16

Relativistic Energy and Momentum

16-1 Relativity and the philosophers

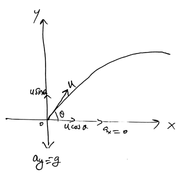
In this chapter we shall continue to discuss the principle of relativity of Einstein and Poincaré, as it affects our ideas of physics and other branches of human thought.

Poincaré made the following statement of the principle of relativity: "According to the principle of relativity, the laws of physical phenomena must be the same for a fixed observer as for an observer who has a uniform motion of translation relative to him, so that we have not, nor can we possibly have, any means of discerning whether or not we are carried along in such a motion."

When this idea descended upon the world, it caused a great stir among philosophers, particularly the "cocktail-party philosophers," who say, "Oh, it is very simple: Einstein's theory says all is relative!" In fact, a surprisingly large number of philosophers, not only those found at cocktail parties (but rather than embarrass them, we shall just call them "cocktail-party philosophers"), will say, "That all is relative is a consequence of Einstein, and it has profound influences on our ideas." In addition, they say "It has been demonstrated in physics that phenomena depend upon your frame of reference." We hear that a great deal, but it is difficult to find out what it means. Probably the frames of reference that were originally referred to were the coordinate systems which we use in the analysis of the theory of relativity. So the fact that "things depend upon your frame of reference" is supposed to have had a profound effect on modern thought. One might well wonder why, because, after all, that things depend upon one's point of view is so simple an idea that it certainly cannot have been necessary to go to all the trouble of the physical relativity theory in order to discover it. That what one sees depends upon his frame of reference is certainly known to anybody who walks around, because he sees an approaching pedestrian first from the front and then from the back; there is nothing deeper in most of the philosophy

PROJECTILE MOTION

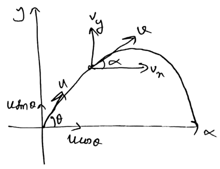
An object that is in fight after being thrown or projected is called a projectile. The motion of a projectile may be thought as the result of two separate, simultaneously occurring perpendicular components of motions. One component is along horizontal direction without any acceleration and other is along vertical direction with constant acceleration due to gravity. It was Galileo who fist stated this independency of the horizontal and the vertical components of projectile motion.



A particle is projected with a velocity u (velocity of projection) making using a angle θ with the horizontal. θ is known as angle of projection. Only force that control the projectile is gravity. We will neglect air resistance. Projectile is subjected to acceleration due to gravity $\vec{a} = -g\hat{j}$ ($a_x = 0$, $a_y = -g$).

$u\cos\theta$ is the horizontal component of velocity which remains constant. $u\sin\theta$ is initial vertical component of velocity. O is the point of projection which is taken as origin.

The velocity of the projectile after t second



which is said to have come from the theory of relativity than the remark that "A person looks different from the front than from the back." The old story about the elephant that several blind men describe in different ways is another example, perhaps, of the theory of relativity from the philosopher's point of view.

But certainly there must be deeper things in the theory of relativity than just this simple remark that "A person looks different from the front than from the back." Of course relativity is deeper than this, because *we can make definite predictions with it.* It certainly would be rather remarkable if we could predict the behavior of nature from such a simple observation alone.

There is another school of philosophers who feel very uncomfortable about the theory of relativity, which asserts that we cannot determine our absolute velocity without looking at something outside, and who would say, "It is obvious that one cannot measure his velocity without looking outside. It is self-evident that it is *meaningless* to talk about the velocity of a thing without looking outside; the physicists are rather stupid for having thought otherwise, but it has just dawned on them that this is the case. If only we philosophers had realized what the problems were that the physicists had, we could have decided immediately by brainwork that it is impossible to tell how fast one is moving without looking outside, and we could have made an enormous contribution to physics." These philosophers are always with us, struggling in the periphery to try to tell us something, but they never really understand the subtleties and depths of the problem.

Our inability to detect absolute motion is a result of *experiment* and not a result of plain thought, as we can easily illustrate. In the first place, Newton believed that it was true that one could not tell how fast he is going if he is moving with uniform velocity in a straight line. In fact, Newton first stated the principle of relativity, and one quotation made in the last chapter was a statement of Newton's. Why then did the philosophers not make all this fuss about "all is relative," or whatever, in Newton's time? Because it was not until Maxwell's theory of electrodynamics was developed that there were physical laws that suggested that one *could* measure his velocity without looking outside; soon it was found *experimentally* that one could *not.*

Now, *is* it absolutely, definitely, philosophically *necessary* that one should not be able to tell how fast he is moving without looking outside? One of the consequences of relativity was the development of a philosophy which said, "You can only define what you can measure! Since it is self-evident that one cannot measure a velocity without seeing what he is measuring it relative to, therefore it is clear that there is no *meaning* to absolute velocity. The physicists should

$$V_y = u_y + a_y t$$

$$V_y = u\sin\theta - gt$$

$$V = \sqrt{V_x^2 + V_y^2} = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2}$$

$$V = \sqrt{u^2 - 2u\sin\theta gt + g^2t^2}$$

Velocity v make an angle α with horizontal such that, $\tan\alpha = \frac{\text{opposite side}}{\text{adj.side}} = \frac{V_y}{V_x} = \frac{u\sin\theta - gt}{u\cos\theta}$

In vector form $\vec{a} = -g\hat{j}$

$$\vec{u} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j}$$

$$\vec{v} = \vec{u} + \vec{a}t$$

$$\vec{v} = u\cos\theta\,\hat{i} + u\sin\theta\,\hat{j} - gt\hat{j}$$

$$\vec{v} = u\cos\theta\,\hat{i} + (u\sin\theta - gt)\hat{j}$$

$$\left|\vec{v}\right| = \sqrt{(u\cos\theta)^2 + (u\sin\theta - gt)^2},\ \tan\alpha = \frac{u\sin\theta - gt}{u\cos\theta}$$

To find the displacement of the projectile after t seconds

along x axis

disp = velocity × time

$$\boxed{x = (u\cos\theta)t}$$

This is the equation for x-coordinate of the projectile at any time t.

along y axis $y = u_y t + \frac{a_y t^2}{2}$

$\boxed{y = u\sin\theta\, t - \frac{1}{2}gt^2}$ Equation for y coordinate or height of projectile at any time t.

have realized that they can talk only about what they can measure." But *that is the whole problem:* whether or not one *can define* absolute velocity is the same as the problem of whether or not one *can detect in an experiment*, without looking outside, whether he is moving. In other words, whether or not a thing is measurable is not something to be decided *a priori* by thought alone, but something that can be decided only by experiment. Given the fact that the velocity of light is 186,000 mi/sec, one will find few philosophers who will calmly state that it is self-evident that if light goes 186,000 mi/sec inside a car, and the car is going 100,000 mi/sec, that the light also goes 186,000 mi/sec past an observer on the ground. That is a shocking fact to them; the very ones who claim it is obvious find, when you give them a specific fact, that it is not obvious.

