

Center of Mass; Moment of Inertia

19-1 Properties of the center of mass

In the previous chapter we found that if a great many forces are acting on a complicated mass of particles, whether the particles comprise a rigid or a nonrigid body, or a cloud of stars, or anything else, and we find the sum of all the forces (that is, of course, the external forces, because the internal forces balance out), then if we consider the body as a whole, and say it has a total mass M , there is a certain point "inside" the body, called the *center of mass*, such that the net resulting external force produces an acceleration of this point, just as though the whole mass were concentrated there. Let us now discuss the center of mass in a little more detail.

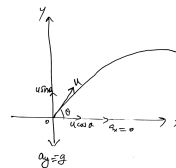
The location of the center of mass (abbreviated CM) is given by the equation

$$\mathbf{R}_{\text{CM}} = \frac{\sum m_i \mathbf{r}_i}{\sum m_i}. \quad (19.1)$$

This is, of course, a vector equation which is really three equations, one for each of the three directions. We shall consider only the x -direction, because if we can understand that one, we can understand the other two. What does $X_{\text{CM}} = \sum m_i x_i / \sum m_i$ mean? Suppose for a moment that the object is divided into little pieces, all of which have the same mass m ; then the total mass is simply the number N of pieces times the mass of one piece, say one gram, or any unit. Then this equation simply says that we add all the x 's, and then divide by the number of things that we have added: $X_{\text{CM}} = m \sum x_i / mN = \sum x_i / N$. In other words, X_{CM} is the average of all the x 's, if the masses are equal. But suppose one of them were twice as heavy as the others. Then in the sum, that x would come in twice. This is easy to understand, for we can think of this double mass as being split into two equal ones, just like the others; then in taking the average, of course, we have to count that x twice because there are two masses

PROJECTILE MOTION

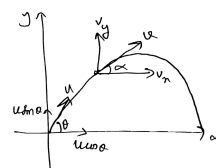
An object that is in flight after being thrown or projected is called a projectile. The motion of a projectile may be thought as the result of two separate, simultaneously occurring perpendicular components of motions. One component is along horizontal direction without any acceleration and other is along vertical direction with constant acceleration due to gravity. It was Galileo who first stated this independency of the horizontal and the vertical components of projectile motion.



A particle is projected with a velocity u (velocity of projection) making using a angle θ with the horizontal. θ is known as angle of projection. Only force that control the projectile is gravity. We will neglect air resistance. Projectile is subjected to acceleration due to gravity $\vec{a} = -g\vec{j}$ ($a_x = 0$, $a_y = -g$).

$u \cos \theta$ is the horizontal component of velocity which remains constant. $u \sin \theta$ is initial vertical component of velocity. O is the point of projection which is taken as origin.

The velocity of the projectile after t second



there. Thus X is the average position, in the x -direction, of all the masses, every mass being counted a number of times proportional to the mass, as though it were divided into "little grams." From this it is easy to prove that X must be somewhere between the largest and the smallest x , and, therefore lies inside the envelope including the entire body. It does not have to be in the *material* of the body, for the body could be a circle, like a hoop, and the center of mass is in the center of the hoop, not in the hoop itself.

Of course, if an object is symmetrical in some way, for instance, a rectangle, so that it has a plane of symmetry, the center of mass lies somewhere on the plane of symmetry. In the case of a rectangle there are two planes, and that locates it uniquely. But if it is just any symmetrical object, then the center of gravity lies somewhere on the axis of symmetry, because in those circumstances there are as many positive as negative x 's.

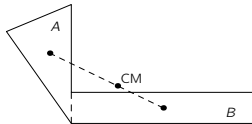


Fig. 19-1. The CM of a compound body lies on the line joining the CM's of the two composite parts.

Another interesting proposition is the following very curious one. Suppose that we imagine an object to be made of two pieces, A and B (Fig. 19-1). Then the center of mass of the whole object can be calculated as follows. First, find the center of mass of piece A , and then of piece B . Also, find the total mass of each piece, M_A and M_B . Then consider a new problem, in which a *point* mass M_A is at the center of mass of object A , and another *point* mass M_B is at the center of mass of object B . The center of mass of these two point masses is then the center of mass of the whole object. In other words, if the centers of mass of various parts of an object have been worked out, we do not have to start all over again to find the center of mass of the whole object; we just have to put the pieces together, treating each one as a point mass situated at the center of mass of that piece. Let us see why that is. Suppose that we wanted to calculate the center of mass of a complete object, some of whose particles are considered to be members of object A and some members of object B . The total sum $\sum m_i x_i$

$$V_y = u_y + a_y t$$

$$V_y = u \sin \theta - gt$$

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{(u \cos \theta)^2 + (u \sin \theta - gt)^2}$$

$$V = \sqrt{u^2 - 2u \sin \theta gt + g^2 t^2}$$

Velocity v make an angle α with horizontal such that, $\tan \alpha = \frac{\text{opposite side}}{\text{adj. side}} = \frac{V_y}{V_x} = \frac{u \sin \theta - gt}{u \cos \theta}$

In vector form $\vec{a} = -g\hat{j}$

$$\vec{u} = u \cos \theta \hat{i} + u \sin \theta \hat{j}$$

$$\vec{v} = \vec{u} + \vec{a} t$$

$$\vec{v} = u \cos \theta \hat{i} + u \sin \theta \hat{j} - g t \hat{j}$$

$$\vec{v} = u \cos \theta \hat{i} + (u \sin \theta - gt) \hat{j}$$

$$|\vec{v}| = \sqrt{(u \cos \theta)^2 + (u \sin \theta - gt)^2}, \tan \alpha = \frac{u \sin \theta - gt}{u \cos \theta}$$

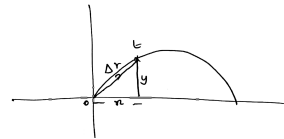
To find the displacement of the projectile after t seconds

along x axis

disp = velocity \times time

$$x = (u \cos \theta) t$$

This is the equation for x -coordinate of the projectile at any time t .



$$\text{along } y \text{ axis } y = u_y t + \frac{a_y t^2}{2}$$

$$y = u \sin \theta t - \frac{1}{2} g t^2 \quad \text{Equation for } y \text{ coordinate or height of projectile at any time } t.$$

can then be split into two pieces—the sum $\sum_A m_i x_i$ for the A object only, and the sum $\sum_B m_i x_i$ for object B only. Now if we were computing the center of mass of object A alone, we would have exactly the first of these sums, and we know that this by itself is $M_A X_A$, the total mass of all the particles in A times the position of the center of mass of A , because that is the theorem of the center of mass, applied to object A . In the same manner, just by looking at object B , we get $M_B X_B$, and of course, adding the two yields $M X_{CM}$:

$$\begin{aligned} M X_{CM} &= \sum_A m_i x_i + \sum_B m_i x_i \\ &= M_A X_A + M_B X_B. \end{aligned} \quad (19.2)$$

Now since M is evidently the sum of M_A and M_B , we see that Eq. (19.2) can be interpreted as a special example of the center of mass formula for two point objects, one of mass M_A located at X_A and the other of mass M_B located at X_B .

