24

Transients

24-1 The energy of an oscillator

Although this chapter is entitled "transients," certain parts of it are, in a way, part of the last chapter on forced oscillation. One of the features of a forced oscillation which we have not yet discussed is the *energy* in the oscillation. Let us now consider that energy.

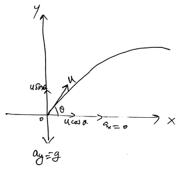
In a mechanical oscillator, how much kinetic energy is there? It is proportional to the square of the velocity. Now we come to an important point. Consider an arbitrary quantity A, which may be the velocity or something else that we want to discuss. When we write $A = \hat{A}e^{i\omega t}$, a complex number, the true and honest A, in the physical world, is only the *real part*; therefore if, for some reason, we want to use the *square* of A, it is not right to square the complex number and then take the real part, because the real part of the square of a complex number is not just the square of the real part, but also involves the *imaginary* part. So when we wish to find the energy we have to get away from the complex notation for a while to see what the inner workings are.

Now the true physical A is the real part of $A_0e^{i(\omega t+\Delta)}$, that is, $A = A_0 \cos(\omega t + \Delta)$, where $\hat{A}$, the complex number, is written as $A_0e^{i\Delta}$. Now the square of this real physical quantity is $A^2 = A_0^2 \cos^2(\omega t + \Delta)$. The square of the quantity, then, goes up and down from a maximum to zero, like the square of the cosine. The square of the cosine has a maximum of 1 and a minimum of 0, and its average value is 1/2.

In many circumstances we are not interested in the energy at any specific moment during the oscillation; for a large number of applications we merely want the average of A^2, the *mean* of the square of A over a period of time large compared with the period of oscillation. In those circumstances, the average of the cosine squared may be used, so we have the following theorem: if A is represented by a complex number, then the mean of A^2 is equal to $\frac{1}{2}A_0^2$. Now A_0^2

PROJECTILE MOTION

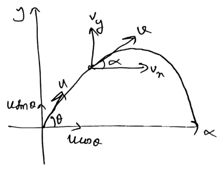
An object that is in fight after being thrown or projected is called a projectile. The motion of a projectile may be thought as the result of two separate, simultaneously occurring perpendicular components of motions. One component is along horizontal direction without any acceleration and other is along vertical direction with constant acceleration due to gravity. It was Galileo who fist stated this independency of the horizontal and the vertical components of projectile motion.



A particle is projected with a velocity u (velocity of projection) making using a angle θ with the horizontal. θ is known as angle of projection. Only force that control the projectile is gravity. We will neglect air resistance. Projectile is subjected to acceleration due to gravity $\vec{a} = -g\hat{j}$ $(a_x = 0,\ a_y = -g)$.

$u\cos\theta$ is the horizontal component of velocity which remains constant. $u\sin\theta$ is initial vertical component of velocity. O is the point of projection which is taken as origin.

The velocity of the projectile after t second



is the square of the magnitude of the complex $\hat{A}$. (This can be written in many ways—some people like to write $|\hat{A}|^2$; others write, $\hat{A}\hat{A}^*$, $\hat{A}$ times its complex conjugate.) We shall use this theorem several times.

Now let us consider the energy in a forced oscillator. The equation for the forced oscillator is

$$m\,d^2x/dt^2 + \gamma m\,dx/dt + m\omega_0^2 x = F(t). \tag{24.1}$$

In our problem, of course, $F(t)$ is a cosine function of t. Now let us analyze the situation: how much work is done by the outside force F? The work done by the force per second, i.e., the power, is the force times the velocity. (We know that the differential work in a time dt is $F\,dx$, and the power is $F\,dx/dt$.) Thus

$$P = F\,\frac{dx}{dt} = m\left[\left(\frac{dx}{dt}\right)\left(\frac{d^2x}{dt^2}\right) + \omega_0^2 x\left(\frac{dx}{dt}\right)\right] + \gamma m\left(\frac{dx}{dt}\right)^2. \tag{24.2}$$

But the first two terms on the right can also be written as $d/dt[\frac{1}{2}m(dx/dt)^2 + \frac{1}{2}m\omega_0^2x^2]$, as is immediately verified by differentiating. That is to say, the term in brackets is a pure derivative of two terms that are easy to understand—one is the kinetic energy of motion, and the other is the potential energy of the spring. Let us call this quantity the *stored energy*, that is, the energy stored in the oscillation. Suppose that we want the average power over many cycles when the oscillator is being forced and has been running for a long time. In the long run, the stored energy does not change—its derivative gives zero average effect. In other words, if we average the power in the long run, *all the energy ultimately ends up in the resistive term* $\gamma m(dx/dt)^2$. There is *some* energy stored in the oscillation, but that does not change with time, if we average over many cycles. Therefore the mean power $\langle P\rangle$ is

$$\langle P\rangle = \langle \gamma m(dx/dt)^2\rangle. \tag{24.3}$$

Using our method of writing complex numbers, and our theorem that $\langle A^2\rangle = \frac{1}{2}A_0^2$, we may find this mean power. Thus if $x = \hat{x}e^{i\omega t}$, then $dx/dt = i\omega\hat{x}e^{i\omega t}$. Therefore, in these circumstances, the average power could be written as

$$\langle P\rangle = \tfrac{1}{2}\gamma m\omega^2 x_0^2. \tag{24.4}$$

In the notation for electrical circuits, dx/dt is replaced by the current I (I is dq/dt, where q corresponds to x), and $m\gamma$ corresponds to the resistance R. Thus

