6

Probability

"The true logic of this world is in the calculus of probabilities."

—James Clerk Maxwell

6-1 Chance and likelihood

"Chance" is a word which is in common use in everyday living. The radio reports speaking of tomorrow's weather may say: "There is a sixty percent chance of rain." You might say: "There is a small chance that I shall live to be one hundred years old." Scientists also use the word chance. A seismologist may be interested in the question: "What is the chance that there will be an earthquake of a certain size in Southern California next year?" A physicist might ask the question: "What is the chance that a particular geiger counter will register twenty counts in the next ten seconds?" A politician or statesman might be interested in the question: "What is the chance that there will be a nuclear war within the next ten years?" You may be interested in the chance that you will learn something from this chapter.

By *chance*, we mean something like a guess. Why do we make guesses? We make guesses when we wish to make a judgment but have incomplete information or uncertain knowledge. We want to make a guess as to what things are, or what things are likely to happen. Often we wish to make a guess because we have to make a decision. For example: Shall I take my raincoat with me tomorrow? For what earth movement should I design a new building? Shall I build myself a fallout shelter? Shall I change my stand in international negotiations? Shall I go to class today?

Sometimes we make guesses because we wish, with our limited knowledge, to say as much as we *can* about some situation. Really, any generalization is in the nature of a guess. Any physical theory is a kind of guesswork. There are good guesses and there are bad guesses. The theory of probability is a

An object that is in fight after being thrown or projected is called a projectile. The motion of a projectile may be thought as the result of two separate, simultaneously occurring perpendicular components of motions. One component is along horizontal direction without any acceleration and other is along vertical direction with constant acceleration due to gravity. It was Galileo who fist stated this independency of the horizontal and the vertical components of projectile motion.
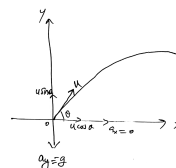
A particle is projected with a velocity u (velocity of projection) making using a angle θ with the horizontal. θ is known as angle of projection. Only force that control the projectile is gravity. We will neglect air resistance. Projectile is subjected to acceleration due to gravity $\vec{a} = -g\hat{j}$ ($a_x = 0$, $a_y = -g$).

$u\cos\theta$ is the horizontal component of velocity which remains constant. $u\sin\theta$ is initial vertical component of velocity. O is the point of projection which is taken as origin.

The velocity of the projectile after t second
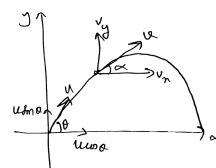
system for making better guesses. The language of probability allows us to speak quantitatively about some situation which may be highly variable, but which does have some consistent average behavior.

Let us consider the flipping of a coin. If the toss—and the coin—are "honest," we have no way of knowing what to expect for the outcome of any particular toss. Yet we would feel that in a large number of tosses there should be about equal numbers of heads and tails. We say: "The probability that a toss will land heads is 0.5."

We speak of probability only for observations that we contemplate being made in the future. *By the "probability" of a particular outcome of an observation we mean our estimate for the most likely fraction of a number of repeated observations that will yield that particular outcome.* If we imagine repeating an observation—such as looking at a freshly tossed coin—N times, and if we call N_A *our estimate* of the most likely number of our observations that will give some specified result A, say the result "heads," then by $P(A)$, the probability of observing A, we mean

$$P(A) = N_A/N. \tag{6.1}$$

Our definition requires several comments. First of all, we may speak of a probability of something happening only if the occurrence is a possible outcome of some *repeatable* observation. It is not clear that it would make any sense to ask: "What is the probability that there is a ghost in that house?"

You may object that no situation is *exactly* repeatable. That is right. Every different observation must at least be at a different time or place. All we can say is that the "repeated" observations should, for our intended purposes, *appear to be equivalent.* We should assume, at least, that each observation was made from an equivalently prepared situation, and especially with the same degree of ignorance at the start. (If we sneak a look at an opponent's hand in a card game, our estimate of our chances of winning are different than if we do not!)

We should emphasize that N and N_A in Eq. (6.1) are *not* intended to represent numbers based on actual observations. N_A is our best *estimate* of what *would* occur in N *imagined* observations. Probability depends, therefore, on our knowledge and on our ability to make estimates. In effect, on our common sense! Fortunately, there is a certain amount of agreement in the common sense of many things, so that different people will make the same estimate. Probabilities need not, however, be "absolute" numbers. Since they depend on our ignorance, they may become different if our knowledge changes.

$V_y = u_y + a_y t$

$V_y = u\sin\theta - gt$

$V = \sqrt{V_x^2 + V_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$

$V = \sqrt{u^2 - 2u\sin\theta gt + g^2 t^2}$

Velocity v make an angle α with horizontal such that, $\tan\alpha = \dfrac{\text{opposite side}}{\text{adj. side}} = \dfrac{V_y}{V_x} = \dfrac{u\sin\theta - gt}{u\cos\theta}$

In vector form $\vec{a} = -g\hat{j}$

$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$

$\vec{v} = \vec{u} + \vec{a}t$

$\vec{v} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} - gt\hat{j}$

$\vec{v} = u\cos\theta\,\hat{i} + (u\sin\theta - gt)\hat{j}$

$\left|\vec{v}\right| = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$, $\tan\alpha = \dfrac{u\sin\theta - gt}{u\cos\theta}$
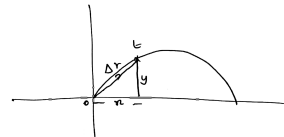
To find the displacement of the projectile after t seconds

along x axis

disp = velocity × time

$x = (u\cos\theta)t$

This is the equation for x-coordinate of the projectile at any time t.

along y axis $y = u_y t + \dfrac{a_y t^2}{2}$

$y = u\sin\theta\, t - \dfrac{1}{2}gt^2$ Equation for y coordinate or height of projectile at any time t.

You may have noticed another rather "subjective" aspect of our definition of probability. We have referred to N_A as "our estimate of the most likely number ..." We do not mean that we expect to observe *exactly* N_A, but that we expect a number *near* N_A, and that the number N_A is *more likely* than any other number in the vicinity. If we toss a coin, say, 30 times, we should expect that the number of heads would not be very likely to be exactly 15, but rather only some number near to 15, say 12, 13, 14, 15, 16, or 17. However, if we *must* choose, we would decide that 15 heads is *more likely* than any other number. We would write $P(\text{heads}) = 0.5$.

Why did we choose 15 as more likely than any other number? We must have argued with ourselves in the following manner: If the most likely number of heads is N_H in a total number of tosses N, then the most likely number of tails N_T is $(N - N_H)$. (We are assuming that every toss gives *either* heads *or* tails, and no "other" result!) But if the coin is "honest," there is no preference for heads or tails. Until we have some reason to think the coin (or toss) is dishonest, we must give equal likelihoods for heads and tails. So we must set $N_T = N_H$. It follows that $N_T = N_H = N/2$, or $P(H) = P(T) = 0.5$.

We can generalize our reasoning to *any* situation in which there are m different but "equivalent" (that is, equally likely) possible results of an observation. If an observation can yield m different results, and we have reason to believe that any one of them is as likely as any other, then the probability of a *particular* outcome A is $P(A) = 1/m$.