The theorem concerning the motion of the center of mass is very interesting, and has played an important part in the development of our understanding of physics. Suppose we assume that Newton's law is right for the small component parts of a much larger object. Then this theorem shows that Newton's law is also correct for the larger object, even if we do not study the details of the object, but only the total force acting on it and its mass. In other words, Newton's law has the peculiar property that if it is right on a certain small scale, then it will be right on a larger scale. If we do not consider a baseball as a tremendously complex thing, made of myriads of interacting particles, but study only the motion of the center of mass and the external forces on the ball, we find $\mathbf{F} = m\mathbf{a}$, where \mathbf{F} is the external force on the baseball, m is its mass, and \mathbf{a} is the acceleration of its center of mass. So $\mathbf{F} = m\mathbf{a}$ is a law which reproduces itself on a larger scale. (There ought to be a good word, out of the Greek, perhaps, to describe a law which reproduces the same law on a larger scale.)

Of course, one might suspect that the first laws that would be discovered by human beings would be those that would reproduce themselves on a larger scale. Why? Because the actual scale of the fundamental gears and wheels of the universe are of atomic dimensions, which are so much finer than our observations that we are nowhere near that scale in our ordinary observations. So the first things that we would discover must be true for objects of no special size relative to an atomic scale. If the laws for small particles did not reproduce themselves on a larger scale, we would not discover those laws very easily. What about the reverse

$$\text{displacement} = r = \sqrt{x^2 + y^2}$$

To find it in vector form we can use the equation $\vec{r} = x\hat{i} + y\hat{j}$ where $\vec{u} = u \sin \theta \hat{j}$, $\vec{a} = -g\hat{j}$

Equation for path of a projectile

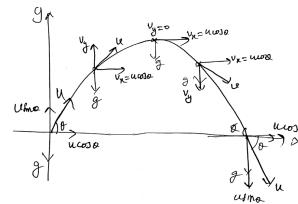
$$x = u \cos \theta t, t = \frac{x}{u \cos \theta}$$

$$y = u \sin \theta t - \frac{1}{2} g t^2$$

$$y = u \sin \theta \frac{x}{u \cos \theta} - \frac{1}{2} g \left(\frac{x}{u \cos \theta} \right)^2$$

$$y = x \tan \theta - \frac{1}{2} g \frac{x^2}{u^2 \cos^2 \theta}$$

This is the equation of a parabola. Thus the path of a projectile is parabola.

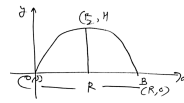


At the highest point of the projectile vertical component of velocity is zero. Horizontal component is $u \cos \theta$ because it remains constant. At the highest point speed of the projectile is minimum and purely horizontal and is equal to $u \cos \theta$. Angle between acceleration and instantaneous velocity decreases from $(90 + \theta)$ to $(90 - \theta)$

problem? Must the laws on a small scale be the same as those on a larger scale? Of course it is not necessarily so in nature, that at an atomic level the laws have to be the same as on a large scale. Suppose that the true laws of motion of atoms were given by some strange equation which does *not* have the property that when we go to a larger scale we reproduce the same law, but instead has the property that if we go to a larger scale, we can *approximate it by a certain expression* such that, if we extend that expression up and up, *it* keeps reproducing itself on a larger and larger scale. That is possible, and in fact that is the way it works. Newton's laws are the "tail end" of the atomic laws, extrapolated to a very large size. The actual laws of motion of particles on a fine scale are very peculiar, but if we take large numbers of them and compound them, they approximate, but *only* approximate, Newton's laws. Newton's laws then permit us to go on to a higher and higher scale, and it still seems to be the same law. In fact, it becomes more and more accurate as the scale gets larger and larger. This self-reproducing factor of Newton's laws is thus really not a fundamental feature of nature, but is an important historical feature. We would never discover the fundamental laws of the atomic particles at first observation because the first observations are much too crude. In fact, it turns out that the fundamental atomic laws, which we call quantum mechanics, are quite different from Newton's laws, and are difficult to understand because all our direct experiences are with large-scale objects and the small-scale atoms behave like nothing we see on a large scale. So we cannot say, "An atom is just like a planet going around the sun," or anything like that. It is like *nothing* we are familiar with because there is *nothing like it*. As we apply quantum mechanics to larger and larger things, the laws about the behavior of many atoms together do *not* reproduce themselves, but produce *new laws*, which are Newton's laws, which then continue to reproduce themselves from, say, micro-microgram size, which still is billions and billions of atoms, on up to the size of the earth, and above.