$V_y = u_y + a_y t$

$V_y = u\sin\theta - gt$

$V = \sqrt{V_x^2 + V_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$

$V = \sqrt{u^2 - 2u\sin\theta gt + g^2t^2}$

Velocity v make an angle α with horizontal such that, $\tan\alpha = \frac{\text{opposite side}}{\text{adj.side}} = \frac{V_y}{V_x} = \frac{u\sin\theta - gt}{u\cos\theta}$

In vector form $\vec{a} = -g\hat{j}$

$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$

$\vec{v} = \vec{u} + \vec{a}t$

$\vec{v} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} - gt\hat{j}$

$\vec{v} = u\cos\theta\,\hat{i} + (u\sin\theta - gt)\hat{j}$

$|\vec{v}| = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$, $\tan\alpha = \frac{u\sin\theta - gt}{u\cos\theta}$

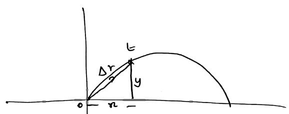
To find the displacement of the projectile after t seconds

along x axis

disp = velocity × time

$\boxed{x = (u\cos\theta)t}$

This is the equation for x-coordinate of the projectile at any time t.



along y axis $y = u_y t + \frac{a_y t^2}{2}$

$\boxed{y = u\sin\theta\,t - \frac{1}{2}gt^2}$ Equation for y coordinate or height of projectile at any time t.

the rate of the energy loss—the power used up by the forcing function—is the resistance in the circuit times the average square of the current:

$$\langle P \rangle = R\langle I^2 \rangle = R \cdot \tfrac{1}{2} I_0^2. \tag{24.5}$$

This energy, of course, goes into heating the resistor; it is sometimes called the heating loss or the Joule heating.

Another interesting feature to discuss is how much energy is *stored*. That is not the same as the power, because although power was at first used to store up some energy, after that the system keeps on absorbing power, insofar as there are any heating (resistive) losses. At any moment there is a certain amount of stored energy, so we would like to calculate the mean stored energy $\langle E \rangle$ also. We have already calculated what the average of $(dx/dt)^2$ is, so we find

$$\begin{aligned} \langle E \rangle &= \tfrac{1}{2} m \langle (dx/dt)^2 \rangle + \tfrac{1}{2} m\omega_0^2 \langle x^2 \rangle \\ &= \tfrac{1}{2} m(\omega^2 + \omega_0^2)\tfrac{1}{2} x_0^2. \end{aligned} \tag{24.6}$$

Now, when an oscillator is very efficient, and if ω is near ω_0, so that $|\hat{x}|$ is large, the stored energy is very high—we can get a large stored energy from a relatively small force. The force does a great deal of work in getting the oscillation going, but then to keep it steady, all it has to do is to fight the friction. The oscillator can have a great deal of energy if the friction is very low, and even though it is oscillating strongly, not much energy is being lost. The efficiency of an oscillator can be measured by how much energy is stored, compared with how much work the force does per oscillation.

How does the stored energy compare with the amount of work that is done in one cycle? This is called the Q of the system, and Q is defined as 2π times the mean stored energy, divided by the work done per cycle. (If we were to say the work done per *radian* instead of per cycle, then the 2π disappears.)

$$Q = 2\pi \frac{\tfrac{1}{2} m(\omega^2 + \omega_0^2) \cdot \langle x^2 \rangle}{\gamma m\omega^2 \langle x^2 \rangle \cdot 2\pi/\omega} = \frac{\omega^2 + \omega_0^2}{2\gamma\omega}. \tag{24.7}$$

Q is not a very useful number unless it is very large. When it is relatively large, it gives a measure of how good the oscillator is. People have tried to define Q in the simplest and most useful way; various definitions differ a bit from one another, but if Q is very large, all definitions are in agreement. The most generally accepted definition is Eq. (24.7), which depends on ω. For a good oscillator, close

displacement = $r = \sqrt{x^2 + y^2}$

To find it in vector form we can use the equation $\vec{r} = x\hat{i} + y\hat{j}$ where $\vec{u} = u\sin\theta\hat{j}$, $\vec{a} = -g\hat{j}$

Equation for path of a projectile

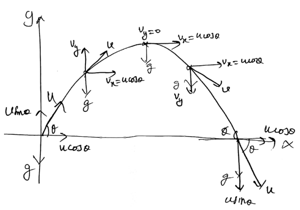
$x = u\cos\theta t$, $t = \dfrac{x}{u\cos\theta}$

$y = u\sin\theta t - \dfrac{1}{2} g t^2$

$y = u\sin\theta \dfrac{x}{u\cos\theta} - \dfrac{1}{2} g\left(\dfrac{x}{u\cos\theta}\right)^2$

$$\boxed{y = x\tan\theta - \frac{1}{2} g \frac{x^2}{u^2\cos^2\theta}}$$

This is the equation of a parabola. Thus the path of a projectile is parabola.