If there are seven different-colored balls in an opaque box and we pick one out "at random" (that is, without looking), the probability of getting a ball of a particular color is $\frac{1}{7}$. The probability that a "blind draw" from a shuffled deck of 52 cards will show the ten of hearts is $\frac{1}{52}$. The probability of throwing a double-one with dice is $\frac{1}{36}$.

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In Chapter 5 we described the size of a nucleus in terms of its apparent area, or "cross section." When we did so we were really talking about probabilities. When we shoot a high-energy particle at a thin slab of material, there is some chance that it will pass right through and some chance that it will hit a nucleus. (Since the nucleus is so small that we cannot *see* it, we cannot aim right at a nucleus. We must "shoot blind.") If there are n atoms in our slab and the nucleus of each atom has a cross-sectional area σ, then the total area "shadowed" by the nuclei is $n\sigma$. In a large number N of random shots, we expect that the number of hits N_C of *some* nucleus will be in the

displacement = $r = \sqrt{x^2 + y^2}$

To find it in vector form we can use the equation $\vec{r} = x\hat{i} + y\hat{j}$ where $\vec{u} = u\sin\theta\hat{j}$, $\vec{a} = -g\hat{j}$
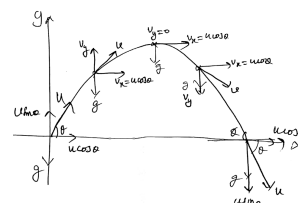
Equation for path of a projectile

$x = u\cos\theta\, t$, $t = \dfrac{x}{u\cos\theta}$

$y = u\sin\theta\, t - \dfrac{1}{2}gt^2$

$y = u\sin\theta\dfrac{x}{u\cos\theta} - \dfrac{1}{2}g\left(\dfrac{x}{u\cos\theta}\right)^2$

$\boxed{y = x\tan\theta - \dfrac{1}{2}g\dfrac{x^2}{u^2\cos^2\theta}}$

This is the equation of a parabola. Thus the path of a projectile is parabola.

At the highest point of the projectile vertical component of velocity is zero. Horizontal component is $u\cos\theta$ because it remains constant. At the highest point speed of the projectile is minimum and purely horizontal and is equal to $u\cos\theta$. Angle between acceleration and instantaneous velocity decreases from $(90 + \theta)$ to $(90 - \theta)$

ratio to N as the shadowed area is to the total area of the slab:

$$N_C/N = n\sigma/A. \tag{6.2}$$

We may say, therefore, that the *probability* that any one projectile particle will suffer a collision in passing through the slab is

$$P_C = \frac{n}{A}\,\sigma, \tag{6.3}$$

where n/A is the number of atoms per unit area in our slab.

6-2 Fluctuations

We would like now to use our ideas about probability to consider in some greater detail the question: "How many heads do I really *expect* to get if I toss a coin N times?" Before answering the question, however, let us look at what does happen in such an "experiment." Figure 6-1 shows the results obtained in the first three "runs" of such an experiment in which $N = 30$. The sequences of "heads" and "tails" are shown just as they were obtained. The first game gave 11 heads; the second also 11; the third 16. In three trials we did not once get 15 heads. Should we begin to suspect the coin? Or were we wrong in thinking that the most likely number of "heads" in such a game is 15? Ninety-seven more runs were made to obtain a total of 100 experiments of 30 tosses each. The results of the experiments are given in Table 6-1.*

```
H                                   11
  x x x              xxx   x  xx x x
  xx x x xxxxxxxxx     xx xx   x x
T                                   19
H                                   11
   x      x   x x xxx      x  x x x
  xxxx xxxx x x    xxxx xx x xx
T                                   19
H                                   16
   x  xxx xx   x   xxx   xx x xxx x
  x xx    x  xx xx    xx  x x  x x
T                                   14
```

Fig. 6-1. Observed sequences of heads and tails in three games of 30 tosses each.

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* After the first three games, the experiment was actually done by shaking 30 pennies violently in a box and then counting the number of heads that showed.

Time of Flight of the projectile (T)
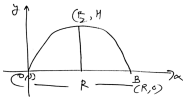
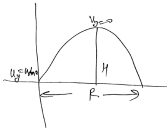Consider the motion of the projectile along y-axis

$$S_y = u_y t + \frac{1}{2} a_y t^2$$

$$t = T$$

$$0 = u\sin\theta\,T - \frac{1}{2}gT^2$$

$$\boxed{T = \frac{2u\sin\theta}{g}} \quad \text{or} \quad \boxed{T = \frac{2u_y}{g}}$$

Time of ascend = Time of descend = $\dfrac{u\sin\theta}{g}$

Maximum height of a projectile H

Consider the motion along y-axis

$$V_y^2 = u_y^2 + 2a_y g_y$$

$$0 = (u\sin\theta)^2 + 2(-gH)$$

$$2gH = (u\sin\theta)^2$$

$$\boxed{H = \frac{u^2\sin^2\theta}{2g}} \quad \text{or} \quad \boxed{H = \frac{u_y^2}{2g}}$$

Table 6-1

Number of heads in successive trials of 30 tosses of a coin.

11	16	17	15	17	16	19	18	15	13	
11	17	17	12	20	23	11	16	17	14	
16	12	15	10	18	17	13	15	14	15	
16	12	11	22	12	20	12	15	16	12	
16	10	15	13	14	16	15	16	13	18	} 100 trials
14	14	13	16	15	19	21	14	12	15	
16	11	16	14	17	14	11	16	17	16	
19	15	14	12	18	15	14	21	11	16	
17	17	12	13	14	17	9	13	19	13	
14	12	15	17	14	10	17	17	12	11	

NUMBER OF GAMES IN WHICH THE SCORE WAS OBTAINED

OBSERVED IN THIS EXPERIMENT
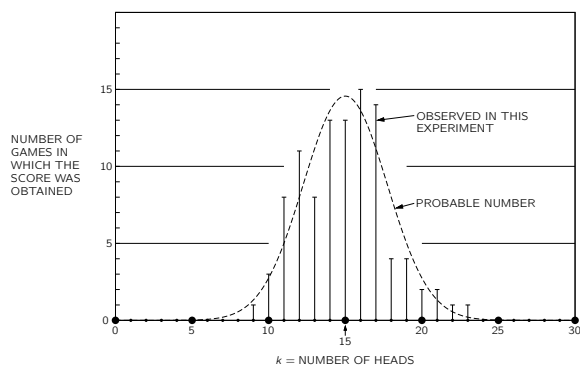
PROBABLE NUMBER

k = NUMBER OF HEADS

Fig. 6-2. Summary of the results of 100 games of 30 tosses each. The vertical bars show the number of games in which a score of k heads was obtained. The dashed curve shows the expected numbers of games with the score k obtained by a probability computation.