Let us now return to the center of mass. The center of mass is sometimes called the center of gravity, for the reason that, in many cases, gravity may be considered uniform. Let us suppose that we have small enough dimensions that the gravitational force is not only proportional to the mass, but is everywhere parallel to some fixed line. Then consider an object in which there are gravitational forces on each of its constituent masses. Let m_i be the mass of one part. Then the gravitational force on that part is m_i times g . Now the question is, where can we apply a single force to balance the gravitational force on the whole thing, so that the entire object, if it is a rigid body, will not turn? The answer is that this force

Time of Flight of the projectile (T)



Consider the motion of the projectile along y-axis

$$S_y = u_y t + \frac{1}{2} a_y t^2$$

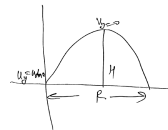
$$t = T$$

$$0 = u \sin \theta T - \frac{1}{2} g T^2$$

$$T = \frac{2u \sin \theta}{g} \quad \text{or} \quad T = \frac{2u_y}{g}$$

$$\text{Time of ascend} = \text{Time of descend} = \frac{u \sin \theta}{g}$$

Maximum height of a projectile H



Consider the motion along y-axis

$$V_y^2 = u_y^2 + 2a_y g,$$

$$0 = (u \sin \theta)^2 + 2(-gH)$$

$$2gH = (u \sin \theta)^2$$

$$H = \frac{u^2 \sin^2 \theta}{2g} \quad \text{or} \quad H = \frac{u_y^2}{2g}$$

must go through the center of mass, and we show this in the following way. In order that the body will not turn, the torque produced by all the forces must add up to zero, because if there is a torque, there is a change of angular momentum, and thus a rotation. So we must calculate the total of all the torques on all the particles, and see how much torque there is about any given axis; it should be zero if this axis is at the center of mass. Now, measuring x horizontally and y vertically, we know that the torques are the forces in the y -direction, times the lever arm x (that is to say, the force times the lever arm around which we want to measure the torque). Now the total torque is the sum

$$\tau = \sum m_i g x_i = g \sum m_i x_i, \quad (19.3)$$

so if the total torque is to be zero, the sum $\sum m_i x_i$ must be zero. But $\sum m_i x_i = MX_{CM}$, the total mass times the distance of the center of mass from the axis. Thus the x -distance of the center of mass from the axis is zero.

Of course, we have checked the result only for the x -distance, but if we use the true center of mass the object will balance in any position, because if we turned it 90 degrees, we would have y 's instead of x 's. In other words, when an object is supported at its center of mass, there is no torque on it because of a parallel gravitational field. In case the object is so large that the nonparallelism of the gravitational forces is significant, then the center where one must apply the balancing force is not simple to describe, and it departs slightly from the center of mass. That is why one must distinguish between the center of mass and the center of gravity. The fact that an object supported exactly at the center of mass will balance in all positions has another interesting consequence. If, instead of gravitation, we have a pseudo force due to acceleration, we may use exactly the same mathematical procedure to find the position to support it so that there are no torques produced by the inertial force of acceleration. Suppose that the object is held in some manner inside a box, and that the box, and everything contained in it, is accelerating. We know that, from the point of view of someone at rest relative to this accelerating box, there will be an effective force due to inertia. That is, to make the object go along with the box, we have to push on it to accelerate it, and this force is "balanced" by the "force of inertia," which is a pseudo force equal to the mass times the acceleration of the box. To the man in the box, this is the same situation as if the object were in a uniform gravitational field whose " g " value is equal to the acceleration a . Thus the inertial force due to accelerating an object has no torque about the center of mass.

Horizontal range (R) of the projectile

Horizontal range = Horizontal velocity \times time of flight

$$R = u \cos \theta T, \quad T = \frac{2u \sin \theta}{g}$$

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}} \quad \text{or} \quad \boxed{R = \frac{2u_x u_y}{g}}$$

Relation connecting R, H, T and angle of projection θ

$$\frac{H}{R} = \frac{u^2 \sin^2 \theta}{2gu^2 \sin \theta \cos \theta} = \frac{\sin \theta}{2 \cos \theta}$$

$$4H = R \tan \theta, \quad \boxed{H = \frac{gT^2}{8}}$$

$$\therefore 4 \frac{gT^2}{8} = R \tan \theta; \quad \boxed{R = \frac{gT^2}{2 \tan \theta}}$$

Angle of projection for maximum range for a given speed of projection

$$R = \frac{u^2 \sin 2\theta}{g}, \quad [\sin 2\theta]_{\max} = 1$$

$$R_{\max} = \frac{u^2}{g} \quad \begin{aligned} \sin 90^\circ &= 1 \\ 2\theta &= 90^\circ \\ \theta &= 45^\circ \end{aligned}$$

To get maximum height we should throw vertically up $\theta = 90^\circ$

$$H_{\max} = \frac{u^2}{2g} \quad (1\text{-dimensional motion})$$

$$H_{\max} = \left(\frac{u^2}{g} \right) = \frac{R_{\max}}{2}$$

Galileo in his book two new sciences stated that for elevations which exceed or fall short of 45° by equal amount, the ranges are equal.

i.e., there are two different angles of projection for same range. If one angle is θ other angle is $90 - \theta$ for same speed of projection.

This fact has a very interesting consequence. In an inertial frame that is not accelerating, the torque is always equal to the rate of change of the angular momentum. However, about an axis through the center of mass of an object which *is* accelerating, it is *still true* that the torque is equal to the rate of change of the angular momentum. Even if the center of mass is accelerating, we may still choose one special axis, namely, one passing through the center of mass, such that it will still be true that the torque is equal to the rate of change of angular momentum around that axis. Thus the theorem that torque equals the rate of change of angular momentum is true in two general cases: (1) a fixed axis in inertial space, (2) an axis through the center of mass, even though the object may be accelerating.

19-2 Locating the center of mass

The mathematical techniques for the calculation of centers of mass are in the province of a mathematics course, and such problems provide good exercise in integral calculus. After one has learned calculus, however, and wants to know how to locate centers of mass, it is nice to know certain tricks which can be used to do so. One such trick makes use of what is called the theorem of Pappus. It works like this: if we take any closed area in a plane and generate a solid by moving it through space such that each point is always moved perpendicular to the plane of the area, the resulting solid has a total volume equal to the area of the cross section times the distance that the center of mass moved! Certainly this is true if we move the area in a straight line perpendicular to itself, but if we move it in a circle or in some other curve, then it generates a rather peculiar volume. For a curved path, the outside goes around farther, and the inside goes around less, and these effects balance out. So if we want to locate the center of mass of a plane sheet of uniform density, we can remember that the volume generated by spinning it about an axis is the distance that the center of mass goes around, times the area of the sheet.