At the highest point of the projectile vertical component of velocity is zero. Horizontal component is $u\cos\theta$ because it remains constant. At the highest point speed of the projectile is minimum and purely horizontal and is equal to $u\cos\theta$. Angle between acceleration and instantaneous velocity decreases from $(90 + \theta)$ to $(90 - \theta)$

to resonance, we can simplify (24.7) a little by setting $\omega = \omega_0$, and we then have $Q = \omega_0/\gamma$, which is the definition of Q that we used before.

What is Q for an electrical circuit? To find out, we merely have to translate L for m, R for $m\gamma$, and $1/C$ for $m\omega_0^2$ (see Table 23-1). The Q at resonance is $L\omega/R$, where ω is the resonance frequency. If we consider a circuit with a high Q, that means that the amount of energy stored in the oscillation is very large compared with the amount of work done per cycle by the machinery that drives the oscillations.

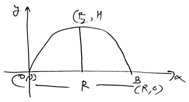
24-2 Damped oscillations

We now turn to our main topic of discussion: transients. By a *transient* is meant a solution of the differential equation when there is no force present, but when the system is not simply at rest. (Of course, if it is standing still at the origin with no force acting, that is a nice problem—it stays there!) Suppose the oscillation starts another way: say it was driven by a force for a while, and then we turn off the force. What happens then? Let us first get a rough idea of what will happen for a very high Q system. So long as a force is acting, the stored energy stays the same, and there is a certain amount of work done to maintain it. Now suppose we turn off the force, and no more work is being done; then the losses which are eating up the energy of the supply are no longer eating up its energy—there *is* no more driver. The losses will have to consume, so to speak, the energy that is stored. Let us suppose that $Q/2\pi = 1000$. Then the work done per cycle is 1/1000 of the stored energy. Is it not reasonable, since it is oscillating with no driving force, that in one cycle the system will still lose a thousandth of its energy E, which ordinarily would have been supplied from the outside, and that it will continue oscillating, always losing 1/1000 of its energy per cycle? So, as a guess, for a relatively high Q system, we would suppose that the following equation might be roughly right (we will later do it exactly, and it will turn out that it *was* right!):

$$dE/dt = -\omega E/Q. \tag{24.8}$$

This is rough because it is true only for large Q. In each radian the system loses a fraction $1/Q$ of the stored energy E. Thus in a given amount of time dt the energy will change by an amount $\omega\,dt/Q$, since the number of radians associated with the time dt is $\omega\,dt$. What is the frequency? Let us suppose that the system moves

Time of Flight of the projectile (T)



Consider the motion of the projectile along y-axis

$$S_y = u_y t + \frac{1}{2} a_y t^2$$

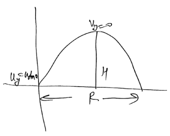
t = T

$$0 = u\sin\theta T - \frac{1}{2} g T^2$$

$$\boxed{T = \frac{2u\sin\theta}{g}} \text{ or } \boxed{T = \frac{2u_y}{g}}$$

Time of ascend = Time of descend = $\frac{u\sin\theta}{g}$

Maximum height of a projectile H



Consider the motion along y-axis

$$V_y^2 = u_y^2 + 2a_y g_y$$

$$0 = (u\sin\theta)^2 + 2(-gH)$$

$$2gH = (u\sin\theta)^2$$

$$\boxed{H = \frac{u^2\sin^2\theta}{2g}} \text{ or } \boxed{H = \frac{u_y^2}{2g}}$$

so nicely, with hardly any force, that if we let go it will oscillate at essentially the same frequency all by itself. So we will guess that ω is the resonant frequency ω_0. Then we deduce from Eq. (24.8) that the stored energy will vary as

$$E = E_0 e^{-\omega_0 t/Q} = E_0 e^{-\gamma t}. \tag{24.9}$$

This would be the measure of the *energy* at any moment. What would the formula be, roughly, for the amplitude of the oscillation as a function of the time? The same? No! The amount of energy in a spring, say, goes as the *square* of the displacement; the kinetic energy goes as the *square* of the velocity; so the total energy goes as the *square* of the displacement. Thus the displacement, the amplitude of oscillation, will decrease half as fast because of the square. In other words, we guess that the solution for the damped transient motion will be an oscillation of frequency close to the resonance frequency ω_0, in which the amplitude of the sine-wave motion will diminish as $e^{-\gamma t/2}$:

$$x = A_0 e^{-\gamma t/2} \cos \omega_0 t. \tag{24.10}$$

This equation and Fig. 24-1 give us an idea of what we should expect; now let us try to analyze the motion *precisely* by solving the differential equation of the motion itself.

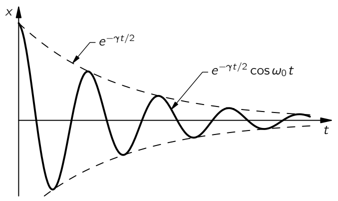


Fig. 24-1. A damped cosine oscillation.