Horizontal range (R) of the projectile

Horizontal range = Horizontal velocity × time of flight

$$R = u\cos\theta\, T, \quad T = \frac{2u\sin\theta}{g}$$

$$\boxed{R = \frac{u^2 \sin 2\theta}{g}} \text{ or } \boxed{R = \frac{2u_x u_y}{g}}$$

Relation connecting R, H, T and angle of projection θ

$$\frac{H}{R} = \frac{u^2 \sin^2\theta}{\dfrac{2gu^2 2\sin\theta\cos\theta}{g}}$$

$$4H = R\tan\theta, \quad \boxed{H = \frac{gT^2}{8}}$$

$$\therefore 4\,\frac{gT^2}{8} = R\tan\theta \; ; \quad \boxed{R = \frac{gT^2}{2\tan\theta}}$$

Angle of projection for maximum range for a given speed of projection

$$R = \frac{u^2 \sin 2\theta}{g}, \quad [\sin 2\theta]_{max} = 1$$

$$R_{max} = \frac{u^2}{g} \qquad \sin 90^0 = 1$$

$$2\theta = 90^0$$

$$\theta = 45^0$$

To get maximum height we should throw vertically up $\theta = 90^0$

$$H_{max} = \frac{u^2}{2g} (1 - \text{dimensional motion})$$

$$H_{max} = \left(\frac{u^2}{g}\right) = \frac{R_{max}}{2}$$

Galileo in his book two new sciences stated that for elevations which exceed or fall short of 45⁰by equal amount, the ranges are equal.

i.e., there are two different angles of projection for same range. If one angle is θ other angle is $90 - \theta$ for same sped of projection.

Looking at the numbers in Table 6-1, we see that most of the results are "near" 15, in that they are between 12 and 18. We can get a better feeling for the details of these results if we plot a graph of the *distribution* of the results. We count the number of games in which a score of k was obtained, and plot this number for each k. Such a graph is shown in Fig. 6-2. A score of 15 heads was obtained in 13 games. A score of 14 heads was also obtained 13 times. Scores of 16 and 17 were each obtained *more* than 13 times. Are we to conclude that there is some bias toward heads? Was our "best estimate" not good enough? Should we conclude now that the "most likely" score for a run of 30 tosses is really 16 heads? But wait! In all the games taken together, there were 3000 tosses. And the total number of heads obtained was 1493. The fraction of tosses that gave heads is 0.498, very nearly, but slightly *less* than half. We should certainly *not* assume that the probability of throwing heads is greater than 0.5! The fact that one *particular* set of observations gave 16 heads most often, is a *fluctuation*. We still expect that the *most likely* number of heads is 15.

We may ask the question: "What *is* the probability that a game of 30 tosses will yield 15 heads—or 16, or any other number?" We have said that in a game of one toss, the probability of obtaining *one* head is 0.5, and the probability of obtaining no head is 0.5. In a game of two tosses there are *four* possible outcomes: HH, HT, TH, TT. Since each of these sequences is equally likely, we conclude that (a) the probability of a score of two heads is $\frac{1}{4}$, (b) the probability of a score of one head is $\frac{2}{4}$, (c) the probability of a zero score is $\frac{1}{4}$. There are *two* ways of obtaining one head, but only one of obtaining either zero or two heads.

Consider now a game of 3 tosses. The third toss is equally likely to be heads or tails. There is only one way to obtain 3 heads: we *must* have obtained 2 heads on the first two tosses, and then heads on the last. There are, however, *three* ways of obtaining 2 heads. We could throw tails after having thrown two heads (one way) or we could throw heads after throwing only one head in the first two tosses (two ways). So for scores of 3-H, 2-H, 1-H, 0-H we have that the number of equally likely ways is 1, 3, 3, 1, with a total of 8 different possible sequences. The probabilities are $\frac{1}{8}$, $\frac{3}{8}$, $\frac{3}{8}$, $\frac{1}{8}$.

The argument we have been making can be summarized by a diagram like that in Fig. 6-3. It is clear how the diagram should be continued for games with a larger number of tosses. Figure 6-4 shows such a diagram for a game of 6 tosses. The number of "ways" to any point on the diagram is just the number of different "paths" (sequences of heads and tails) which can be taken from the starting point. The vertical position gives us the total number of heads thrown. The set of

Let θ_1 and θ_2 to be two different angles of projection for same range.

$H_2 = \dfrac{u^2 \sin^2 45^\circ}{2g}$

$H_2 = \dfrac{u^2}{4g}$

$H_2 = \dfrac{R_{max}}{4}$
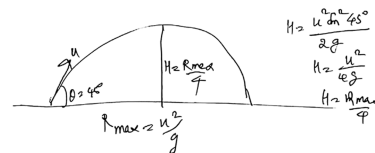
$R = \dfrac{u^2 \sin 2\theta}{g}$, since $R_1 = R_2$
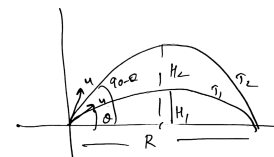
$\sin 2\theta_1 = \sin 2\theta_2 \quad \sin(180 - A) = \sin A$

$\sin 2\theta_1 = \sin(180 - 2\theta_2)$

$2\theta_1 = 180 - 2\theta_2 \; ; \; \theta_1 = 90 - \theta_2$

$\theta_1 + \theta_2 = 90^0$

e.g. for same speed at angles of projection 30^0 and 60^0 range is same. Also at 15^0 and 75^0 range is same.

$\Rightarrow$ In the above situation when $R_1 = R_2$

$T_1 = \dfrac{2u \sin \theta}{g} \; ; \; T_2 = \dfrac{2u \sin(90 - \theta)}{g}$

$\dfrac{T_1}{T_2} = \dfrac{\sin \theta}{\cos \theta} = \tan \theta \qquad [\sin(90 - \theta) = \cos \theta]$

$H_1 = \dfrac{u^2 \sin^2 \theta}{2g}; H_2 = \dfrac{u^2 \sin^2(90 - \theta)}{2g} \; ; \quad \dfrac{H_1}{H_2} = \dfrac{\sin^2 \theta}{\cos^2 \theta} = \tan^2 \theta$
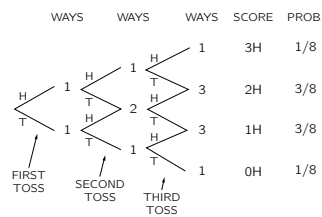
Fig. 6-3. A diagram for showing the number of ways a score of 0, 1, 2, or 3 heads can be obtained in a game of 3 tosses.
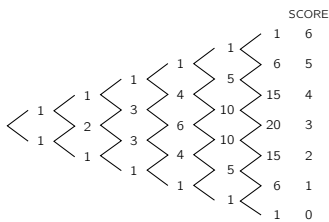

Fig. 6-4. A diagram like that of Fig. 6-3, for a game of 6 tosses.

numbers which appears in such a diagram is known as *Pascal's triangle*. The numbers are also known as the *binomial coefficients*, because they also appear in the expansion of $(a+b)^n$. If we call n the number of tosses and k the number of heads thrown, then the numbers in the diagram are usually designated by the symbol $\binom{n}{k}$. We may remark in passing that the binomial coefficients can also be computed from

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \tag{6.4}$$

where $n!$, called "n-factorial," represents the product $(n)(n-1)(n-2)\cdots(3)(2)(1)$.