Finally, there is even a philosophy which says that one cannot detect *any* motion except by looking outside. It is simply not true in physics. True, one cannot perceive a *uniform* motion in a *straight line*, but if the whole room were *rotating* we would certainly know it, for everybody would be thrown to the wall—there would be all kinds of "centrifugal" effects. That the earth is turning on its axis can be determined without looking at the stars, by means of the so-called Foucault pendulum, for example. Therefore it is not true that "all is relative"; it is only *uniform velocity* that cannot be detected without looking outside. Uniform *rotation* about a fixed axis *can* be. When this is told to a philosopher, he is very upset that he did not really understand it, because to him it seems impossible that one should be able to determine rotation about an axis without looking outside. If the philosopher is good enough, after some time he may come back and say, "I understand. We really do not have such a thing as absolute rotation; we are really rotating *relative to the stars*, you see. And so some influence exerted by the stars on the object must cause the centrifugal force."

Now, for all we know, that is true; we have no way, at the present time, of telling whether there would have been centrifugal force if there were no stars and nebulae around. We have not been able to do the experiment of removing all the nebulae and then measuring our rotation, so we simply do not know. We must admit that the philosopher may be right. He comes back, therefore, in delight and says, "It is absolutely necessary that the world ultimately turn out to be this way: *absolute* rotation means nothing; it is only *relative* to the nebulae." Then we say to him, "*Now*, my friend, is it or is it not obvious that uniform velocity in a straight line, *relative to the nebulae* should produce no effects inside a car?" Now that the motion is no longer absolute, but is a motion *relative to the nebulae*, it becomes a mysterious question, and a question that can be answered only by experiment.

displacement = $r = \sqrt{x^2 + y^2}$

To find it in vector form we can use the equation $\vec{r} = x\hat{i} + y\hat{j}$ where $\vec{u} = u\sin\theta\hat{j}$, $\vec{a} = -g\hat{j}$

Equation for path of a projectile

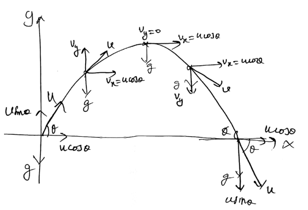
$$x = u\cos\theta t,\ t = \frac{x}{u\cos\theta}$$

$$y = u\sin\theta t - \frac{1}{2}gt^2$$

$$y = u\sin\theta\frac{x}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$$\boxed{y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}}$$

This is the equation of a parabola. Thus the path of a projectile is parabola.



At the highest point of the projectile vertical component of velocity is zero. Horizontal component is $u\cos\theta$ because it remains constant. At the highest point speed of the projectile is minimum and purely horizontal and is equal to $u\cos\theta$. Angle between acceleration and instantaneous velocity decreases from $(90+\theta)$ to $(90-\theta)$

What, then, *are* the philosophic influences of the theory of relativity? If we limit ourselves to influences in the sense of *what kind of new ideas and suggestions* are made to the physicist by the principle of relativity, we could describe some of them as follows. The first discovery is, essentially, that even those ideas which have been held for a very long time and which have been very accurately verified might be wrong. It was a shocking discovery, of course, that Newton's laws are wrong, after all the years in which they seemed to be accurate. Of course it is clear, not that the experiments were wrong, but that they were done over only a limited range of velocities, so small that the relativistic effects would not have been evident. But nevertheless, we now have a much more humble point of view of our physical laws—everything *can* be wrong!

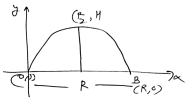
Secondly, if we have a set of "strange" ideas, such as that time goes slower when one moves, and so forth, whether we *like* them or do *not* like them is an irrelevant question. The only relevant question is whether the ideas are consistent with what is found experimentally. In other words, the "strange ideas" need only agree with *experiment*, and the only reason that we have to discuss the behavior of clocks and so forth is to demonstrate that although the notion of the time dilation is strange, it is *consistent* with the way we measure time.

Finally, there is a third suggestion which is a little more technical but which has turned out to be of enormous utility in our study of other physical laws, and that is to *look at the symmetry of the laws* or, more specifically, to look for the ways in which the laws can be transformed and leave their form the same. When we discussed the theory of vectors, we noted that the fundamental laws of motion are not changed when we rotate the coordinate system, and now we learn that they are not changed when we change the space and time variables in a particular way, given by the Lorentz transformation. So this idea of studying the patterns or operations under which the fundamental laws are not changed has proved to be a very useful one.