So, starting with Eq. (24.1), with no outside force, how do we solve it? Being physicists, we do not have to worry about the *method* as much as we do about what the solution *is.* Armed with our previous experience, let us try as a solution an exponential curve, $x = Ae^{i\alpha t}$. (Why do we try this? It is the easiest thing to differentiate!) We put this into (24.1) (with $F(t) = 0$), using the rule that each

Horizontal range (R) of the projectile

Horizontal range = Horizontal velocity × time of flight

$R = u\cos\theta T,\ T = \frac{2u\sin\theta}{g}$

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}} \text{ or } \boxed{R = \frac{2u_x u_y}{g}}$$

Relation connecting R, H, T and angle of projection θ

$$\frac{H}{R} = \frac{\frac{u^2 \sin^2\theta}{2g}}{\frac{u^2\, 2\sin\theta\cos\theta}{g}}$$

$$4H = R\tan\theta,\ \boxed{H = \frac{gT^2}{8}}$$

$$\therefore 4\,\frac{gT^2}{8} = R\tan\theta\ ;\ \boxed{R = \frac{gT^2}{2\tan\theta}}$$

Angle of projection for maximum range for a given speed of projection

$$R = \frac{u^2 \sin 2\theta}{g},\quad [\sin 2\theta]_{max} = 1$$

$$R_{max} = \frac{u^2}{g} \qquad \sin 90^0 = 1$$

$$2\theta = 90^0$$

$$\theta = 45^0$$

To get maximum height we should throw vertically up $\theta = 90^0$

$$H_{max} = \frac{u^2}{2g} (1-\text{dimensional motion})$$

$$H_{max} = \left(\frac{u^2}{g}\right) = \frac{R_{max}}{2}$$

Galileo in his book two new sciences stated that for elevations which exceed or fall short of $45^0$by equal amount, the ranges are equal.

i.e., there are two different angles of projection for same range. If one angle is θ other angle is $90-\theta$ for same sped of projection.

time we differentiate x with respect to time, we multiply by $i\alpha$. So it is really quite simple to substitute. Thus our equation looks like this:

$$(-\alpha^2 + i\gamma\alpha + \omega_0^2)Ae^{i\alpha t} = 0. \tag{24.11}$$

The net result must be zero for *all times*, which is impossible unless (a) $A = 0$, which is no solution at all—it stands still, or (b)

$$-\alpha^2 + i\alpha\gamma + \omega_0^2 = 0. \tag{24.12}$$

If we can solve this and find an α, then we will have a solution in which A need not be zero!

$$\alpha = i\gamma/2 \pm \sqrt{\omega_0^2 - \gamma^2/4}. \tag{24.13}$$

For a while we shall assume that γ is fairly small compared with ω_0, so that $\omega_0^2 - \gamma^2/4$ is definitely positive, and there is nothing the matter with taking the square root. The only bothersome thing is that we get *two* solutions! Thus

$$\alpha_1 = i\gamma/2 + \sqrt{\omega_0^2 - \gamma^2/4} = i\gamma/2 + \omega_\gamma \tag{24.14}$$

and

$$\alpha_2 = i\gamma/2 - \sqrt{\omega_0^2 - \gamma^2/4} = i\gamma/2 - \omega_\gamma. \tag{24.15}$$

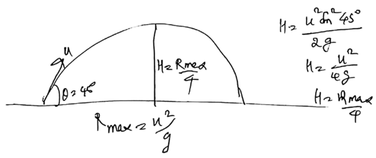
Let us consider the first one, supposing that we had not noticed that the square root has two possible values. Then we know that a solution for x is $x_1 = Ae^{i\alpha_1 t}$, where A is any constant whatever. Now, in substituting α_1, because it is going to come so many times and it takes so long to write, we shall call $\sqrt{\omega_0^2 - \gamma^2/4} = \omega_\gamma$. Thus $i\alpha_1 = -\gamma/2 + i\omega_\gamma$, and we get $x = Ae^{(-\gamma/2 + i\omega_\gamma)t}$, or what is the same, because of the wonderful properties of an exponential,

$$x_1 = Ae^{-\gamma t/2}e^{i\omega_\gamma t}. \tag{24.16}$$

First, we recognize this as an oscillation, an oscillation at a frequency ω_γ, which is not *exactly* the frequency ω_0, but is rather close to ω_0 if it is a good system. Second, the amplitude of the oscillation is decreasing exponentially! If we take, for instance, the real part of (24.16), we get

$$x_1 = Ae^{-\gamma t/2}\cos\omega_\gamma t. \tag{24.17}$$

Let θ_1 and θ_2 to be two different angles of projection for same range.



$R = \dfrac{u^2 \sin 2\theta}{g}$, since $R_1 = R_2$

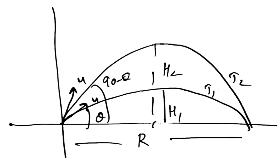
$\sin 2\theta_1 = \sin 2\theta_2 \quad \sin(180 - A) = \sin A$

$\sin 2\theta_1 = \sin(180 - 2\theta_2)$

$2\theta_1 = 180 - 2\theta_2 \;;\; \theta_1 = 90 - \theta_2$

$\theta_1 + \theta_2 = 90^0$

e.g. for same speed at angles of projection 30^0 and 60^0 range is same. Also at 15^0 and 75^0 range is same.