We are now ready to compute the probability $P(k,n)$ of throwing k heads in n tosses, using our definition Eq. (6.1). The total number of possible sequences is 2^n (since there are 2 outcomes for each toss), and the number of ways of
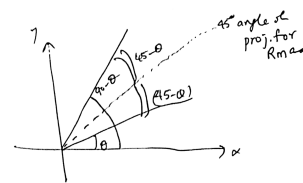
Two different angles of projection for same range are equally inclined to the angle of projection for maximum range i.e. 45^0.
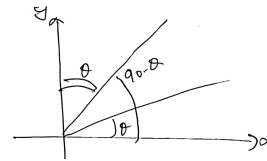
The different angles of projection for same range are equally inclined to the vertical and horizontal

Equation for path of a projectile

$$y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$

$$y = x\tan\theta\left[1 - \frac{x}{\frac{(2u^2\sin\theta\cos\theta)}{g}}\right]$$

$$y = x\tan\theta\left[1 - \frac{x}{R}\right]$$

6-7

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obtaining k heads is $\binom{n}{k}$, all equally likely, so we have

$$P(k,n) = \frac{\binom{n}{k}}{2^n}. \tag{6.5}$$

Since $P(k,n)$ is the fraction of games which we expect to yield k heads, then in 100 games we should expect to find k heads $100 \cdot P(k,n)$ times. The dashed curve in Fig. 6-2 passes through the points computed from $100 \cdot P(k,30)$. We see that we *expect* to obtain a score of 15 heads in 14 or 15 games, whereas this score was observed in 13 games. We *expect* a score of 16 in 13 or 14 games, but we obtained that score in 16 games. Such fluctuations are "part of the game."

The method we have just used can be applied to the most general situation in which there are only two possible outcomes of a single observation. Let us designate the two outcomes by W (for "win") and L (for "lose"). In the general case, the probability of W or L in a single event need not be equal. Let p be the probability of obtaining the result W. Then q, the probability of L, is necessarily $(1-p)$. In a set of n trials, the probability $P(k,n)$ that W will be obtained k times is

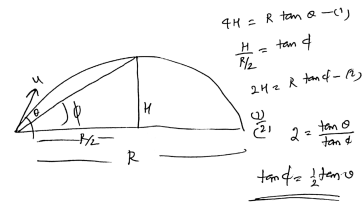$$P(k,n) = \binom{n}{k}p^k q^{n-k}. \tag{6.6}$$

This probability function is called the *Bernoulli* or, also, the *binomial* probability.
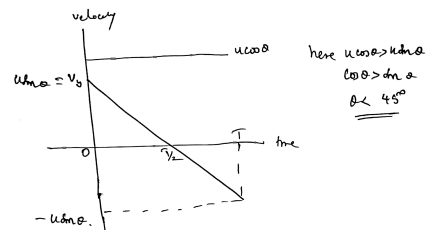
6-3 The random walk

There is another interesting problem in which the idea of probability is required. It is the problem of the "random walk." In its simplest version, we imagine a "game" in which a "player" starts at the point $x=0$ and at each "move" is required to take a step *either* forward (toward $+x$) *or* backward (toward $-x$). The choice is to be made *randomly*, determined, for example, by the toss of a coin. How shall we describe the resulting motion? In its general form the problem is related to the motion of atoms (or other particles) in a gas—called Brownian motion—and also to the combination of errors in measurements. You will see that the random-walk problem is closely related to the coin-tossing problem we have already discussed.

First, let us look at a few examples of a random walk. We may characterize the walker's progress by the net distance D_N traveled in N steps. We show in the graph of Fig. 6-5 three examples of the path of a random walker. (We have
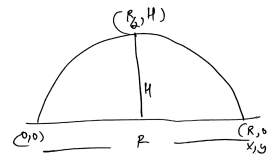

Relation between angle of projection 'θ' and angle of elevation φ, at maximum height

$$4H = R \tan \alpha \quad -(1)$$
$$\frac{H}{R/2} = \tan \phi$$
$$2H = R \tan \phi \quad -(2)$$
$$\frac{(1)}{(2)} \quad 2 = \frac{\tan \theta}{\tan \phi}$$
$$\tan \phi = \tfrac{1}{2}\tan \theta$$

Velocity time graph

$u\cos\theta$
$u\sin\theta = v_y$
here $u\cos\theta > u\sin\theta$
$\cos\theta > \sin\theta$
$\theta < 45°$

If the path of a projectile is given by the equation $y = ax - bx^2$ find R, H, T

$(\tfrac{R}{2}, H)$
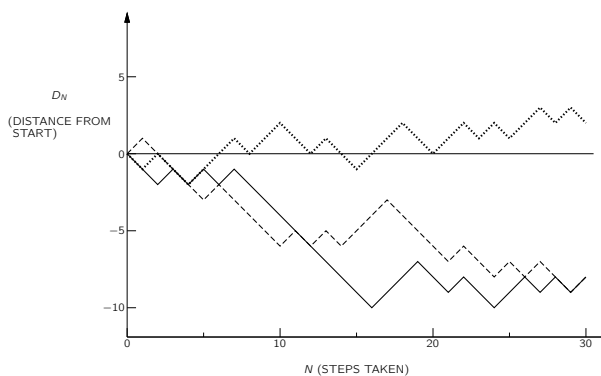$(0,0)$
$(R, 0)$

6-8

Fig. 6-5. The progress made in a random walk. The horizontal coordinate N is the total number of steps taken; the vertical coordinate D_N is the net distance moved from the starting position.

used for the random sequence of choices the results of the coin tosses shown in Fig. 6-1.)

What can we say about such a motion? We might first ask: "How far does he get on the average?" We must *expect* that his average progress will be zero, since he is equally likely to go either forward or backward. But we have the feeling that as N increases, he is more likely to have strayed farther from the starting point. We might, therefore, ask what is his average distance travelled in *absolute value*, that is, what is the average of $|D|$. It is, however, more convenient to deal with another measure of "progress," the square of the distance: D^2 is positive for either positive or negative motion, and is therefore a reasonable *measure* of such random wandering.

We can show that the expected value of D_N^2 is just N, the number of steps taken. By "expected value" we mean the probable value (our best guess), which we can think of as the *expected* average behavior in *many repeated* sequences. We represent such an expected value by $\langle D_N^2 \rangle$, and may refer to it also as the "mean square distance." After one step, D^2 is always $+1$, so we have certainly $\langle D_1^2 \rangle = 1$.

6-9

$y = ax - bx^2$, if $y = 0$, $x = R$

$0 = ax - bx^2$

$ax = bx^2$
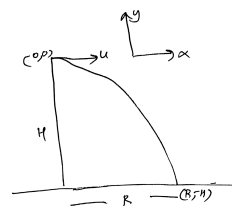
$x = R = a/b$

Compare $y = x \tan\theta[1 - x/R]$

$\tan\theta = a$

$4H = R\tan\theta$

$4H = \dfrac{a}{b}a$, $\qquad H = \dfrac{a^2}{4b}$

$H = \dfrac{T^2 g}{8} = \dfrac{a^2}{4b}$

Horizontal projection

A particle is projected horizontally with a velocity u from a height H. It follows a parabolic path and stroke the ground, horizontal component of velocity u remains constant vertical component is subjected to acceleration due to gravity.