16-2 The twin paradox

To continue our discussion of the Lorentz transformation and relativistic effects, we consider a famous so-called "paradox" of Peter and Paul, who are supposed to be twins, born at the same time. When they are old enough to drive a space ship, Paul flies away at very high speed. Because Peter, who is left on the ground, sees Paul going so fast, all of Paul's clocks appear to go slower, his heart beats go slower, his thoughts go slower, everything goes slower, from

Time of Flight of the projectile (T)



Consider the motion of the projectile along y-axis

$$S_y = u_y t + \frac{1}{2} a_y t^2$$

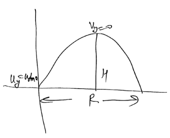
$t = T$

$$0 = u\sin\theta T - \frac{1}{2} g T^2$$

$$\boxed{T = \frac{2u\sin\theta}{g}} \text{ or } \boxed{T = \frac{2u_y}{g}}$$

Time of ascend = Time of descend = $\frac{u\sin\theta}{g}$

Maximum height of a projectile H



Consider the motion along y-axis

$$V_y^2 = u_y^2 + 2a_y g_y$$

$$0 = (u\sin\theta)^2 + 2(-gH)$$

$$2gH = (u\sin\theta)^2$$

$$\boxed{H = \frac{u^2\sin^2\theta}{2g}} \text{ or } \boxed{H = \frac{u_y^2}{2g}}$$

Peter's point of view. Of course, Paul notices nothing unusual, but if he travels around and about for a while and then comes back, he will be younger than Peter, the man on the ground! That is actually right; it is one of the consequences of the theory of relativity which has been clearly demonstrated. Just as the mu-mesons last longer when they are moving, so also will Paul last longer when he is moving. This is called a "paradox" only by the people who believe that the principle of relativity means that *all motion* is relative; they say, "Heh, heh, heh, from the point of view of Paul, can't we say that *Peter* was moving and should therefore appear to age more slowly? By symmetry, the only possible result is that both should be the same age when they meet." But in order for them to come back together and make the comparison, Paul must either stop at the end of the trip and make a comparison of clocks or, more simply, he has to come back, and the one who comes back must be the man who was moving, and he knows this, because he had to turn around. When he turned around, all kinds of unusual things happened in his space ship—the rockets went off, things jammed up against one wall, and so on—while Peter felt nothing.

So the way to state the rule is to say that *the man who has felt the accelerations*, who has seen things fall against the walls, and so on, is the one who would be the younger; that is the difference between them in an "absolute" sense, and it is certainly correct. When we discussed the fact that moving mu-mesons live longer, we used as an example their straight-line motion in the atmosphere. But we can also make mu-mesons in a laboratory and cause them to go in a curve with a magnet, and even under this accelerated motion, they last exactly as much longer as they do when they are moving in a straight line. Although no one has arranged an experiment explicitly so that we can get rid of the paradox, one could compare a mu-meson which is left standing with one that had gone around a complete circle, and it would surely be found that the one that went around the circle lasted longer. Although we have not actually carried out an experiment using a complete circle, it is really not necessary, of course, because everything fits together all right. This may not satisfy those who insist that every single fact be demonstrated directly, but we confidently predict the result of the experiment in which Paul goes in a complete circle.

16-3 Transformation of velocities

The main difference between the relativity of Einstein and the relativity of Newton is that the laws of transformation connecting the coordinates and times

Horizontal range (R) of the projectile

Horizontal range = Horizontal velocity × time of flight

$$R = u\cos\theta T,\ T = \frac{2u\sin\theta}{g}$$

$$\boxed{R = \frac{u^2\sin 2\theta}{g}} \text{ or } \boxed{R = \frac{2u_x u_y}{g}}$$

Relation connecting R, H, T and angle of projection θ

$$\frac{H}{R} = \frac{u^2\sin^2\theta}{\dfrac{2gu^2 2\sin\theta\cos\theta}{g}}$$

$$4H = R\tan\theta,\ \boxed{H = \frac{gT^2}{8}}$$

$$\therefore 4\,\frac{gT^2}{8} = R\tan\theta\ ;\ \boxed{R = \frac{gT^2}{2\tan\theta}}$$

Angle of projection for maximum range for a given speed of projection

$$R = \frac{u^2\sin 2\theta}{g},\quad [\sin 2\theta]_{max} = 1$$

$$R_{max} = \frac{u^2}{g} \qquad \sin 90^0 = 1$$

$$2\theta = 90^0$$

$$\theta = 45^0$$

To get maximum height we should throw vertically up $\theta = 90^0$

$$H_{max} = \frac{u^2}{2g}(1-\text{dimensional motion})$$

$$H_{max} = \left(\frac{u^2}{g}\right) = \frac{R_{max}}{2}$$

Galileo in his book two new sciences stated that for elevations which exceed or fall short of $45^0$by equal amount, the ranges are equal.

i.e., there are two different angles of projection for same range. If one angle is θ other angle is $90-\theta$ for same sped of projection.

between relatively moving systems are different. The correct transformation law, that of Lorentz, is

$$
\begin{aligned}
x' &= \frac{x - ut}{\sqrt{1 - u^2/c^2}},\\
y' &= y,\\
z' &= z,\\
t' &= \frac{t - ux/c^2}{\sqrt{1 - u^2/c^2}}.
\end{aligned}
\tag{16.1}
$$

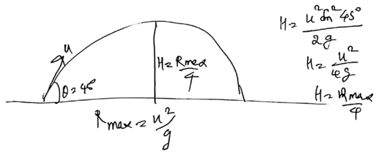
These equations correspond to the relatively simple case in which the relative motion of the two observers is along their common x-axes. Of course other directions of motion are possible, but the most general Lorentz transformation is rather complicated, with all four quantities mixed up together. We shall continue to use this simpler form, since it contains all the essential features of relativity.

Let us now discuss more of the consequences of this transformation. First, it is interesting to solve these equations in reverse. That is, here is a set of linear equations, four equations with four unknowns, and they can be solved in reverse, for x, y, z, t in terms of x', y', z', t'. The result is very interesting, since it tells us how a system of coordinates "at rest" looks from the point of view of one that is "moving." Of course, since the motions are relative and of uniform velocity, the man who is "moving" can say, if he wishes, that it is really the other fellow who is moving and he himself who is at rest. And since he is moving in the opposite direction, he should get the same transformation, but with the opposite sign of velocity. That is precisely what we find by manipulation, so that is consistent. If it did not come out that way, we would have real cause to worry!

$$
\begin{aligned}
x &= \frac{x' + ut'}{\sqrt{1 - u^2/c^2}},\\
y &= y',\\
z &= z',\\
t &= \frac{t' + ux'/c^2}{\sqrt{1 - u^2/c^2}}.
\end{aligned}
\tag{16.2}
$$

Next we discuss the interesting problem of the addition of velocities in relativity. We recall that one of the original puzzles was that light travels at 186,000 mi/sec in all systems, even when they are in relative motion. This is a special case of

Let θ_1 and θ_2 to be two different angles of projection for same range.