$\Rightarrow$ In the above situation when $R_1 = R_2$

$T_1 = \dfrac{2u\sin\theta}{g} \;;\; T_2 = \dfrac{2u\sin(90-\theta)}{g}$

$\dfrac{T_1}{T_2} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta \qquad [\sin(90-\theta) = \cos\theta]$

$H_1 = \dfrac{u^2\sin^2\theta}{2g}; \; H_2 = \dfrac{u^2\sin^2(90-\theta)}{2g}; \quad \dfrac{H_1}{H_2} = \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

This is very much like our guessed-at solution (24.10), except that the frequency really is ω_γ. This is the only error, so it is the same thing—we have the right idea. But everything is *not* all right! What is not all right is that *there is another solution.*

The other solution is α_2, and we see that the difference is only that the sign of ω_γ is reversed:

$$x_2 = Be^{-\gamma t/2}e^{-i\omega_\gamma t}. \tag{24.18}$$

What does this mean? We shall soon prove that if x_1 and x_2 are each a possible solution of Eq. (24.1) with $F = 0$, then $x_1 + x_2$ is also a solution of the same equation! So the general solution x is of the mathematical form

$$x = e^{-\gamma t/2}(Ae^{i\omega_\gamma t} + Be^{-i\omega_\gamma t}). \tag{24.19}$$

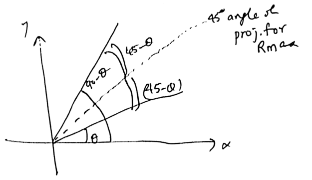
Now we may wonder why we bother to give this other solution, since we were so happy with the first one all by itself. What is the extra one for, because of course we know we should only take the real part? *We* know that we must take the real part, but how did the *mathematics* know that we only wanted the real part? When we had a nonzero driving force $F(t)$, we put in an *artificial* force to go with it, and the *imaginary* part of the equation, so to speak, was driven in a definite way. But when we put $F(t) \equiv 0$, our convention that x should be only the real part of whatever we write down is purely our own, and the mathematical equations do not know it yet. The physical world *has* a real solution, but the answer that we were so happy with before is not real, it is *complex.* The *equation* does not know that we are arbitrarily going to take the real part, so it has to present us, so to speak, with a complex conjugate type of solution, so that by putting them together we can *make a truly real* solution; that is what α_2 is doing for us. In order for x to be real, $Be^{-\omega_\gamma t}$ will have to be the complex conjugate of $Ae^{\omega_\gamma t}$ that the imaginary parts disappear. So it turns out that B is the complex conjugate of A, and our real solution is

$$x = e^{-\gamma t/2}(Ae^{i\omega_\gamma t} + A^*e^{-i\omega_\gamma t}). \tag{24.20}$$

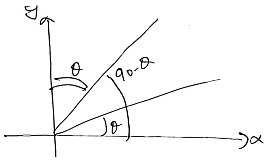
So our real solution is an oscillation with a *phase shift* and a damping—just as advertised.

24-3 Electrical transients

Now let us see if the above really works. We construct the electrical circuit shown in Fig. 24-2, in which we apply to an oscilloscope the voltage across the



Two different angles of projection for same range are equally inclined to the angle of projection for maximum range i.e. 45⁰.



The different angles of projection for same range are equally inclined to the vertical and horizontal

Equation for path of a projectile

$$y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$

$$y = x\tan\theta\left[1 - \frac{x}{\frac{(2u^2\sin\theta\cos\theta}{g}}\right]$$

$$y = x\tan\theta\left[1 - \frac{x}{R}\right]$$

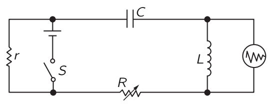


Fig. 24-2. An electrical circuit for demonstrating transients.

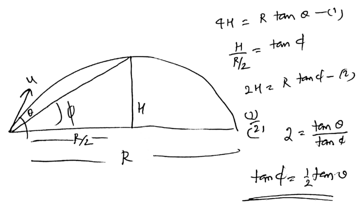
inductance L after we suddenly turn on a voltage by closing the switch S. It is an oscillatory circuit, and it generates a transient of some kind. It corresponds to a circumstance in which we suddenly apply a force and the system starts to oscillate. It is the electrical analog of a damped mechanical oscillator, and we watch the oscillation on an oscilloscope, where we should see the curves that we were trying to analyze. (The horizontal motion of the oscilloscope is driven at a uniform speed, while the vertical motion is the voltage across the inductor. The rest of the circuit is only a technical detail. We would like to repeat the experiment many, many times, since the persistence of vision is not good enough to see only one trace on the screen. So we do the experiment again and again by closing the switch 60 times a second; each time we close the switch, we also start the oscilloscope horizontal sweep, and it draws the curve over and over.) In Figs. 24-3 to 24-6 we see examples of damped oscillations, actually photographed on an oscilloscope screen. Figure 24-3 shows a damped oscillation in a circuit which has a high Q, a small γ. It does not die out very fast; it oscillates many times on the way down.

But let us see what happens as we decrease Q, so that the oscillation dies out more rapidly. We can decrease Q by increasing the resistance R in the circuit. When we increase the resistance in the circuit, it dies out faster (Fig. 24-4). Then if we increase the resistance in the circuit still more, it dies out faster still (Fig. 24-5). But when we put in more than a certain amount, we cannot see any oscillation at all! The question is, is this because our eyes are not good enough? If we increase the resistance still more, we get a curve like that of Fig. 24-6, which does not appear to have any oscillations, except perhaps one. Now, how can we explain that by mathematics?