$u_x = u$, $u_y = 0$ $\quad \vec{u} = u\hat{i}$

$a_x = 0, a_y = -g$ $\quad \vec{a} = -g\hat{i}$

To find time of flight we consider the motion along y-axis

$S_y = u_y t + \dfrac{1}{2} a_y t^2$

$-H = 0 \times t + \dfrac{1}{2}(-g)t^2$

9

(All distances will be measured in terms of a unit of one step. We shall not continue to write the units of distance.)

The expected value of D_N^2 for $N > 1$ can be obtained from D_{N-1}. If, after $(N-1)$ steps, we have D_{N-1}, then after N steps we have $D_N = D_{N-1} + 1$ or $D_N = D_{N-1} - 1$. For the squares,

$$D_N^2 = \begin{cases} D_{N-1}^2 + 2D_{N-1} + 1, \\ \qquad or \\ D_{N-1}^2 - 2D_{N-1} + 1. \end{cases} \qquad (6.7)$$

In a number of independent sequences, we expect to obtain each value one-half of the time, so our average expectation is just the average of the two possible values. The expected value of D_N^2 is then $D_{N-1}^2 + 1$. *In general*, we should *expect* for D_{N-1}^2 its "expected value" $\langle D_{N-1}^2 \rangle$ (by definition!). So

$$\langle D_N^2 \rangle = \langle D_{N-1}^2 \rangle + 1. \qquad (6.8)$$

We have already shown that $\langle D_1^2 \rangle = 1$; it follows then that

$$\langle D_N^2 \rangle = N, \qquad (6.9)$$

a particularly simple result!

If we wish a number like a distance, rather than a distance squared, to represent the "progress made away from the origin" in a random walk, we can use the "root-mean-square distance" D_{rms}:

$$D_{\mathrm{rms}} = \sqrt{\langle D^2 \rangle} = \sqrt{N}. \qquad (6.10)$$

We have pointed out that the random walk is closely similar in its mathematics to the coin-tossing game we considered at the beginning of the chapter. If we imagine the direction of each step to be in correspondence with the appearance of heads or tails in a coin toss, then D is just $N_H - N_T$, the difference in the number of heads and tails. Since $N_H + N_T = N$, the total number of steps (and tosses), we have $D = 2N_H - N$. We have derived earlier an expression for the expected distribution of N_H (also called k) and obtained the result of Eq. (6.5). Since N is just a constant, we have the corresponding distribution for D. (Since for every head more than $N/2$ there is a tail "missing," we have the factor of 2 between
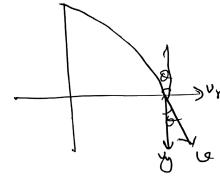
$H = \dfrac{1}{2}gt^2$

$t = \sqrt{\dfrac{2}{g}H}$

This is same as the time take by a dropped body to reach the ground dropped from rest.

To find range R, consider the horizontal motion,

$S_x = u_x t$

$R = ut \qquad R = u\sqrt{\dfrac{2H}{g}}$

Velocity with which it hits the ground

$V_x = u$

$V_y^2 = u_y^2 + 2a_y y$

$Vy^2 = 0 + 2(-g)(-H)$

$V_y = \sqrt{2gH} \qquad V = \sqrt{V_x^2 + V_y^2} \ ; \ \tan\theta = \dfrac{V_x}{V_y}$

θ is the angle made by the velocity with vertical.

N_H and D.) The graph of Fig. 6-2 represents the distribution of distances we might get in 30 random steps (where $k = 15$ is to be read $D = 0$; $k = 16$, $D = 2$; etc.).

The variation of N_H from its expected value $N/2$ is

$$N_H - \frac{N}{2} = \frac{D}{2}. \qquad (6.11)$$

The rms deviation is

$$\left(N_H - \frac{N}{2}\right)_{\mathrm{rms}} = \tfrac{1}{2}\sqrt{N}. \qquad (6.12)$$

According to our result for D_{rms}, we expect that the "typical" distance in 30 steps ought to be $\sqrt{30} = 5.5$, or a typical k should be about $5.5/2 = 2.8$ units from 15. We see that the "width" of the curve in Fig. 6-2, measured from the center, is just about 3 units, in agreement with this result.

We are now in a position to consider a question we have avoided until now. How shall we tell whether a coin is "honest" or "loaded"? We can give now at least a partial answer. For an honest coin, we expect the fraction of the times heads appears to be 0.5, that is,

$$\frac{\langle N_H \rangle}{N} = 0.5. \qquad (6.13)$$

We *also* expect an actual N_H to deviate from $N/2$ by about $\sqrt{N}/2$, or the *fraction* to deviate by

$$\frac{1}{N}\;\frac{\sqrt{N}}{2} = \frac{1}{2\sqrt{N}}.$$

The larger N is, the closer we *expect* the fraction N_H/N to be to one-half.

In Fig. 6-6 we have plotted the fraction N_H/N for the coin tosses reported earlier in this chapter. We see the tendency for the fraction of heads to approach 0.5 for large N. Unfortunately, for any given run or combination of runs there is no *guarantee* that the observed deviation will be even *near* the *expected* deviation. There is always the finite chance that a large fluctuation—a long string of heads or tails—will give an arbitrarily large deviation. All we can say is that *if* the deviation is near the expected $1/2\sqrt{N}$ (say within a factor of 2 or 3), we have no reason to suspect the honesty of the coin. If it is much larger, we may be suspicious, but cannot prove, that the coin is loaded (or that the tosser is clever!).

6-11

Three projectiles one is dropped, other two are thrown with some velocities are shown below. Position are drawn at different intervals.
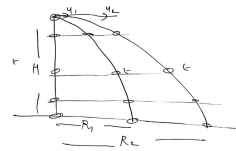

t is same for all $R_1 = u_1 t$ $R_2 = u_2 t$

$$\frac{R_1}{R_2} = \frac{u_1}{u_2}$$

All of them reach the ground at the same time. Their vertical motion are identical because they have same initial vertical velocity (zero in this case) and same acceleration.

Path of a projectile with respect to another projectile is a straight line. Their relative acceleration is zero.

Projectile Projected from the top of a building (Projected upwards)
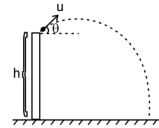
Horizontal motion	Vertical motion
$u_x = u\cos\theta$	$u_y = u\sin\theta$
$a_x = 0$	$a_y = -g$

Time of flight (T)

at $t = T$, $s_y = -h$

$$s_y = u_y t + \frac{1}{2}a_y t^2$$

$$-h = -u\sin\theta\, T + \frac{1}{2}(-g)T^2$$
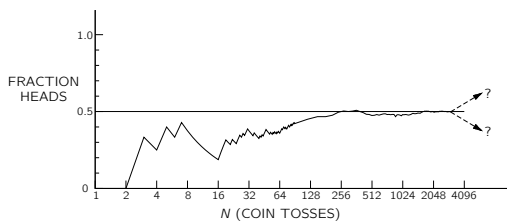
Solving this equation 'T' will be obtained

Fig. 6-6. The fraction of the tosses that gave heads in a particular sequence of N tosses of a penny.