For example, if we wish to find the center of mass of a right triangle of base D and height H (Fig. 19-2), we might solve the problem in the following way. Imagine an axis along H , and rotate the triangle about that axis through a full 360 degrees. This generates a cone. The distance that the x -coordinate of the center of mass has moved is $2\pi x$. The area which is being moved is the area of the triangle, $\frac{1}{2}HD$. So the x -distance of the center of mass times the area of the triangle is the volume swept out, which is of course $\pi D^2 H/3$. Thus

Let θ_1 and θ_2 to be two different angles of projection for same range.



$$R = \frac{u^2 \sin 2\theta}{g}, \text{ since } R_1 = R_2$$

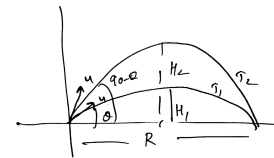
$$\sin 2\theta_1 = \sin 2\theta_2 \quad \sin(180 - A) = \sin A$$

$$\sin 2\theta_1 = \sin(180 - 2\theta_2)$$

$$2\theta_1 = 180 - 2\theta_2 \quad ; \quad \theta_1 = 90 - \theta_2$$

$$\theta_1 + \theta_2 = 90^\circ$$

e.g. for same speed at angles of projection 30° and 60° range is same. Also at 15° and 75° range is same.



\Rightarrow In the above situation when $R_1 = R_2$

$$T_1 = \frac{2u \sin \theta}{g} \quad ; \quad T_2 = \frac{2u \sin(90 - \theta)}{g}$$

$$\frac{T_1}{T_2} = \frac{\sin \theta}{\cos \theta} = \tan \theta \quad [\sin(90 - \theta) = \cos \theta]$$

$$H_1 = \frac{u^2 \sin^2 \theta}{2g} \quad ; \quad H_2 = \frac{u^2 \sin^2(90 - \theta)}{2g} \quad ; \quad \frac{H_1}{H_2} = \frac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta$$

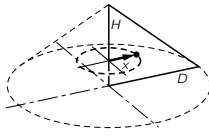


Fig. 19-2. A right triangle and a right circular cone generated by rotating the triangle.

$(2\pi x)(\frac{1}{2}HD) = \pi D^2 H/3$, or $x = D/3$. In a similar manner, by rotating about the other axis, or by symmetry, we find $y = H/3$. In fact, the center of mass of any uniform triangular area is where the three medians, the lines from the vertices through the centers of the opposite sides, all meet. That point is $1/3$ of the way along each median. *Clue:* Slice the triangle up into a lot of little pieces, each parallel to a base. Note that the median line bisects every piece, and therefore the center of mass must lie on this line.

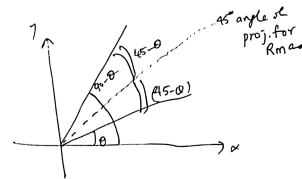
Now let us try a more complicated figure. Suppose that it is desired to find the position of the center of mass of a uniform semicircular disc—a disc sliced in half. Where is the center of mass? For a full disc, it is at the center, of course, but a half-disc is more difficult. Let r be the radius and x be the distance of the center of mass from the straight edge of the disc. Spin it around this edge as axis to generate a sphere. Then the center of mass has gone around $2\pi x$, the area is $\pi r^2/2$ (because it is only half a circle). The volume generated is, of course, $4\pi r^3/3$, from which we find that

$$(2\pi x)(\frac{1}{2}\pi r^2) = 4\pi r^3/3,$$

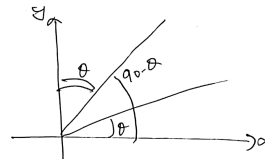
or

$$x = 4r/3\pi.$$

There is another theorem of Pappus which is a special case of the above one, and therefore equally true. Suppose that, instead of the solid semicircular disc, we have a semicircular piece of wire with uniform mass density along the wire, and we want to find its center of mass. In this case there is no mass in the interior, only on the wire. Then it turns out that the *area* which is swept by a plane curved line, when it moves as before, is the distance that the center of mass moves times the *length* of the line. (The line can be thought of as a very narrow area, and the previous theorem can be applied to it.)



Two different angles of projection for same range are equally inclined to the angle of projection for maximum range i.e. 45° .



The different angles of projection for same range are equally inclined to the vertical and horizontal

Equation for path of a projectile

$$y = x \tan \theta - \frac{1}{2} g \frac{x^2}{u^2 \cos^2 \theta}$$

$$y = x \tan \theta \left[1 - \frac{x}{\frac{2u^2 \sin \theta \cos \theta}{g}} \right]$$

$$y = x \tan \theta \left[1 - \frac{x}{R} \right]$$

19-3 Finding the moment of inertia

Now let us discuss the problem of finding the *moments of inertia* of various objects. The formula for the moment of inertia about the z -axis of an object is

$$I = \sum m_i(x_i^2 + y_i^2)$$

or

$$I = \int (x^2 + y^2) dm = \int (x^2 + y^2) \rho dV. \quad (19.4)$$

That is, we must sum the masses, each one multiplied by the square of its distance $(x_i^2 + y_i^2)$ from the axis. Note that it is not the three-dimensional distance, only the two-dimensional distance squared, even for a three-dimensional object. For the most part, we shall restrict ourselves to two-dimensional objects, but the formula for rotation about the z -axis is just the same in three dimensions.

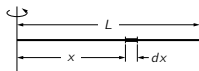


Fig. 19-3. A straight rod of length L rotating about an axis through one end.

As a simple example, consider a rod rotating about a perpendicular axis through one end (Fig. 19-3). Now we must sum all the masses times the x -distances squared (the y 's being all zero in this case). What we mean by "the sum," of course, is the integral of x^2 times the little elements of mass. If we divide the rod into small elements of length dx , the corresponding elements of mass are proportional to dx , and if dx were the length of the whole rod the mass would be M . Therefore

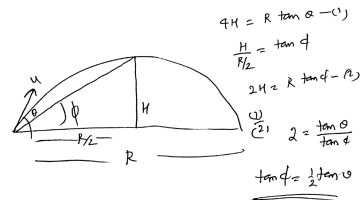
$$dm = M dx/L$$

and so

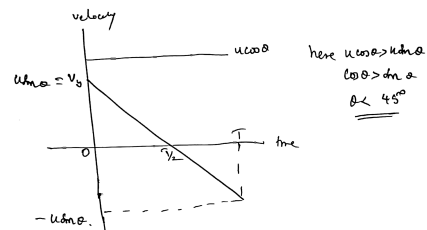
$$I = \int_0^L x^2 \frac{M dx}{L} = \frac{M}{L} \int_0^L x^2 dx = \frac{ML^2}{3}. \quad (19.5)$$

The dimensions of moment of inertia are always mass times length squared, so all we really had to work out was the factor $1/3$.

