= a\ln(\lambda x_1^1 + (1-\lambda)x_1^2) \\ &\quad + b\ln(\lambda x_2^1 + (1-\lambda)x_2^2) \\ &> a(\lambda \ln(x_1^1) + (1-\lambda)\ln(x_1^2)) \\ &\quad + b(\lambda \ln(x_2^1) + (1-\lambda)\ln(x_2^2)) \\ &= \lambda(a\ln(x_1^1) + b\ln(x_2^1)) \\ &\quad + (1-\lambda)(a\ln(x_1^2) + b\ln(x_2^2)) \\ &= \lambda u(x^1) + (1-\lambda)u(x^2) \end{aligned}$$

so that $u(x)$ satisfies Jensen's inequality for concave functions and is therefore concave in x.

f) Suppose that a second consumer has utility function $u(x) = x_1^a x_2^b$ and solves (21) with the same parameters. Explain why both consumers will have the same demands for goods 1 and 2.
Notice that

$$\begin{aligned} \ln(x_1^a x_2^b) &= \ln(x_1^a) + \ln(x_2^b) \\ &= a\ln(x_1) + b\ln(x_2). \end{aligned}$$

Now $\ln(t)$ is monotone increasing in t and so preserves the preference ordering determined by the utility function $u(x) = x_1^a x_2^b$. It follows that the two consumers will prefer the same optimal bundle to all others within the feasible set.

E.2. Suppose that a consumer receives income y at each date $t \in \{0, 1\}$. At date 1, the consumer incurs a loss of size $L > 0$ with probability $p > 0$. At date 0, he buys an insurance contract that provides c units of coverage at date 1 if the loss occurs. The consumer pays π per contracted unit of coverage and solves

$$\max_c v(c) = u(y - \pi c) + \beta[pu(y - L + c) + (1-p)u(y)],$$

where $u(w)$ is strictly increasing and strictly concave in w, and $\beta \in (0, 1)$ is his intertemporal discount factor.

a) Derive the first-order condition for the optimal level of coverage c^*. Show that the second-order condition holds.
The optimal level of coverage satisfies the first-order condition

$$\begin{aligned} 0 &= v'(c^*) \\ &= -\pi u(y - \pi c^*) + \beta p u'(y - L + c^*) \end{aligned}$$

so that

$$\pi u'(y-\pi c^*) = \beta p u'(y-L+c^*). \tag{26}$$

The second derivative

$$v''(c) = \pi^2 u''(y-\pi c) + \beta p u''(y-L+c)$$

is strictly negative because $u(w)$ is strictly concave. Hence $v(c)$ is strictly concave in c and therefore c^* is indeed a maximiser.

b) Show that c^* is increasing in p.
Differentiating (26) with respect to p gives

$$-\pi^2 u''(y-\pi c^*)\frac{\partial c^*}{\partial p} = \beta u'(y-L+c^*) + \beta p u''(y-L+c^*)\frac{\partial c^*}{\partial p}$$

so that

$$\frac{\partial c^*}{\partial p} = -\frac{\beta u'(y-L+c^*)}{\pi^2 u''(y-\pi c^* + \beta p u''(y-L+c^*)}.$$

But the denominator is strictly negative from the second-order condition and therefore c^* is increasing in p.

c) Show that c^* is increasing in y if and only if

$$A(y-L+c) < A(y-\pi c), \tag{27}$$

where $A(w) = -u''(w)/u'(w)$ is the Arrow-Pratt measure of absolute risk aversion.
Differentiating (26) with respect to y gives

$$\left(1-\pi\frac{\partial c^*}{\partial y}\right)\pi u''(y-\pi c^*) = \left(1+\frac{\partial c^*}{\partial y}\right)\beta p u''(y-L+c^*)$$

so that

$$\frac{\partial c^*}{\partial y} = \frac{\pi u''(y-\pi c^*) - \beta p u''(y-L+c^*)}{\pi^2 u''(y-\pi c^*) + \beta p u''(y-L+c)}$$

and therefore

$$\text{sign}\left(\frac{\partial c^*}{\partial y}\right) = \text{sign}(\beta p u''(y-L+c^*) - \pi u''(y-\pi c^*))$$

from the second-order condition. It follows from (26) that

$$\begin{aligned}\text{sign}\left(\frac{\partial c^*}{\partial y}\right) &= \text{sign}\left(\frac{\pi u'(y-\pi c^*)}{u'(y-L+c^*)}u''(y-L+c^*) - \pi u''(y-\pi c^*)\right)\\ &= \text{sign}\left(\frac{u''(y-L+c^*)}{u'(y-L+c^*)} - \frac{u''(y-\pi c^*)}{u'(y-\pi c^*)}\right)\end{aligned}$$

since $u(w)$ is strictly increasing. Hence

$$\text{sign}\left(\frac{\partial c^*}{\partial y}\right) = \text{sign}(A(y-\pi c) - A(y-L+c))$$

so that c^* is increasing in y if and only if (27) holds.

d) Let π_L denote the per-unit price of coverage at which $c^* = L$. Show that π_L is increasing in β.

Differentiating (26) with respect to β gives

$$u'(y - \pi_L L)\frac{\partial \pi_L}{\partial \beta} - \pi_L u''(y - \pi_L L)L\frac{\partial \pi_L}{\partial \beta} = pu'(y)$$

so that

$$\frac{\partial \pi_L}{\partial \beta} = \frac{pu'(y)}{u'(y - \pi_L L) - \pi_L L u''(y - \pi_L L)}.$$

Now $u(w)$ is strictly increasing and concave in w, and therefore π_0 is increasing in p.

e) Assume that $u(w) = \ln(w)$. Find an expression for c^* in terms of the parameters y, π, β, p and L.

If $u(w) = \ln(w)$ then $u'(w) = 1/w$. Hence from (26) we have

$$\frac{\pi}{y - \pi c^*} = \frac{\beta p}{y - L + c^*}$$

so that

$$\pi(y - L + c^*) = \beta p(y - \pi c^*)$$

and therefore

$$c^* = \frac{\pi L - (\pi - \beta p)y}{\pi(1 + \beta p)}.$$