$R = \frac{u^2 \sin 2\theta}{g}$, since $R_1 = R_2$

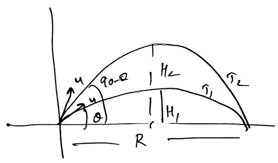
$\sin 2\theta_1 = \sin 2\theta_2 \quad \sin(180 - A) = \sin A$

$\sin 2\theta_1 = \sin(180 - 2\theta_2)$

$2\theta_1 = 180 - 2\theta_2 \;;\; \theta_1 = 90 - \theta_2$

$\theta_1 + \theta_2 = 90^0$

e.g. for same speed at angles of projection 30^0 and 60^0 range is same. Also at 15^0 and 75^0 range is same.



$\Rightarrow$ In the above situation when $R_1 = R_2$

$T_1 = \frac{2u \sin\theta}{g} \;;\; T_2 = \frac{2u \sin(90 - \theta)}{g}$

$\frac{T_1}{T_2} = \frac{\sin\theta}{\cos\theta} = \tan\theta \qquad [\sin(90 - \theta) = \cos\theta]$

$H_1 = \frac{u^2 \sin^2\theta}{2g}; H_2 = \frac{u^2 \sin^2(90 - \theta)}{2g}; \quad \frac{H_1}{H_2} = \frac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

the more general problem exemplified by the following. Suppose that an object inside a space ship is going at 100,000 mi/sec and the space ship itself is going at 100,000 mi/sec; how fast is the object inside the space ship moving from the point of view of an observer outside? We might want to say 200,000 mi/sec, which is faster than the speed of light. This is very unnerving, because it is not supposed to be going faster than the speed of light! The general problem is as follows.

Let us suppose that the object inside the ship, from the point of view of the man inside, is moving with velocity v, and that the space ship itself has a velocity u with respect to the ground. We want to know with what velocity v_x this object is moving from the point of view of the man on the ground. This is, of course, still but a special case in which the motion is in the x-direction. There will also be a transformation for velocities in the y-direction, or for any angle; these can be worked out as needed. Inside the space ship the velocity is $v_{x'}$, which means that the displacement x' is equal to the velocity times the time:

$$x' = v_{x'}t'. \tag{16.3}$$

Now we have only to calculate what the position and time are from the point of view of the outside observer for an object which has the relation (16.2) between x' and t'. So we simply substitute (16.3) into (16.2), and obtain

$$x = \frac{v_{x'}t' + ut'}{\sqrt{1 - u^2/c^2}}. \tag{16.4}$$

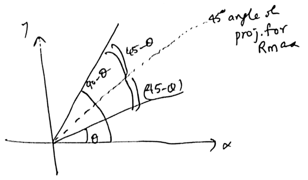
But here we find x expressed in terms of t'. In order to get the velocity as seen by the man on the outside, we must divide *his distance* by *his time*, not by the *other man's time!* So we must also calculate the *time* as seen from the outside, which is

$$t = \frac{t' + u(v_{x'}t')/c^2}{\sqrt{1 - u^2/c^2}}. \tag{16.5}$$

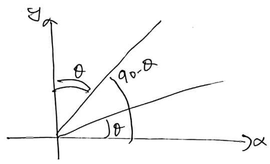
Now we must find the ratio of x to t, which is

$$v_x = \frac{x}{t} = \frac{u + v_{x'}}{1 + uv_{x'}/c^2}, \tag{16.6}$$

the square roots having cancelled. This is the law that we seek: the resultant velocity, the "summing" of two velocities, is not just the algebraic sum of two



Two different angles of projection for same range are equally inclined to the angle of projection for maximum range i.e. 45°.



The different angles of projection for same range are equally inclined to the vertical and horizontal

Equation for path of a projectile

$$y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}$$

$$y = x\tan\theta\left[1 - \frac{x}{\frac{(2u^2\sin\theta\cos\theta}{g}}\right]$$

$$y = x\tan\theta\left[1 - \frac{x}{R}\right]$$

velocities (we know that it cannot be or we get in trouble), but is "corrected" by $1+uv/c^2$.

Now let us see what happens. Suppose that you are moving inside the space ship at half the speed of light, and that the space ship itself is going at half the speed of light. Thus u is $\frac{1}{2}c$ and v is $\frac{1}{2}c$, but in the denominator uv is one-fourth, so that

$$v=\frac{\frac{1}{2}c+\frac{1}{2}c}{1+\frac{1}{4}}=\frac{4c}{5}.$$

So, in relativity, "half" and "half" does not make "one," it makes only "4/5." Of course low velocities can be added quite easily in the familiar way, because so long as the velocities are small compared with the speed of light we can forget about the $(1+uv/c^2)$ factor; but things are quite different and quite interesting at high velocity.

Let us take a limiting case. Just for fun, suppose that inside the space ship the man was observing *light itself*. In other words, $v=c$, and yet the space ship is moving. How will it look to the man on the ground? The answer will be

$$v=\frac{u+c}{1+uc/c^2}=c\,\frac{u+c}{u+c}=c.$$

Therefore, if something is moving at the speed of light inside the ship, it will appear to be moving at the speed of light from the point of view of the man on the ground too! This is good, for it is, in fact, what the Einstein theory of relativity was designed to do in the first place—so it had *better* work!