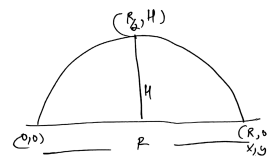
Relation between angle of projection ' θ ' and angle of elevation ϕ , at maximum height



Velocity time graph



If the path of a projectile is given by the equation $y = ax - bx^2$ find R , H , T



Now what is I if the rotation axis is at the center of the rod? We could just do the integral over again, letting x range from $-\frac{1}{2}L$ to $+\frac{1}{2}L$. But let us notice a few things about the moment of inertia. We can imagine the rod as two rods, each of mass $M/2$ and length $L/2$; the moments of inertia of the two small rods are equal, and are both given by the formula (19.5). Therefore the moment of inertia is

$$I = \frac{2(M/2)(L/2)^2}{3} = \frac{ML^2}{12}. \quad (19.6)$$

Thus it is much easier to turn a rod about its center, than to swing it around an end.

Of course, we could go on to compute the moments of inertia of various other bodies of interest. However, while such computations provide a certain amount of important exercise in the calculus, they are not basically of interest to us as such. There is, however, an interesting theorem which is very useful. Suppose we have an object, and we want to find its moment of inertia around some axis. That means we want the inertia needed to carry it by rotation about that axis. Now if we support the object on pivots at the center of mass, so that the object does not turn as it rotates about the axis (because there is no torque on it from inertial effects, and therefore it will not turn when we start moving it), then the forces needed to swing it around are the same as though all the mass were concentrated at the center of mass, and the moment of inertia would be simply $I_1 = MR_{CM}^2$, where R_{CM} is the distance from the axis to the center of mass. But of course that is not the right formula for the moment of inertia of an object which is really being rotated as it revolves, because not only is the center of it moving in a circle, which would contribute an amount I_1 to the moment of inertia, but also we must turn it about its center of mass. So it is not unreasonable that we must add to I_1 the moment of inertia I_c about the center of mass. So it is a good guess that the total moment of inertia about any axis will be

$$I = I_c + MR_{CM}^2. \quad (19.7)$$

This theorem is called the *parallel-axis theorem*, and may be easily proved. The moment of inertia about any axis is the mass times the sum of the x_i 's and the y_i 's, each squared: $I = \sum (x_i^2 + y_i^2)m_i$. We shall concentrate on the x 's, but of course the y 's work the same way. Now x is the distance of a particular point mass from the origin, but let us consider how it would look if we measured x' from the CM, instead of x from the origin. To get ready for this analysis, we

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$$y = ax - bx^2, \text{ if } y = 0, x = R$$

$$0 = ax - bx^2$$

$$ax = bx^2$$

$$x = R = a/b$$

$$\text{Compare } y = x \tan \theta [1 - x/R]$$

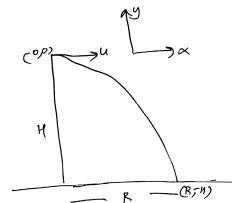
$$\tan \theta = a$$

$$4H = R \tan \theta$$

$$4H = \frac{a}{b}, \quad H = \frac{a^2}{4b}$$

$$H = \frac{T^2 g}{8} = \frac{a^2}{4b}$$

Horizontal projection



A particle is projected horizontally with a velocity u from a height H . It follows a parabolic path and strikes the ground, horizontal component of velocity u remains constant vertical component is subjected to acceleration due to gravity.

$$u_x = u, \quad u_y = 0 \quad \vec{u} = u\hat{i}$$

$$a_x = 0, a_y = -g \quad \vec{a} = -g\hat{j}$$

To find time of flight we consider the motion along y-axis

$$S_y = u_y t + \frac{1}{2} a_y t^2$$

$$-H = 0 \cdot t + \frac{1}{2} (-g) t^2$$

write

$$x_i = x'_i + X_{\text{CM}}.$$

Then we just square this to find

$$x_i^2 = x_i'^2 + 2X_{\text{CM}}x'_i + X_{\text{CM}}^2.$$

So, when this is multiplied by m_i and summed over all i , what happens? Taking the constants outside the summation sign, we get

$$I_x = \sum m_i x_i'^2 + 2X_{\text{CM}} \sum m_i x'_i + X_{\text{CM}}^2 \sum m_i.$$

The third sum is easy; it is just MX_{CM}^2 . In the second sum there are two pieces, one of them is $\sum m_i x'_i$, which is the total mass times the x' -coordinate of the center of mass. But this contributes nothing, because x' is *measured* from the center of mass, and in these axes the average position of all the particles, weighted by the masses, is zero. The first sum, of course, is the x part of I_c . Thus we arrive at Eq. (19.7), just as we guessed.

Let us check (19.7) for one example. Let us just see whether it works for the rod. For an axis through one end, the moment of inertia should be $ML^2/3$, for we calculated that. The center of mass of a rod, of course, is in the center of the rod, at a distance $L/2$. Therefore we should find that $ML^2/3 = ML^2/12 + M(L/2)^2$. Since one-quarter plus one-twelfth is one-third, we have made no fundamental error.

Incidentally, we did not really need to use an integral to find the moment of inertia (19.5). If we simply assume that it is ML^2 times γ , an unknown coefficient, and then use the argument about the two halves to get $\frac{1}{4}\gamma$ for (19.6), then from our argument about transferring the axes we could prove that $\gamma = \frac{1}{4}\gamma + \frac{1}{4}$, so γ must be $1/3$. There is always another way to do it!

In applying the parallel-axis theorem, it is of course important to remember that the axis for I_c *must be parallel* to the axis about which the moment of inertia is wanted.

One further property of the moment of inertia is worth mentioning because it is often helpful in finding the moment of inertia of certain kinds of objects. This property is that if one has a *plane figure* and a set of coordinate axes with origin in the plane and z -axis perpendicular to the plane, then the moment of inertia of this figure about the z -axis is equal to the sum of the moments of inertia about

$$H = \frac{1}{2}gt^2$$

$$t = \sqrt{\frac{2H}{g}}$$

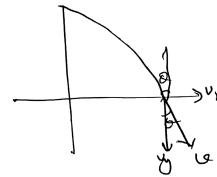
This is same as the time take by a dropped body to reach the ground dropped from rest.