We have also not considered how we should treat the case of a "coin" or some similar "chancy" object (say a stone that always lands in either of two positions) that we have good reason to believe *should* have a different probability for heads and tails. We have defined $P(H) = \langle N_H \rangle / N$. How shall we know what to *expect* for N_H? In some cases, the best we can do is to observe the number of heads obtained in large numbers of tosses. For want of anything better, we must set $\langle N_H \rangle = N_H(\text{observed})$. (How could we expect anything else?) We must understand, however, that in such a case a different experiment, or a different observer, might conclude that $P(H)$ was different. We would *expect*, however, that the various answers should agree within the deviation $1/2\sqrt{N}$ [if $P(H)$ is near one-half]. An experimental physicist usually says that an "experimentally determined" probability has an "error," and writes

$$P(H) = \frac{N_H}{N} \pm \frac{1}{2\sqrt{N}}. \qquad (6.14)$$

There is an implication in such an expression that there *is* a "true" or "correct" probability which *could* be computed if we knew enough, and that the observation may be in "error" due to a fluctuation. There is, however, no way to make such thinking logically consistent. It is probably better to realize that the probability concept is in a sense subjective, that it is always based on uncertain knowledge, and that its quantitative evaluation is subject to change as we obtain more information.
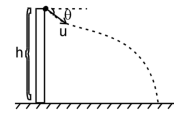
Range (R)

at t = T, s_x = R

$s_x = u_x t + \dfrac{1}{2}a_x t^2$

$R = u\cos\theta \times T + 0$

$\boxed{R = u\cos\theta \times T}$

Projectile Projected from the top of a building (Projected downwards)

Horizontal motion	Vertical motion
$u_x = u\cos\theta$	$u_y = u\sin\theta$
$a_x = 0$	$a_y = -g$

Time of flight (T)

at t = T, s_y = –h
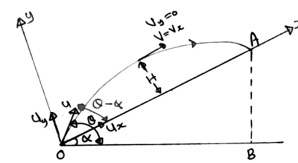
$s_y = u_y t + \dfrac{1}{2}a_y t^2$

$-h = -u\sin\theta\, T + \dfrac{1}{2}(-g)T^2$

Solving this equation 'T' will be obtained

Range $R = u\cos\theta \times T$

Projection From Inclined Plane

6-4 A probability distribution

Let us return now to the random walk and consider a modification of it. Suppose that in addition to a random choice of the *direction* ($+$ or $-$) of each step, the *length* of each step also varied in some unpredictable way, the only condition being that *on the average* the step length was one unit. This case is more representative of something like the thermal motion of a molecule in a gas. If we call the length of a step S, then S may have any value at all, but most often will be "near" 1. To be specific, we shall let $\langle S^2 \rangle = 1$ or, equivalently, $S_{\text{rms}} = 1$. Our derivation for $\langle D^2 \rangle$ would proceed as before except that Eq. (6.8) would be changed now to read

$$\langle D_N^2 \rangle = \langle D_{N-1}^2 \rangle + \langle S^2 \rangle = \langle D_{N-1}^2 \rangle + 1. \qquad (6.15)$$

We have, as before, that

$$\langle D_N^2 \rangle = N. \qquad (6.16)$$

What would we expect now for the distribution of distances D? What is, for example, the probability that $D = 0$ after 30 steps? The answer is zero! The probability is zero that D will be *any particular* value, since there is no chance at all that the sum of the backward steps (of varying lengths) would exactly equal the sum of forward steps. We cannot plot a graph like that of Fig. 6-2.

We can, however, obtain a representation similar to that of Fig. 6-2, if we ask, not what is the probability of obtaining D exactly equal to 0, 1, or 2, but instead what is the probability of obtaining D *near* 0, 1, or 2. Let us define $P(x, \Delta x)$ as the probability that D will lie in the interval Δx located at x (say from x to $x + \Delta x$). We expect that for small Δx the chance of D landing in the interval is proportional to Δx, the width of the interval. So we can write

$$P(x, \Delta x) = p(x)\, \Delta x. \qquad (6.17)$$

The function $p(\text{x})$ is called the *probability density*.

The form of $p(x)$ will depend on N, the number of steps taken, and also on the distribution of individual step lengths. We cannot demonstrate the proofs here, but for large N, $p(x)$ is the *same* for all reasonable distributions in individual step lengths, and depends only on N. We plot $p(x)$ for three values of N in Fig. 6-7. You will notice that the "half-widths" (typical spread from $x = 0$) of these curves is $\sqrt{N}$, as we have shown it should be.

$u_x = u\cos(\theta - \alpha)$

$u_y = u\sin(\theta - \alpha)$

$a_x = -g\sin\alpha$

$a_y = -g\cos\alpha$

For motion from 'O' to 'A' the displacement along the y-direction is zero.

$$\therefore\ y = u_y t + \frac{a_y t^2}{2}$$

$$O = u\sin(\theta - \alpha)T - \frac{g\cos\alpha\, T^2}{2}$$

$$\boxed{T = \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}} \quad \text{or} \quad \boxed{T = \frac{2U_y}{|a_y|}}$$

Maximum Height from inclined surface (H)

$V_y^2 - U_y^2 = 2a_y y$

$\therefore O - U_y^2 = -2a_y H$

$$\boxed{H = \frac{U_y^2}{2a_y}}$$

$$\boxed{H = \frac{U^2 \sin^2(\theta - \alpha)}{2g\cos\alpha}}$$

Horizontal displacement $OB = (u\cos\theta)T$

$$OB = u\cos\theta \times \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}$$
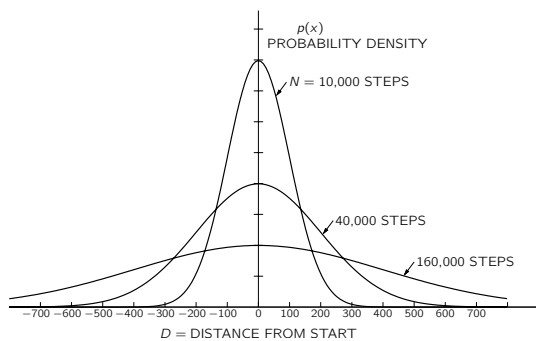
Range Along the inclined surface

Fig. 6-7. The probability density for ending up at the distance D from the starting place in a random walk of N steps. (D is measured in units of the rms step length.)