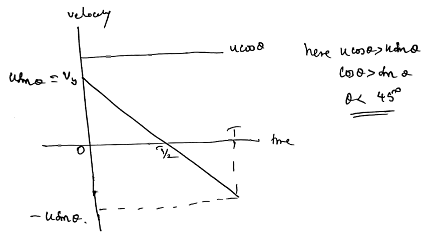
The resistance is, of course, proportional to the γ term in the mechanical device. Specifically, γ is R/L. Now if we increase the γ in the solutions (24.14) and (24.15) that we were so happy with before, chaos sets in when $\gamma/2$ exceeds ω_0; we must write it a different way, as

$$i\gamma/2 + i\sqrt{\gamma^2/4 - \omega_0^2} \qquad \text{and} \qquad i\gamma/2 - i\sqrt{\gamma^2/4 - \omega_0^2}.$$

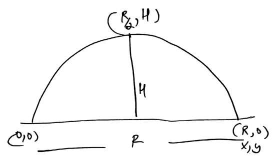
Relation between angle of projection 'θ' and angle of elevation ϕ, at maximum height



Velocity time graph



If the path of a projectile is given by the equation $y = ax - bx^2$ find R, H, T



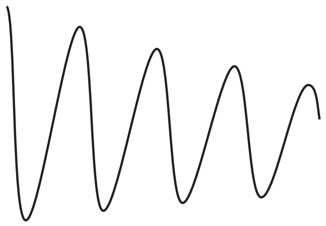


Figure 24-3

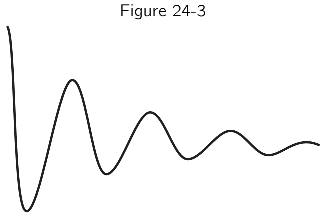


Figure 24-4

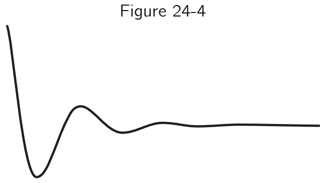


Figure 24-5

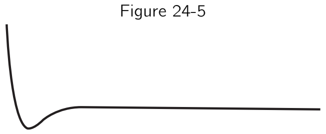


Figure 24-6

$y = ax - bx^2$, if $y = 0$, $x = R$

$0 = ax - bx^2$

$ax = bx^2$

$x = R = a/b$

Compare $y = x\tan\theta[1 - x/R]$

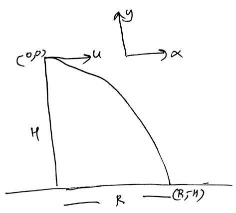
$\tan\theta = a$

$4H = R\tan\theta$

$4H = \frac{a}{b}a, \qquad H = \frac{a^2}{4b}$

$H = \frac{T^2 g}{8} = \frac{a^2}{4b}$

Horizontal projection



A particle is projected horizontally with a velocity u from a height H. It follows a parabolic path and stroke the ground, horizontal component of velocity u remains constant vertical component is subjected to acceleration due to gravity.

$u_x = u, \; u_y = 0 \quad \vec{u} = u\hat{i}$

$a_x = 0, a_y = -g \quad \vec{a} = -g\hat{i}$

To find time of flight we consider the motion along y-axis

$S_y = u_y t + \frac{1}{2}a_y t^2$

$-H = oxt + \frac{1}{2}(-g)t^2$

Those are now the two solutions and, following the same line of mathematical reasoning as previously, we again find two solutions: $e^{i\alpha_1 t}$ and $e^{i\alpha_2 t}$. If we now substitute for α_1, we get

$$x = Ae^{-(\gamma/2+\sqrt{\gamma^2/4-\omega_0^2})t},$$

a nice exponential decay with no oscillations. Likewise, the other solution is

$$x = Be^{-(\gamma/2-\sqrt{\gamma^2/4-\omega_0^2})t}.$$

Note that the square root cannot exceed $\gamma/2$, because even if $\omega_0 = 0$, one term just equals the other. But ω_0^2 is taken away from $\gamma^2/4$, so the square root is less than $\gamma/2$, and the term in parentheses is, therefore, always a positive number. Thank goodness! Why? Because if it were negative, we would find e raised to a *positive* factor times t, which would mean it was exploding! In putting more and more resistance into the circuit, we know it is not going to explode—quite the contrary. So now we have two solutions, each one by itself a dying exponential, but one having a much faster "dying rate" than the other. The general solution is of course a combination of the two; the coefficients in the combination depending upon how the motion starts—what the initial conditions of the problem are. In the particular way this circuit happens to be starting, the A is negative and the B is positive, so we get the difference of two exponential curves.

Now let us discuss how we can find the two coefficients A and B (or A and A^*), if we know how the motion was started.

Suppose that at $t = 0$ we know that $x = x_0$, and $dx/dt = v_0$. If we put $t = 0$, $x = x_0$, and $dx/dt = v_0$ into the expressions

$$\begin{aligned} x &= e^{-\gamma t/2}(Ae^{i\omega_\gamma t} + A^* e^{-i\omega_\gamma t}),\\ dx/dt &= e^{-\gamma t/2}[(-\gamma/2 + i\omega_\gamma)Ae^{i\omega_\gamma t} + (-\gamma/2 - i\omega_\gamma)A^* e^{-i\omega_\gamma t}], \end{aligned}$$

we find, since $e^0 = e^{i0} = 1$,

$$\begin{aligned} x_0 &= A + A^* = 2A_R,\\ v_0 &= -(\gamma/2)(A + A^*) + i\omega_\gamma(A - A^*)\\ &= -\gamma x_0/2 + i\omega_\gamma(2iA_I), \end{aligned}$$

where $A = A_R + iA_I$, and $A^* = A_R - iA_I$. Thus we find

$$A_R = x_0/2$$

$H = \frac{1}{2}gt^2$

$t = \sqrt{\frac{2}{g}H}$

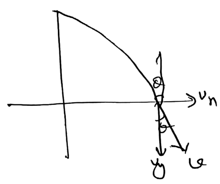
This is same as the time take by a dropped body to reach the ground dropped from rest.