Of course, there are cases in which the motion is not in the direction of the uniform translation. For example, there may be an object inside the ship which is just moving "upward" with the velocity $v_{y'}$ with respect to the ship, and the ship is moving "horizontally." Now, we simply go through the same thing, only using y's instead of x's, with the result

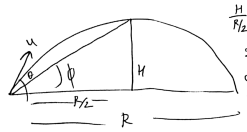
$$y=y'=v_{y'}t',$$

so that if $v_{x'}=0$,

$$v_y=\frac{y}{t}=v_{y'}\sqrt{1-u^2/c^2}. \tag{16.7}$$

Thus a sidewise velocity is no longer $v_{y'}$, but $v_{y'}\sqrt{1-u^2/c^2}$. We found this result by substituting and combining the transformation equations, but we can

Relation between angle of projection 'θ' and angle of elevation ϕ, at maximum height



$4H = R\tan\theta$ —(1)

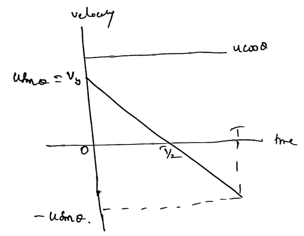
$\frac{H}{R/2} = \tan\phi$

$2H = R\tan\phi$ —(2)

$\frac{(1)}{(2)}$ $2 = \frac{\tan\theta}{\tan\phi}$

$\tan\phi = \frac{1}{2}\tan\theta$

Velocity time graph

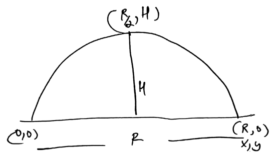


here $u\cos\theta > u\sin\theta$

$\cos\theta > \sin\theta$

$\theta < 45^\circ$

If the path of a projectile is given by the equation $y=ax-bx^2$ find R, H, T



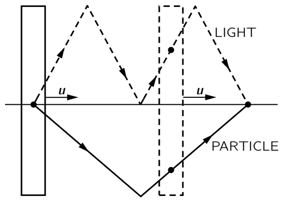


Fig. 16-1. Trajectories described by a light ray and particle inside a moving clock.

also see the result directly from the principle of relativity for the following reason (it is always good to look again to see whether we can see the reason). We have already (Fig. 15-3) seen how a possible clock might work when it is moving; the light appears to travel at an angle at the speed c in the fixed system, while it simply goes vertically with the same speed in the moving system. We found that the *vertical component* of the velocity in the fixed system is less than that of light by the factor $\sqrt{1-u^2/c^2}$ (see Eq. 15.3). But now suppose that we let a material particle go back and forth in this same "clock," but at some integral fraction $1/n$ of the speed of light (Fig. 16-1). Then when the particle has gone back and forth once, the light will have gone exactly n times. That is, each "click" of the "particle" clock will coincide with each nth "click" of the light clock. *This fact must still be true when the whole system is moving*, because the physical phenomenon of coincidence will be a coincidence in any frame. Therefore, since the speed c_y is less than the speed of light, the speed v_y of the particle must be slower than the corresponding speed by the same square-root ratio! That is why the square root appears in any vertical velocity.

16-4 Relativistic mass

We learned in the last chapter that the mass of an object increases with velocity, but no demonstration of this was given, in the sense that we made no arguments analogous to those about the way clocks have to behave. However, we *can* show that, as a consequence of relativity plus a few other reasonable assumptions, the mass must vary in this way. (We have to say "a few other assumptions" because we cannot prove anything unless we have some laws which

$y = ax - bx^2$, if $y = 0$, $x = R$

$0 = ax - bx^2$

$ax = bx^2$

$x = R = a/b$

Compare $y = x\tan\theta[1 - x/R]$

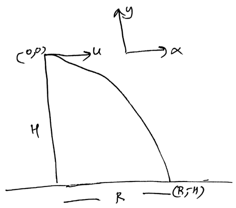
$\tan\theta = a$

$4H = R\tan\theta$

$4H = \frac{a}{b}a, \qquad H = \frac{a^2}{4b}$

$H = \frac{T^2 g}{8} = \frac{a^2}{4b}$

Horizontal projection



A particle is projected horizontally with a velocity u from a height H. It follows a parabolic path and stroke the ground, horizontal component of velocity u remains constant vertical component is subjected to acceleration due to gravity.

$u_x = u, \; u_y = 0 \quad \vec{u} = u\hat{i}$

$a_x = 0, a_y = -g \quad \vec{a} = -g\hat{i}$

To find time of flight we consider the motion along y-axis

$S_y = u_y t + \frac{1}{2}a_y t^2$

$-H = oxt + \frac{1}{2}(-g)t^2$

we assume to be true, if we expect to make meaningful deductions.) To avoid the need to study the transformation laws of force, we shall analyze a *collision*, where we need know nothing about the laws of force, except that we shall assume the conservation of momentum and energy. Also, we shall assume that the momentum of a particle which is moving is a vector and is always directed in the direction of the velocity. However, we shall not assume that the momentum is a *constant* times the velocity, as Newton did, but only that it is some *function* of velocity. We thus write the momentum vector as a certain coefficient times the vector velocity:

$$\boldsymbol{p} = m_v \boldsymbol{v}. \tag{16.8}$$

We put a subscript v on the coefficient to remind us that it is a function of velocity, and we shall agree to call this coefficient m_v the "mass." Of course, when the velocity is small, it is the same mass that we would measure in the slow-moving experiments that we are used to. Now we shall try to demonstrate that the formula for m_v must be $m_0/\sqrt{1 - v^2/c^2}$, by arguing from the principle of relativity that the laws of physics must be the same in every coordinate system.

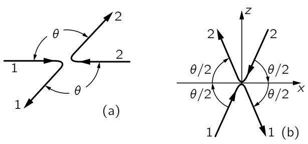


Fig. 16-2. Two views of an elastic collision between equal objects moving at the same speed in opposite directions.

Suppose that we have two particles, like two protons, that are absolutely equal, and they are moving toward each other with exactly equal velocities. Their total momentum is zero. Now what can happen? After the collision, their directions of motion must be exactly opposite to each other, because if they are not exactly opposite, there will be a nonzero total vector momentum, and momentum would not have been conserved. Also they must have the same speeds, since they are exactly similar objects; in fact, they must have the same speed they started with, since we suppose that the energy is conserved in these collisions. So the diagram of an elastic collision, a reversible collision, will look like Fig. 16-2(a): all the arrows are the same length, all the speeds are equal. We shall suppose that such

$H = \frac{1}{2}gt^2$

$t = \sqrt{\frac{2}{g}H}$

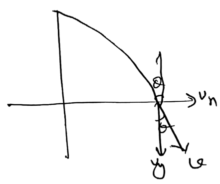
This is same as the time take by a dropped body to reach the ground dropped from rest.