To find range R, consider the horizontal motion,

$$S_x = u_x t$$

$$R = ut \quad R = u \sqrt{\frac{2H}{g}}$$

Velocity with which it hits the ground



$$V_x = u$$

$$V_y^2 = u_y^2 + 2a_y y$$

$$V_y^2 = 0 + 2(-g)(-H)$$

$$V_y = \sqrt{2gH} \quad V = \sqrt{V_x^2 + V_y^2} \quad ; \quad \tan \theta = \frac{V_y}{V_x}$$

θ is the angle made by the velocity with vertical.

the x - and y -axes. This is easily proved by noting that

$$I_x = \sum m_i(y_i^2 + z_i^2) = \sum m_i y_i^2$$

(since $z_i = 0$). Similarly,

$$I_y = \sum m_i(x_i^2 + z_i^2) = \sum m_i x_i^2,$$

but

$$I_z = \sum m_i(x_i^2 + y_i^2) = \sum m_i x_i^2 + \sum m_i y_i^2 \\ = I_x + I_y.$$

As an example, the moment of inertia of a uniform rectangular plate of mass M , width w , and length L , about an axis perpendicular to the plate and through its center is simply

$$I = M(w^2 + L^2)/12,$$

because its moment of inertia about an axis in its plane and parallel to its length is $Mw^2/12$, i.e., just as for a rod of length w , and the moment of inertia about the other axis in its plane is $ML^2/12$, just as for a rod of length L .

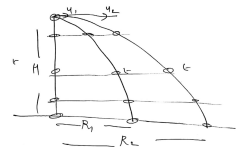
To summarize, the moment of inertia of an object about a given axis, which we shall call the z -axis, has the following properties:

- (1) The moment of inertia is

$$I_z = \sum_i m_i(x_i^2 + y_i^2) = \int (x^2 + y^2) dm.$$

- (2) If the object is made of a number of parts, each of whose moment of inertia is known, the total moment of inertia is the sum of the moments of inertia of the pieces.
- (3) The moment of inertia about any given axis is equal to the moment of inertia about a parallel axis through the CM plus the total mass times the square of the distance from the axis to the CM.
- (4) If the object is a plane figure, the moment of inertia about an axis perpendicular to the plane is equal to the sum of the moments of inertia about any two mutually perpendicular axes lying in the plane and intersecting at the perpendicular axis.

Three projectiles one is dropped, other two are thrown with some velocities are shown below. Position are drawn at different intervals.



t is same for all

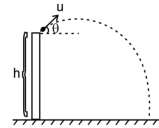
$$R_1 = u_1 t \quad R_2 = u_2 t$$

$$\frac{R_1}{R_2} = \frac{u_1}{u_2}$$

All of them reach the ground at the same time. Their vertical motion are identical because they have same initial vertical velocity (zero in this case) and same acceleration.

Path of a projectile with respect to another projectile is a straight line. Their relative acceleration is zero.

Projectile Projected from the top of a building (Projected upwards)



Horizontal motion

$$u_x = u \cos \theta$$

$$a_x = 0$$

Vertical motion

$$u_y = u \sin \theta$$

$$a_y = -g$$

Time of flight (T)

at $t = T$, $s_y = -h$

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$-h = -u \sin \theta T + \frac{1}{2} (-g) T^2$$

Solving this equation 'T' will be obtained

The moments of inertia of a number of elementary shapes having uniform mass densities are given in Table 19-1, and the moments of inertia of some other objects, which may be deduced from Table 19-1, using the above properties, are given in Table 19-2.

Table 19-1

Object	z-axis	I_z
Thin rod, length L	\perp rod at center	$ML^2/12$
Thin concentric circular ring, radii r_1 and r_2	\perp ring at center	$M(r_1^2 + r_2^2)/2$
Sphere, radius r	through center	$2Mr^2/5$

Table 19-2

Object	z-axis	I_z
Rect. sheet, sides a, b	$\parallel b$ at center	$Ma^2/12$
Rect. sheet, sides a, b	\perp sheet at center	$M(a^2 + b^2)/12$
Thin annular ring, radii r_1, r_2	any diameter	$M(r_1^2 + r_2^2)/4$
Rect. parallelepiped, sides a, b, c	$\parallel c$, through center	$M(a^2 + b^2)/12$
Rt. circ. cyl., radius r , length L	$\parallel L$, through center	$Mr^2/2$
Rt. circ. cyl., radius r , length L	$\perp L$, through center	$M(r^2/4 + L^2/12)$

19-4 Rotational kinetic energy

Now let us go on to discuss dynamics further. In the analogy between linear motion and angular motion that we discussed in Chapter 18, we used the work theorem, but we did not talk about kinetic energy. What is the kinetic energy of a rigid body, rotating about a certain axis with an angular velocity ω ? We can immediately guess the correct answer by using our analogies. The moment of inertia corresponds to the mass, angular velocity corresponds to velocity, and so

Range (R)

at $t = T, s_x = R$

$$s_x = u_x t + \frac{1}{2} a_x t^2$$

$$R = u \cos \theta \times T + 0$$

$R = u \cos \theta \times T$

Projectile Projected from the top of a building (Projected downwards)

Horizontal motion

$u_x = u \cos \theta$
 $a_x = 0$

Vertical motion

$u_y = u \sin \theta$
 $a_y = -g$

Time of flight (T)

at $t = T, s_y = -h$

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$-h = -u \sin \theta T + \frac{1}{2} (-g) T^2$$

Solving this equation 'T' will be obtained

Range $R = u \cos \theta \times T$

Projection From Inclined Plane

the kinetic energy ought to be $\frac{1}{2}I\omega^2$, and indeed it is, as will now be demonstrated. Suppose the object is rotating about some axis so that each point has a velocity whose magnitude is ωr_i , where r_i is the radius from the particular point to the axis. Then if m_i is the mass of that point, the total kinetic energy of the whole thing is just the sum of the kinetic energies of all of the little pieces:

$$T = \frac{1}{2} \sum m_i v_i^2 = \frac{1}{2} \sum m_i (r_i \omega)^2.$$

Now ω^2 is a constant, the same for all points. Thus

$$T = \frac{1}{2} \omega^2 \sum m_i r_i^2 = \frac{1}{2} I \omega^2. \quad (19.8)$$