To find range R, consider the horizontal motion,

$S_x = u_x t$

$R = ut \quad R = u\sqrt{\frac{2H}{g}}$

Velocity with which it hits the ground



$V_x = u$

$V_y^2 = u_y^2 + 2a_y y$

$Vy^2 = 0 + 2(-g)(-H)$

$V_y = \sqrt{2gH} \qquad V = \sqrt{V_x^2 + V_y^2}\ ;\ \tan\theta = \frac{V_x}{V_y}$

θ is the angle made by the velocity with vertical.

and

$$A_I = -(v_0 + \gamma x_0/2)/2\omega_\gamma. \tag{24.21}$$

This completely determines A and A^*, and therefore the complete curve of the transient solution, in terms of how it begins. Incidentally, we can write the solution another way if we note that

$$e^{i\theta} + e^{-i\theta} = 2\cos\theta \qquad \text{and} \qquad e^{i\theta} - e^{-i\theta} = 2i\sin\theta.$$

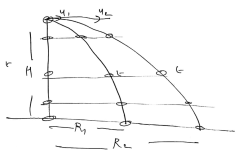
We may then write the complete solution as

$$x = e^{-\gamma t/2}\left[x_0 \cos\omega_\gamma t + \frac{v_0 + \gamma x_0/2}{\omega_\gamma}\sin\omega_\gamma t\right], \tag{24.22}$$

where $\omega_\gamma = +\sqrt{\omega_0^2 - \gamma^2/4}$. This is the mathematical expression for the way an oscillation dies out. We shall not make direct use of it, but there are a number of points we should like to emphasize that are true in more general cases.

First of all the behavior of such a system with no external force is expressed by a sum, or superposition, of pure exponentials in time (which we wrote as $e^{i\alpha t}$). This is a good solution to try in such circumstances. The values of α may be complex in general, the imaginary parts representing damping. Finally the intimate mathematical relation of the sinusoidal and exponential function discussed in Chapter 22 often appears physically as a change from oscillatory to exponential behavior when some physical parameter (in this case resistance, γ) exceeds some critical value.

Three projectiles one is dropped, other two are thrown with some velocities are shown below. Position are drawn at different intervals.



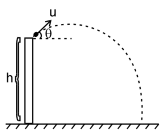
t is same for all $\quad R_1 = u_1 t \quad R_2 = u_2 t$

$$\frac{R_1}{R_2} = \frac{u_1}{u_2}$$

All of them reach the ground at the same time. Their vertical motion are identical because they have same initial vertical velocity (zero in this case) and same acceleration.

Path of a projectile with respect to another projectile is a straight line. Their relative acceleration is zero.

Projectile Projected from the top of a building (Projected upwards)



Horizontal motion	Vertical motion
$u_x = u\cos\theta$	$u_y = u\sin\theta$
$a_x = 0$	$a_y = -g$

Time of flight (T)

at t = T, $s_y = -h$

$$s_y = u_y t + \frac{1}{2}a_y t^2$$

$$-h = -u\sin\theta T + \frac{1}{2}(-g)T^2$$

Solving this equation 'T' will be obtained

Range (R)

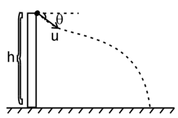
at t = T, s_x = R

$s_x = u_x t + \frac{1}{2} a_x t^2$

$R = u\cos\theta \times T + 0$

$\boxed{R = u\cos\theta \times T}$

Projectile Projected from the top of a building (Projected downwards)



Horizontal motion

$u_x = u\cos\theta$

$a_x = 0$

Vertical motion

$u_y = u\sin\theta$

$a_y = -g$

Time of flight (T)

at t = T, $s_y = -h$

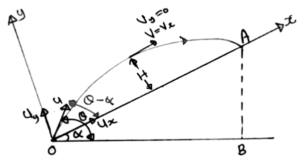
$s_y = u_y t + \frac{1}{2} a_y t^2$

$-h = -u\sin\theta T + \frac{1}{2}(-g)T^2$

Solving this equation 'T' will be obtained

Range $R = u\cos\theta \times T$

Projection From Inclined Plane



$u_x = u\cos(\theta - \alpha)$

$u_y = u\sin(\theta - \alpha)$

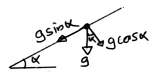
$a_x = -g\sin\alpha$

$a_y = -g\cos\alpha$

For motion from 'O' to 'A' the displacement along the y-direction is zero.