To find range R, consider the horizontal motion,

$S_x = u_x t$

$R = ut \qquad R = u\sqrt{\frac{2H}{g}}$

Velocity with which it hits the ground



$V_x = u$

$V_y^2 = u_y^2 + 2a_y y$

$Vy^2 = 0 + 2(-g)(-H)$

$V_y = \sqrt{2gH} \qquad V = \sqrt{V_x^2 + V_y^2}$; $\tan\theta = \frac{V_x}{V_y}$

θ is the angle made by the velocity with vertical.

collisions can always be arranged, that any angle θ can occur, and that any speed could be used in such a collision. Next, we notice that this same collision can be viewed differently by turning the axes, and just for convenience we *shall* turn the axes, so that the horizontal splits it evenly, as in Fig. 16-2(b). It is the same collision redrawn, only with the axes turned.

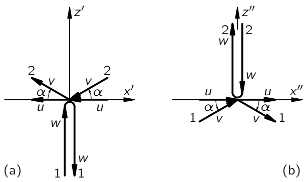
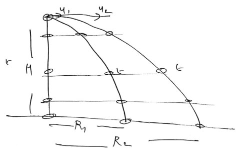


Fig. 16-3. Two more views of the collision, from moving cars.

Now here is the real trick: let us look at this collision from the point of view of someone riding along in a car that is moving with a speed equal to the horizontal component of the velocity of one particle. Then how does the collision look? It looks as though particle 1 is just going straight up, because it has lost its horizontal component, and it comes straight down again, also because it does not have that component. That is, the collision appears as shown in Fig. 16-3(a). Particle 2, however, was going the other way, and as we ride past it appears to fly by at some terrific speed and at a smaller angle, but we can appreciate that the angles before and after the collision are the *same*. Let us denote by u the horizontal component of the velocity of particle 2, and by w the vertical velocity of particle 1.

Now the question is, what is the vertical velocity $u \tan \alpha$? If we knew that, we could get the correct expression for the momentum, using the law of conservation of momentum in the vertical direction. Clearly, the horizontal component of the momentum is conserved: it is the same before and after the collision for both particles, and is zero for particle 1. So we need use the conservation law only for the upward velocity $u \tan \alpha$. But we *can* get the upward velocity, simply by looking at the same collision going the other way! If we look at the collision of Fig. 16-3(a) from a car moving to the left with speed u, we see the same collision, except "turned over," as shown in Fig. 16-3(b). Now particle 2 is the one that goes up and down with speed w, and particle 1 has picked up the horizontal speed u.

Three projectiles one is dropped, other two are thrown with some velocities are shown below. Position are drawn at different intervals.



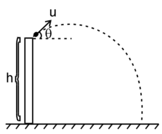
t is same for all $\qquad R_1 = u_1 t \qquad R_2 = u_2 t$

$$\frac{R_1}{R_2} = \frac{u_1}{u_2}$$

All of them reach the ground at the same time. Their vertical motion are identical because they have same initial vertical velocity (zero in this case) and same acceleration.

Path of a projectile with respect to another projectile is a straight line. Their relative acceleration is zero.

Projectile Projected from the top of a building (Projected upwards)



Horizontal motion	Vertical motion
$u_x = u\cos\theta$	$u_y = u\sin\theta$
$a_x = 0$	$a_y = -g$

Time of flight (T)

at t = T, $s_y = -h$

$$s_y = u_y t + \frac{1}{2} a_y t^2$$

$$-h = -u\sin\theta T + \frac{1}{2}(-g)T^2$$

Solving this equation 'T' will be obtained

Of course, now we *know* what the velocity $u\tan\alpha$ is: it is $w\sqrt{1-u^2/c^2}$ (see Eq. 16.7). We know that the change in the vertical momentum of the vertically moving particle is

$$\Delta p = 2m_w w$$

(2, because it moves up and back down). The obliquely moving particle has a certain velocity v whose components we have found to be u and $w\sqrt{1-u^2/c^2}$, and whose mass is m_v. The change in *vertical* momentum of this particle is therefore $\Delta p' = 2m_v w\sqrt{1-u^2/c^2}$ because, in accordance with our assumed law (16.8), the momentum component is always the mass corresponding to the magnitude of the velocity times the component of the velocity in the direction of interest. Thus in order for the total momentum to be zero the vertical momenta must cancel and the ratio of the mass moving with speed v and the mass moving with speed w must therefore be

$$\frac{m_w}{m_v} = \sqrt{1-u^2/c^2}. \tag{16.9}$$

Let us take the limiting case that w is infinitesimal. If w is very tiny indeed, it is clear that v and u are practically equal. In this case, $m_w \to m_0$ and $m_v \to m_u$. The grand result is

$$m_u = \frac{m_0}{\sqrt{1-u^2/c^2}}. \tag{16.10}$$

It is an interesting exercise now to check whether or not Eq. (16.9) is indeed true for arbitrary values of w, assuming that Eq. (16.10) is the right formula for the mass. Note that the velocity v needed in Eq. (16.9) can be calculated from the right-angle triangle:

$$v^2 = u^2 + w^2(1-u^2/c^2).$$

It will be found to check out automatically, although we used it only in the limit of small w.