At the end of Chapter 18 we pointed out that there are some interesting phenomena associated with an object which is not rigid, but which changes from one rigid condition with a definite moment of inertia, to another rigid condition. Namely, in our example of the turntable, we had a certain moment of inertia I_1 with our arms stretched out, and a certain angular velocity ω_1 . When we pulled our arms in, we had a different moment of inertia, I_2 , and a different angular velocity, ω_2 , but again we were "rigid." The angular momentum remained constant, since there was no torque about the vertical axis of the turntable. This means that $I_1 \omega_1 = I_2 \omega_2$. Now what about the energy? That is an interesting question. With our arms pulled in, we turn faster, but our moment of inertia is less, and it looks as though the energies might be equal. But they are not, because what *does* balance is $I\omega$, not $I\omega^2$. So if we compare the kinetic energy before and after, the kinetic energy before is $\frac{1}{2}I_1\omega_1^2 = \frac{1}{2}L\omega_1$, where $L = I_1\omega_1 = I_2\omega_2$ is the angular momentum. Afterward, by the same argument, we have $T = \frac{1}{2}L\omega_2$ and since $\omega_2 > \omega_1$ the kinetic energy of rotation is greater than it was before. So we had a certain energy when our arms were out, and when we pulled them in, we were turning faster and had more kinetic energy. What happened to the theorem of the conservation of energy? Somebody must have done some work. We did work! When did we do any work? When we move a weight horizontally, we do not do any work. If we hold a thing out and pull it in, we do not do any work. But that is when we are not rotating! When we *are* rotating, there is centrifugal force on the weights. They are trying to fly out, so when we are going around we have to pull the weights in against the centrifugal force. So, the work we do against the centrifugal force ought to agree with the difference in rotational energy, and of course it does. That is where the extra kinetic energy comes from.

$$u_x = u \cos(\theta - \alpha)$$

$$u_y = u \sin(\theta - \alpha)$$

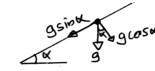
$$a_x = -g \sin \alpha$$

$$a_y = -g \cos \alpha$$

For motion from 'O' to 'A' the displacement along the y-direction is zero.

$$\therefore y = u_y t + \frac{a_y t^2}{2}$$

$$0 = u \sin(\theta - \alpha) T - \frac{g \cos \alpha T^2}{2}$$



$$T = \frac{2u \sin(\theta - \alpha)}{g \cos \alpha} \quad \text{or} \quad T = \frac{2U_y}{|a_y|}$$

Maximum Height from inclined surface (H)

$$V_y^2 - U_y^2 = 2a_y y$$

$$\therefore 0 - U_y^2 = -2a_y H$$

$$H = \frac{U_y^2}{2a_y}$$

$$H = \frac{U^2 \sin^2(\theta - \alpha)}{2g \cos \alpha}$$

Horizontal displacement OB = (u cos θ) T

$$OB = u \cos \theta \times \frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$$

Range Along the inclined surface

There is still another interesting feature which we can treat only descriptively, as a matter of general interest. This feature is a little more advanced, but is worth pointing out because it is quite curious and produces many interesting effects.

Consider that turntable experiment again. Consider the body and the arms separately, from the point of view of the man who is rotating. After the weights are pulled in, the whole object is spinning faster, but observe, *the central part of the body is not changed*, yet it is turning faster after the event than before. So, if we were to draw a circle around the inner body, and consider only objects inside the circle, *their* angular momentum would *change*; they are going faster. Therefore there must be a torque exerted on the body while we pull in our arms. No torque can be exerted by the centrifugal force, because that is radial. So that means that among the forces that are developed in a rotating system, centrifugal force is not the entire story, *there is another force*. This other force is called *Coriolis force*, and it has the very strange property that when we move something in a rotating system, it seems to be pushed sidewise. Like the centrifugal force, it is an apparent force. But if we live in a system that is rotating, and move something radially, we find that we must also push it sidewise to move it radially. This sidewise push which we have to exert is what turned our body around.

Now let us develop a formula to show how this Coriolis force really works. Suppose Moe is sitting on a carousel that appears to him to be stationary. But from the point of view of Joe, who is standing on the ground and who knows the right laws of mechanics, the carousel is going around. Suppose that we have drawn a radial line on the carousel, and that Moe is moving some mass radially along this line. We would like to demonstrate that a sidewise force is required to do that. We can do this by paying attention to the angular momentum of the mass. It is always going around with the same angular velocity ω , so that the angular momentum is

$$L = mv_{\text{tang}}r = m\omega r \cdot r = m\omega r^2.$$

So when the mass is close to the center, it has relatively little angular momentum, but if we move it to a new position farther out, if we increase r , m has more angular momentum, so a *torque must be exerted* in order to move it along the radius. (To walk along the radius in a carousel, one has to lean over and push sidewise. Try it sometime.) The torque that is required is the rate of change of L with time as m moves along the radius. If m moves only along the radius, omega

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$$R = OA = \frac{OB}{\cos \alpha}$$

$$R = \frac{2u^2 \cos \theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$$

$$R = \frac{u^2 2 \cos \theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$$

$$R = \frac{u^2 [\sin(2\theta - \alpha) - \sin \alpha]}{g \cos^2 \alpha}$$

$$2 \cos A \sin B = \sin(A + B) - \sin(A - B)$$

Range R is maximum, when $\sin(2\theta - \alpha) = 1$

$$2\theta - \alpha = \frac{\pi}{2}$$

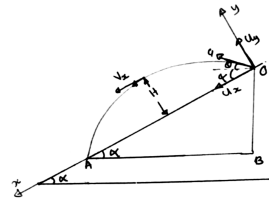
$$\theta = \frac{\pi}{4} + \frac{\alpha}{2}$$

$$\theta - \alpha = \frac{\pi}{4} + \frac{\alpha}{2} - \alpha = \frac{\pi - 2\alpha}{4}$$

$$R_{\text{max}} = \frac{u^2 [1 - \sin \alpha]}{g \cos^2 \alpha} = \frac{u^2 (1 - \sin \alpha)}{g (1 - \sin^2 \alpha)}; \quad R_{\text{max}} = \frac{u^2 [1 - \sin \alpha]}{g [1 + \sin \alpha] [1 - \sin \alpha]}$$

$$R_{\text{max}} = \frac{u^2}{g [1 + \sin \alpha]}$$

Projectile Motion from an Inclined Plane



$$u_x = u \cos(\theta + \alpha)$$

$$u_y = u \sin(\theta + \alpha)$$

$$a_x = g \sin \alpha, a_y = -g \cos \alpha$$

stays constant, so that the torque is

$$\tau = F_c r = \frac{dL}{dt} = \frac{d(m\omega r^2)}{dt} = 2m\omega r \frac{dr}{dt},$$

where F_c is the Coriolis force. What we really want to know is what sidewise force has to be exerted by Moe in order to move m out at speed $v_r = dr/dt$. This is $F_c = \tau/r = 2m\omega v_r$.