$\therefore\ y = u_y t + \frac{a_y t^2}{2}$

$O = u\sin(\theta - \alpha)T - \frac{g\cos\alpha\, T^2}{2}$



$\boxed{T = \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}}$ or $\boxed{T = \frac{2U_y}{|a_y|}}$

Maximum Height from inclined surface (H)

$V_y^2 - U_y^2 = 2a_y y$

$\therefore O - U_y^2 = -2a_y H$

$$\boxed{H = \frac{U_y^2}{2a_y} \qquad H = \frac{U^2\sin^2(\theta - \alpha)}{2g\cos\alpha}}$$

Horizontal displacement $OB = (u\cos\theta)T$

$OB = u\cos\theta \times \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}$

Range Along the inclined surface

$$R = OA = \frac{OB}{\cos\alpha}$$

$$\boxed{R = \frac{2u^2 \cos\theta \sin(\theta-\alpha)}{g\cos^2\alpha}}$$

$$R = \frac{u^2\, 2\cos\theta \sin(\theta-\alpha)}{g\cos^2\alpha}$$

$$\boxed{R = \frac{u^2[\sin(2\theta-\alpha)-\sin\alpha]}{g\cos^2\alpha}} \qquad 2\cos A \sin B = \sin(A+B) - \sin(A-B)$$

Range R is maximum, when $\sin(2\theta-\alpha)=1$

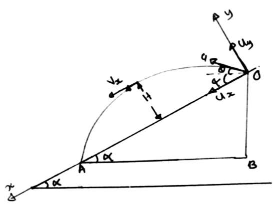
$$2\theta - \alpha = \frac{\pi}{2}$$

$$\boxed{\theta = \frac{\pi}{4} + \frac{\alpha}{2}} \qquad \theta - \alpha = \frac{\pi}{4} + \frac{\alpha}{2} - \alpha = \frac{\pi - 2\alpha}{4}$$

$$R_{max} = \frac{u^2[1-\sin\alpha]}{g\cos^2\alpha} = \frac{u^2(1-\sin\alpha)}{g(1-\sin^2\alpha)}\ ; \quad R_{max} = \frac{u^2[1-\sin\alpha]}{g[1+\sin\alpha]\,[1-\sin\alpha]}$$

$$\boxed{R_{max} = \frac{u^2}{g[1+\sin\alpha]}}$$

Projectile Motion from an Inclined Plane



$$u_x = u\cos(\theta+\alpha)$$

$$u_y = u\sin(\theta+\alpha)$$

$$a_x = g\sin\alpha,\ a_y = -g\cos\alpha$$

When the object hits on the inclined plane

$$y = 0, \quad \therefore y = u_y t + \frac{a_y t^2}{2}$$

$$O = u\sin(\theta+\alpha)T - \frac{g\cos\alpha}{2}T^2 \qquad \boxed{T = \frac{2u\sin(\theta+\alpha)}{g\cos\alpha} = \frac{2U_y}{a_y}}$$

$$BA = (U_x \cos\theta)T = u\cos\theta \times \frac{2u\sin(\theta+\alpha)}{g\cos\alpha}$$

Range along the inclined surface

$$R = OA = \frac{AB}{\cos\alpha} = \frac{2u^2\cos\theta \sin(\theta+\alpha)}{g\cos^2\alpha} \qquad \boxed{R = \frac{2u^2\cos\theta\sin(\theta+\alpha)}{g\cos^2\alpha}}$$

$$R = \frac{u^2}{g\cos^2\alpha}[\sin(2\theta+\alpha) - \sin(-\alpha)]$$

$$\boxed{R = \frac{u^2[\sin(2\theta+\alpha)+\sin\alpha]}{g\cos^2\alpha}}$$

For maximum range $\sin(2\theta+\alpha)=1$, $\quad 2\theta+\alpha = \frac{\pi}{2}$

$$\boxed{\begin{array}{l} \theta+\alpha = \frac{\pi-2\alpha}{4}+\alpha \\ \theta+\alpha = \frac{\pi+2\alpha}{4} \end{array}} \qquad \theta = \frac{\pi-2\alpha}{4} \qquad \boxed{R_{max} = \frac{u^2[1+\sin\alpha]}{g\cos^2\alpha}}$$

$$R_{max} = \frac{u^2(1+\sin\alpha)}{g(1-\sin^2\alpha)} = \frac{u^2(1+\sin\alpha)}{g[1+\sin\alpha][1-\sin\alpha]}$$

$$\boxed{R_{max} = \frac{u^2}{g\,[1-\sin\alpha]}}$$

Maximum height (H) from the inclined surface.

At maximum height $V_y = 0$

$$\therefore V_y^2 - u_y^2 = 2a_y y \text{ becomes } 0 - u_y^2 = 2a_y H$$

$$H = \frac{U_y^2}{2a_y} = \frac{u^2\sin^2(\theta+\alpha)}{2g\cos\alpha}$$

Note : For a given speed, the direction which gives the maximum range of the projectile on an inclined plane, bisects the angle between the incline and the vertical, for upward or downward projection.

Standard results for projectile motion on an incline plane

	Up the incline	**Down the incline**
Range	$\frac{2u^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}$	$\frac{2u^2\cos\theta\sin(\theta+\alpha)}{g\cos^2\alpha}$
Time of flight	$\frac{2u\sin(\theta-\alpha)}{g\cos\alpha}$	$\frac{2u\sin(\theta+\alpha)}{g\cos\alpha}=\frac{2u_y}{a_y}$
Maximum Range	$\frac{u^2}{g\,[1+\sin\alpha]}$	$\frac{u^2}{g\,[1-\sin\alpha]}$
Angle of projection for maximum range (from inclined surface)	$\frac{\pi-2\alpha}{4}$	$\frac{\pi+2\alpha}{4}$