You may notice also that the value of $p(x)$ near zero is inversely proportional to $\sqrt{N}$. This comes about because the curves are all of a similar shape and their areas under the curves must all be equal. Since $p(x)\,\Delta x$ is the probability of finding D in Δx when Δx is small, we can determine the chance of finding D *somewhere* inside an arbitrary interval from x_1 to x_2, by cutting the interval in a number of small increments Δx and evaluating the sum of the terms $p(x)\,\Delta x$ for each increment. The probability that D lands somewhere between x_1 and x_2, which we may write $P(x_1 < D < x_2)$, is equal to the shaded area in Fig. 6-8. The smaller we take the increments Δx, the more correct is our result. We can write, therefore,

$$P(x_1 < D < x_2) = \sum p(x)\,\Delta x = \int_{x_1}^{x_2} p(x)\,dx. \qquad (6.18)$$

The area under the whole curve is the probability that D lands somewhere (that is, has *some* value between $x = -\infty$ and $x = +\infty$). That probability is

6-14

$$R = OA = \frac{OB}{\cos \alpha}$$

$$\boxed{R = \frac{2u^2 \cos \theta \sin(\theta - \alpha)}{g \cos^2 \alpha}}$$

$$R = \frac{u^2\, 2 \cos \theta \sin(\theta - \alpha)}{g \cos^2 \alpha}$$

$$\boxed{R = \frac{u^2 [\sin(2\theta - \alpha) - \sin \alpha]}{g \cos^2 \alpha}} \qquad 2\cos A \sin B = \sin(A + B) - \sin(A - B)$$

Range R is maximum, when $\sin(2\theta - \alpha) = 1$
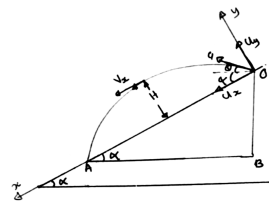
$$2\theta - \alpha = \frac{\pi}{2}$$

$$\boxed{\theta = \frac{\pi}{4} + \frac{\alpha}{2}} \qquad \theta - \alpha = \frac{\pi}{4} + \frac{\alpha}{2} - \alpha = \frac{\pi - 2\alpha}{4}$$

$$R_{max} = \frac{u^2[1 - \sin \alpha]}{g \cos^2 \alpha} = \frac{u^2(1 - \sin \alpha)}{g(1 - \sin^2 \alpha)}\,; \; R_{max} = \frac{u^2[1 - \sin \alpha]}{g[1 + \sin \alpha]\,[1 - \sin \alpha]}$$

$$\boxed{R_{max} = \frac{u^2}{g[1 + \sin \alpha]}}$$

Projectile Motion from an Inclined Plane

$$u_x = u \cos(\theta + \alpha)$$

$$u_y = u \sin(\theta + \alpha)$$

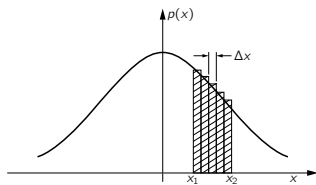$$a_x = g \sin \alpha,\; a_y = -g \cos \alpha$$

Fig. 6-8. The probability that the distance D traveled in a random walk is between x_1 and x_2 is the area under the curve of $p(x)$ from x_1 to x_2.

surely 1. We must have that

$$\int_{-\infty}^{+\infty} p(x)\, dx = 1. \tag{6.19}$$

Since the curves in Fig. 6-7 get wider in proportion to $\sqrt{N}$, their heights must be proportional to $1/\sqrt{N}$ to maintain the total area equal to 1.

The probability density function we have been describing is one that is encountered most commonly. It is known as the *normal* or *gaussian* probability density. It has the mathematical form

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-x^2/2\sigma^2}, \tag{6.20}$$

where σ is called the *standard deviation* and is given, in our case, by $\sigma = \sqrt{N}$ or, if the rms step size is different from 1, by $\sigma = \sqrt{N} S_{\mathrm{rms}}$.

We remarked earlier that the motion of a molecule, or of any particle, in a gas is like a random walk. Suppose we open a bottle of an organic compound and let some of its vapor escape into the air. If there are air currents, so that the air is circulating, the currents will also carry the vapor with them. But even in *perfectly still air*, the vapor will gradually spread out—will diffuse—until it has penetrated throughout the room. We might detect it by its color or odor. The individual molecules of the organic vapor spread out in still air because of the molecular motions caused by collisions with other molecules. If we know the average "step" size, and the number of steps taken per second, we can find the

6-15

When the object hits on the inclined plane

$$y = 0, \quad \therefore y = u_y t + \frac{a_y t^2}{2}$$

$$O = u\sin(\theta + \alpha) T - \frac{g\cos\alpha}{2} T^2 \qquad \boxed{T = \frac{2u\sin(\theta + \alpha)}{g\cos\alpha} = \frac{2U_y}{a_y}}$$

$$BA = (U_x \cos\theta) T = u\cos\theta \times \frac{2u\sin(\theta + \alpha)}{g\cos\alpha}$$

Range along the inclined surface

$$R = OA = \frac{AB}{\cos\alpha} = \frac{2u^2 \cos\theta \sin(\theta + \alpha)}{g\cos^2\alpha} \qquad \boxed{R = \frac{2u^2 \cos\theta \sin(\theta + \alpha)}{g\cos^2\alpha}}$$

$$R = \frac{u^2}{g\cos^2\alpha}[\sin(2\theta + \alpha) - \sin(-\alpha)]$$

$$\boxed{R = \frac{u^2[\sin(2\theta + \alpha) + \sin\alpha]}{g\cos^2\alpha}}$$

For maximum range $\sin(2\theta + \alpha) = 1, \quad 2\theta + \alpha = \frac{\pi}{2}$

$$\boxed{\begin{array}{c}\theta + \alpha = \dfrac{\pi - 2\alpha}{4} + \alpha\\[4pt]\theta + \alpha = \dfrac{\pi + 2\alpha}{4}\end{array}} \qquad \theta = \frac{\pi - 2\alpha}{4} \quad \boxed{R_{max} = \frac{u^2[1 + \sin\alpha]}{g\cos^2\alpha}}$$

$$R_{max} = \frac{u^2(1 + \sin\alpha)}{g(1 - \sin^2\alpha)} = \frac{u^2(1 + \sin\alpha)}{g[1 + \sin\alpha][1 - \sin\alpha]}$$

$$\boxed{R_{max} = \frac{u^2}{g\,[1 - \sin\alpha]}}$$

Maximum height (H) from the inclined surface.

At maximum height $V_y = 0$

$$\therefore V_y^2 - u_y^2 = 2a_y y \text{ becomes } 0 - u_y^2 = 2a_y H$$

$$H = \frac{U_y^2}{2a_y} = \frac{u^2 \sin^2(\theta + \alpha)}{2g\cos\alpha}$$

probability that one, or several, molecules will be found at some distance from their starting point after any particular passage of time. As time passes, more steps are taken and the gas spreads out as in the successive curves of Fig. 6-7. In a later chapter, we shall find out how the step sizes and step frequencies are related to the temperature and pressure of a gas.

Earlier, we said that the pressure of a gas is due to the molecules bouncing against the walls of the container. When we come later to make a more quantitative description, we will wish to know how fast the molecules are going when they bounce, since the impact they make will depend on that speed. We cannot, however, speak of *the* speed of the molecules. It is necessary to use a probability description. A molecule may have any speed, but some speeds are more likely than others. We describe what is going on by saying that the probability that any particular molecule will have a speed between v and $v + \Delta v$ is $p(v)\,\Delta v$, where $p(v)$, a probability density, is a given function of the speed v. We shall see later how Maxwell, using common sense and the ideas of probability, was able to find a mathematical expression for $p(v)$. The form* of the function $p(v)$ is shown in Fig. 6-9. Velocities may have any value, but are most likely to be near the most probable value v_p.
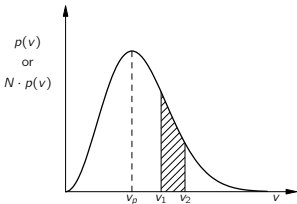
Fig. 6-9. The distribution of velocities of the molecules in a gas.