Now, let us accept that momentum is conserved and that the mass depends upon the velocity according to (16.10) and go on to find what else we can conclude. Let us consider what is commonly called an *inelastic collision*. For simplicity, we shall suppose that two objects of the same kind, moving oppositely with equal speeds w, hit each other and stick together, to become some new, stationary object, as shown in Fig. 16-4(a). The mass m of each corresponds to w, which, as we know, is $m_0/\sqrt{1-w^2/c^2}$. If we assume the conservation of momentum and

Range (R)

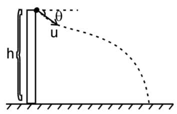
at t = T, s_x = R

$s_x = u_x t + \frac{1}{2}a_x t^2$

$R = u\cos\theta \times T + 0$

$\boxed{R = u\cos\theta \times T}$

Projectile Projected from the top of a building (Projected downwards)



Horizontal motion

$u_x = u\cos\theta$

$a_x = 0$

Vertical motion

$u_y = u\sin\theta$

$a_y = -g$

Time of flight (T)

at t = T, s_y = –h

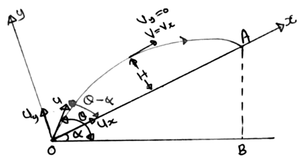
$s_y = u_y t + \frac{1}{2}a_y t^2$

$-h = -u\sin\theta T + \frac{1}{2}(-g)T^2$

Solving this equation 'T' will be obtained

Range $R = u\cos\theta \times T$

Projection From Inclined Plane



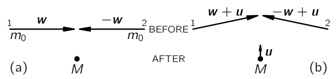


Fig. 16-4. Two views of an inelastic collision between equally massive objects.

the principle of relativity, we can demonstrate an interesting fact about the mass of the new object which has been formed. We imagine an infinitesimal velocity u at right angles to w (we can do the same with finite values of u, but it is easier to understand with an infinitesimal velocity), then look at this same collision as we ride by in an elevator at the velocity $-u$. What we see is shown in Fig. 16-4(b). The composite object has an unknown mass M. Now object 1 moves with an upward component of velocity u and a horizontal component which is practically equal to w, and so also does object 2. After the collision we have the mass M moving upward with velocity u, considered very small compared with the speed of light, and also small compared with w. Momentum must be conserved, so let us estimate the momentum in the upward direction before and after the collision. Before the collision we have $p \approx 2m_w u$, and after the collision the momentum is evidently $p' = M_u u$, but M_u is essentially the same as M_0 because u is so small. These momenta must be equal because of the conservation of momentum, and therefore

$$M_0 = 2m_w. \tag{16.11}$$

The mass of the object which is formed when two equal objects collide must be twice the mass of the objects which come together. You might say, "Yes, of course, that is the conservation of mass." But not "Yes, of course," so easily, because *these masses have been enhanced* over the masses that they would be if they were standing still, yet they still contribute, to the total M, not the mass they have when standing still, but *more*. Astonishing as that may seem, in order for the conservation of momentum to work when two objects come together, the mass that they form must be greater than the rest masses of the objects, even though the objects are at rest after the collision!

16-5 Relativistic energy

In the last chapter we demonstrated that as a result of the dependence of the mass on velocity and Newton's laws, the changes in the kinetic energy of an

$u_x = u\cos(\theta - \alpha)$

$u_y = u\sin(\theta - \alpha)$

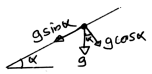
$a_x = -g\sin\alpha$

$a_y = -g\cos\alpha$

For motion from 'O' to 'A' the displacement along the y-direction is zero.

$\therefore\ y = u_y t + \frac{a_y t^2}{2}$

$O = u\sin(\theta - \alpha)T - \frac{g\cos\alpha\, T^2}{2}$



$\boxed{T = \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}}$ or $\boxed{T = \frac{2U_y}{|a_y|}}$

Maximum Height from inclined surface (H)

$V_y^2 - U_y^2 = 2a_y y$

$\therefore O - U_y^2 = -2a_y H$

$$\boxed{H = \frac{U_y^2}{2a_y} \qquad H = \frac{U^2\sin^2(\theta - \alpha)}{2g\cos\alpha}}$$

Horizontal displacement $OB = (u\cos\theta)T$

$OB = u\cos\theta \times \frac{2u\sin(\theta - \alpha)}{g\cos\alpha}$

Range Along the inclined surface

object resulting from the total work done by the forces on it always comes out to be

$$\Delta T = (m_u - m_0)c^2 = \frac{m_0c^2}{\sqrt{1-u^2/c^2}} - m_0c^2. \tag{16.12}$$

We even went further, and guessed that the total energy is the total mass times c^2. Now we continue this discussion.

Suppose that our two equally massive objects that collide can still be "seen" inside M. For instance, a proton and a neutron are "stuck together," but are still moving about inside of M. Then, although we might at first expect the mass M to be $2m_0$, we have found that it is not $2m_0$, but $2m_w$. Since $2m_w$ is what is put in, but $2m_0$ are the rest masses of the things inside, the *excess* mass of the composite object is equal to the kinetic energy brought in. This means, of course, that *energy has inertia*. In the last chapter we discussed the heating of a gas, and showed that because the gas molecules are moving and moving things are heavier, when we put energy into the gas its molecules move faster and so the gas gets heavier. But in fact the argument is completely general, and our discussion of the inelastic collision shows that the mass is there whether or not it is *kinetic* energy. In other words, if two particles come together and produce potential or any other form of energy; if the pieces are slowed down by climbing hills, doing work against internal forces, or whatever; then it is still true that the mass is the total energy that has been put in. So we see that the conservation of mass which we have deduced above is equivalent to the conservation of energy, and therefore there is no place in the theory of relativity for strictly inelastic collisions, as there was in Newtonian mechanics. According to Newtonian mechanics it is all right for two things to collide and so form an object of mass $2m_0$ which is in no way distinct from the one that would result from putting them together slowly. Of course we know from the law of conservation of energy that there is more kinetic energy inside, but that does not affect the mass, according to Newton's laws. But now we see that this is impossible; because of the kinetic energy involved in the collision, the resulting object will be *heavier;* therefore, it will be a *different* object. When we put the objects together gently they make something whose mass is $2m_0$; when we put them together forcefully, they make something whose mass is greater. When the mass is different, we can *tell* that it is different. So, necessarily, the conservation of energy must go along with the conservation of momentum in the theory of relativity.