Now that we have a formula for the Coriolis force, let us look at the situation a little more carefully, to see whether we can understand the origin of this force from a more elementary point of view. We note that the Coriolis force is the same at every radius, and is evidently present even at the origin! But it is especially easy to understand it at the origin, just by looking at what happens from the inertial system of Joe, who is standing on the ground. Figure 19-4 shows three successive views of m just as it passes the origin at $t = 0$. Because of the rotation of the carousel, we see that m does not move in a straight line, but in a *curved path* tangent to a diameter of the carousel where $r = 0$. In order for m to go in a curve, there must be a force to accelerate it in absolute space. This is the Coriolis force.

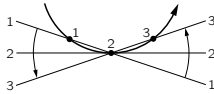


Fig. 19-4. Three successive views of a point moving radially on a rotating turntable.

This is not the only case in which the Coriolis force occurs. We can also show that if an object is moving with constant speed around the circumference of a circle, there is also a Coriolis force. Why? Moe sees a velocity v_M around the circle. On the other hand, Joe sees m going around the circle with the velocity $v_J = v_M + \omega r$, because m is also carried by the carousel. Therefore we know what the force really is, namely, the total centripetal force due to the velocity v_J , or mv_J^2/r ; that is the actual force. Now from Moe's point of view, this centripetal force has three pieces. We may write it all out as follows:

$$F_r = -\frac{mv_J^2}{r} = -\frac{mv_M^2}{r} - 2mv_M\omega - m\omega^2 r.$$

19-15

When the object hits on the inclined plane

$$y=0, \therefore y=U_y t + \frac{a_y t^2}{2}$$

$$O = u \sin(\theta + \alpha) T - \frac{g \cos \alpha}{2} T^2$$

$$T = \frac{2u \sin(\theta + \alpha)}{g \cos \alpha} = \frac{2U_y}{a_y}$$

$$BA = (U_x \cos \theta) T = u \cos \theta \times \frac{2u \sin(\theta + \alpha)}{g \cos \alpha}$$

Range along the inclined surface

$$R = OA = \frac{AB}{\cos \alpha} = \frac{2u^2 \cos \theta \sin(\theta + \alpha)}{g \cos^2 \alpha}$$

$$R = \frac{2u^2 \cos \theta \sin(\theta + \alpha)}{g \cos^2 \alpha}$$

$$R = \frac{u^2}{g \cos^2 \alpha} [\sin(2\theta + \alpha) - \sin(-\alpha)]$$

$$R = \frac{u^2 [\sin(2\theta + \alpha) + \sin \alpha]}{g \cos^2 \alpha}$$

$$\text{For maximum range } \sin(2\theta + \alpha) = 1, \quad 2\theta + \alpha = \frac{\pi}{2}$$

$$\begin{aligned} \theta + \alpha &= \frac{\pi - 2\alpha}{4} + \alpha \\ \theta + \alpha &= \frac{\pi + 2\alpha}{4} \end{aligned}$$

$$\theta = \frac{\pi - 2\alpha}{4} \quad R_{\max} = \frac{u^2 [1 + \sin \alpha]}{g \cos^2 \alpha}$$

$$R_{\max} = \frac{u^2 (1 + \sin \alpha)}{g (1 - \sin^2 \alpha)} = \frac{u^2 (1 + \sin \alpha)}{g [1 + \sin \alpha][1 - \sin \alpha]}$$

$$R_{\max} = \frac{u^2}{g [1 - \sin \alpha]}$$

Maximum height (H) from the inclined surface.

At maximum height $V_y = 0$

$$\therefore V_y^2 - u_y^2 = 2a_y y \text{ becomes } 0 - u_y^2 = 2a_y H$$

$$H = \frac{U_y^2}{2a_y} = \frac{u^2 \sin^2(\theta + \alpha)}{2g \cos \alpha}$$

Now, F_r is the force that Moe would see. Let us try to understand it. Would Moe appreciate the first term? “Yes,” he would say, “even if I were not turning, there would be a centripetal force if I were to run around a circle with velocity v_M .” This is simply the centripetal force that Moe would expect, having nothing to do with rotation. In addition, Moe is quite aware that there is another centripetal force that would act even on objects which are standing still on his carousel. This is the third term. But there is another term in addition to these, namely the second term, which is again $2m\omega v$. The Coriolis force F_c was tangential when the velocity was radial, and now it is radial when the velocity is tangential. In fact, one expression has a minus sign relative to the other. The force is always in the same direction, relative to the velocity, no matter in which direction the velocity is. The force is at right angles to the velocity, and of magnitude $2m\omega v$.

Note : For a given speed, the direction which gives the maximum range of the projectile on an inclined plane, bisects the angle between the incline and the vertical, for upward or downward projection.

Standard results for projectile motion on an incline plane

	Up the incline	Down the incline
Range	$\frac{2u^2 \cos \theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$	$\frac{2u^2 \cos \theta \sin(\theta + \alpha)}{g \cos^2 \alpha}$
Time of flight	$\frac{2u \sin(\theta - \alpha)}{g \cos \alpha}$	$\frac{2u \sin(\theta + \alpha)}{g \cos \alpha} = \frac{2u_r}{a_y}$
Maximum Range	$\frac{u^2}{g [1 + \sin \alpha]}$	$\frac{u^2}{g [1 - \sin \alpha]}$
Angle of projection for maximum range (from inclined surface)	$\frac{\pi - 2\alpha}{4}$	$\frac{\pi + 2\alpha}{4}$