We often think of the curve of Fig. 6-9 in a somewhat different way. If we consider the molecules in a typical container (with a volume of, say, one liter), then there are a very large number N of molecules present ($N \approx 10^{22}$). Since $p(v)\,\Delta v$ is the probability that *one* molecule will have its velocity in Δv,

---

* Maxwell's expression is $p(v) = Cv^2 e^{-av^2}$, where a is a constant related to the temperature and C is chosen so that the total probability is one.

Standard results for projectile motion on an incline plane

	Up the incline	Down the incline
Range	$\dfrac{2u^2 \cos\theta \sin(\theta - \alpha)}{g\cos^2\alpha}$	$\dfrac{2u^2 \cos\theta \sin(\theta + \alpha)}{g\cos^2\alpha}$
Time of flight	$\dfrac{2u\sin(\theta - \alpha)}{g\cos\alpha}$	$\dfrac{2u\sin(\theta + \alpha)}{g\cos\alpha} = \dfrac{2u_y}{a_y}$
Maximum Range	$\dfrac{u^2}{g\,[1 + \sin\alpha]}$	$\dfrac{u^2}{g\,[1 - \sin\alpha]}$
Angle of projection for maximum range (from inclined surface)	$\dfrac{\pi - 2\alpha}{4}$	$\dfrac{\pi + 2\alpha}{4}$

by our definition of probability we mean that the *expected* number $\langle \Delta N \rangle$ to be found with a velocity in the interval Δv is given by

$$\langle \Delta N \rangle = N \, p(v) \, \Delta v. \tag{6.21}$$

We call $N \, p(v)$ the "distribution in velocity." The area under the curve between two velocities v_1 and v_2, for example the shaded area in Fig. 6-9, represents [for the curve $N \, p(v)$] the expected number of molecules with velocities between v_1 and v_2. Since with a gas we are usually dealing with large numbers of molecules, we expect the deviations from the expected numbers to be small (like $1/\sqrt{N}$), so we often neglect to say the "expected" number, and say instead: "The number of molecules with velocities between v_1 and v_2 *is* the area under the curve." We should remember, however, that such statements are always about *probable* numbers.

6-5 The uncertainty principle

The ideas of probability are certainly useful in describing the behavior of the 10^{22} or so molecules in a sample of a gas, for it is clearly impractical even to attempt to write down the position or velocity of each molecule. When probability was first applied to such problems, it was considered to be a *convenience*—a way of dealing with very complex situations. We now believe that the ideas of probability are *essential* to a description of atomic happenings. According to quantum mechanics, the mathematical theory of particles, there is always some uncertainty in the *specification* of positions and velocities. We can, at best, say that there is a certain probability that any particle will have a position near some coordinate x.

We can give a probability density $p_1(x)$, such that $p_1(x) \, \Delta x$ is the probability that the particle will be found between x and $x + \Delta x$. If the particle is reasonably well localized, say near x_0, the function $p_1(x)$ might be given by the graph of Fig. 6-10(a). Similarly, we must specify the velocity of the particle by means of a probability density $p_2(v)$, with $p_2(v) \, \Delta v$ the probability that the velocity will be found between v and $v + \Delta v$.

It is one of the fundamental results of quantum mechanics that the two functions $p_1(x)$ and $p_2(v)$ cannot be chosen independently and, in particular, cannot both be made arbitrarily narrow. If we call the typical "width" of the $p_1(x)$ curve $[\Delta x]$, and that of the $p_2(v)$ curve $[\Delta v]$ (as shown in the figure), nature demands that the *product* of the two widths be at least as big as the
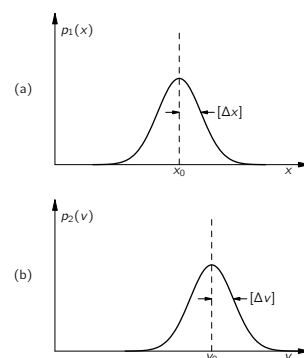
Fig. 6-10. Probability densities for observation of the position and velocity of a particle.

number $\hbar/2m$, where m is the mass of the particle. We may write this basic relationship as

$$[\Delta x] \cdot [\Delta v] \geq \hbar/2m. \tag{6.22}$$

This equation is a statement of the *Heisenberg uncertainty principle* that we mentioned earlier.

Since the right-hand side of Eq. (6.22) is a constant, this equation says that if we try to "pin down" a particle by forcing it to be at a particular place, it ends up by having a high speed. Or if we try to force it to go very slowly, or at a precise velocity, it "spreads out" so that we do not know very well just where it is. Particles behave in a funny way!

The uncertainty principle describes an inherent fuzziness that must exist in any attempt to describe nature. Our most precise description of nature *must* be in terms of *probabilities*. There are some people who do not like this way of describing nature. They feel somehow that if they could only tell what is *really* going on with a particle, they could know its speed and position simultaneously. In the early days of the development of quantum mechanics, Einstein was quite worried about this problem. He used to shake his head and say, "But, surely God
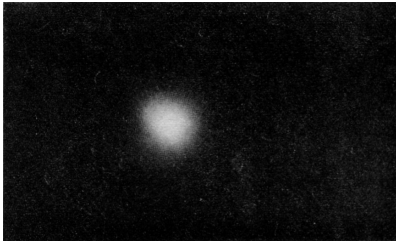
Fig. 6-11. A way of visualizing a hydrogen atom. The density (whiteness) of the cloud represents the probability density for observing the electron.

does not throw dice in determining how electrons should go!" He worried about that problem for a long time and he probably never really reconciled himself to the fact that this is the best description of nature that one can give. There are still one or two physicists who are working on the problem who have an intuitive conviction that it is possible somehow to describe the world in a different way and that all of this uncertainty about the way things are can be removed. No one has yet been successful.

The necessary uncertainty in our specification of the position of a particle becomes most important when we wish to describe the structure of atoms. In the hydrogen atom, which has a nucleus of one proton with one electron outside of the nucleus, the uncertainty in the position of the electron is as large as the atom itself! We cannot, therefore, properly speak of the electron moving in some "orbit" around the proton. The most we can say is that there is a certain *chance* $p(r)\,\Delta V$, of observing the electron in an element of volume ΔV at the distance r from the proton. The probability density $p(r)$ is given by quantum mechanics. For an undisturbed hydrogen atom $p(r) = Ae^{-2r/a}$. The number a is the "typical" radius, where the function is decreasing rapidly. Since there is a small probability of finding the electron at distances from the nucleus much greater than a, we may think of a as "the radius of the atom," about 10^{-10} meter.

We can form an image of the hydrogen atom by imagining a "cloud" whose density is proportional to the probability density for observing the electron. A sample of such a cloud is shown in Fig. 6-11. Thus our best "picture" of a hydrogen atom is a nucleus surrounded by an "electron cloud" (although we *really* mean a "probability cloud"). The electron is there somewhere, but nature permits us to know only the chance of finding it at any particular place.

In its efforts to learn as much as possible about nature, modern physics has found that certain things can never be "known" with certainty. Much of our knowledge must always remain uncertain. The *most* we can know is in terms of probabilities.