This has interesting consequences. For example, suppose that we have an object whose mass M is measured, and suppose something happens so that it flies

$$R = OA = \frac{OB}{\cos\alpha}$$

$$\boxed{R = \frac{2u^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}}$$

$$R = \frac{u^2\, 2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}$$

$$\boxed{R = \frac{u^2[\sin(2\theta-\alpha)-\sin\alpha}{g\cos^2\alpha}} \qquad 2\cos A\sin B = \sin(A+B) - \sin(A-B)$$

Range R is maximum, when $\sin(2\theta-\alpha)=1$

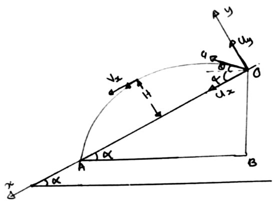
$$2\theta-\alpha = \frac{\pi}{2}$$

$$\boxed{\theta = \frac{\pi}{4}+\frac{\alpha}{2}} \qquad \theta-\alpha = \frac{\pi}{4}+\frac{\alpha}{2}-\alpha = \frac{\pi-2\alpha}{4}$$

$$R_{max} = \frac{u^2[1-\sin\alpha]}{g\cos^2\alpha} = \frac{u^2(1-\sin\alpha)}{g(1-\sin^2\alpha)}; \quad R_{max} = \frac{u^2[1-\sin\alpha]}{g[1+\sin\alpha][1-\sin\alpha]}$$

$$\boxed{R_{max} = \frac{u^2}{g[1+\sin\alpha]}}$$

Projectile Motion from an Inclined Plane



$$u_x = u\cos(\theta+\alpha)$$

$$u_y = u\sin(\theta+\alpha)$$

$$a_x = g\sin\alpha,\ a_y = -g\cos\alpha$$

into two equal pieces moving with speed w, so that they each have a mass m_w. Now suppose that these pieces encounter enough material to slow them up until they stop; then they will have mass m_0. How much energy will they have given to the material when they have stopped? Each will give an amount $(m_w - m_0)c^2$, by the theorem that we proved before. This much energy is left in the material in some form, as heat, potential energy, or whatever. Now $2m_w = M$, so the liberated energy is $E = (M - 2m_0)c^2$. This equation was used to estimate how much energy would be liberated under fission in the atomic bomb, for example. (Although the fragments are not exactly equal, they are nearly equal.) The mass of the uranium atom was known—it had been measured ahead of time—and the atoms into which it split, iodine, xenon, and so on, all were of known mass. By masses, we do not mean the masses while the atoms are moving, we mean the masses when the atoms are *at rest*. In other words, both M and m_0 are known. So by subtracting the two numbers one can calculate how much energy will be released if M can be made to split in "half." For this reason poor old Einstein was called the "father" of the atomic bomb in all the newspapers. Of course, all that meant was that he could tell us ahead of time how much energy would be released if we told him what process would occur. The energy that should be liberated when an atom of uranium undergoes fission was estimated about six months before the first direct test, and as soon as the energy was in fact liberated, someone measured it directly (and if Einstein's formula had not worked, they would have measured it anyway), and the moment they measured it they no longer needed the formula. Of course, we should not belittle Einstein, but rather should criticize the newspapers and many popular descriptions of what causes what in the history of physics and technology. The problem of how to get the thing to occur in an effective and rapid manner is a completely different matter.

The result is just as significant in chemistry. For instance, if we were to weigh the carbon dioxide molecule and compare its mass with that of the carbon and the oxygen, we could find out how much energy would be liberated when carbon and oxygen form carbon dioxide. The only trouble here is that the differences in masses are so small that it is technically very difficult to do.

Now let us turn to the question of whether we should add m_0c^2 to the kinetic energy and say from now on that the total energy of an object is mc^2. First, if we can still *see* the component pieces of rest mass m_0 inside M, then we could say that some of the mass M of the compound object is the mechanical rest mass of the parts, part of it is kinetic energy of the parts, and part of it is potential energy of the parts. But we have discovered, in nature, particles of various kinds which

When the object hits on the inclined plane

$$y=0, \quad \therefore y=u_y t+\frac{a_y t^2}{2}$$

$$O=u\sin(\theta+\alpha)T-\frac{g\cos\alpha}{2}T^2 \qquad \boxed{T=\frac{2u\sin(\theta+\alpha)}{g\cos\alpha}=\frac{2U_y}{a_y}}$$

$$BA=(U_x\cos\theta)T=u\cos\theta\times\frac{2u\sin(\theta+\alpha)}{g\cos\alpha}$$

Range along the inclined surface

$$R=OA=\frac{AB}{\cos\alpha}=\frac{2u^2\cos\theta\sin(\theta+\alpha)}{g\cos^2\alpha} \qquad \boxed{R=\frac{2u^2\cos\theta\sin(\theta+\alpha)}{g\cos^2\alpha}}$$

$$R=\frac{u^2}{g\cos^2\alpha}[\sin(2\theta+\alpha)-\sin(-\alpha)]$$

$$\boxed{R=\frac{u^2[\sin(2\theta+\alpha)+\sin\alpha]}{g\cos^2\alpha}}$$

For maximum range $\sin(2\theta+\alpha)=1, \quad 2\theta+\alpha=\frac{\pi}{2}$

$$\boxed{\begin{aligned}\theta+\alpha&=\frac{\pi-2\alpha}{4}+\alpha\\ \theta+\alpha&=\frac{\pi+2\alpha}{4}\end{aligned}} \qquad \theta=\frac{\pi-2\alpha}{4} \qquad \boxed{R_{max}=\frac{u^2[1+\sin\alpha]}{g\cos^2\alpha}}$$

$$R_{max}=\frac{u^2(1+\sin\alpha)}{g(1-\sin^2\alpha)}=\frac{u^2(1+\sin\alpha)}{g[1+\sin\alpha][1-\sin\alpha]}$$

$$\boxed{R_{max}=\frac{u^2}{g[1-\sin\alpha]}}$$

Maximum height (H) from the inclined surface.

At maximum height $V_y = 0$

$\therefore V_y^2-u_y^2=2a_y y$ becomes $0-u_y^2=2a_y H$

$$H=\frac{U_y^2}{2a_y}=\frac{u^2\sin^2(\theta+\alpha)}{2g\cos\alpha}$$

undergo reactions just like the one we have treated above, in which with all the study in the world, we *cannot see the parts inside*. For instance, when a K-meson disintegrates into two pions it does so according to the law (16.11), but the idea that a K is made out of 2 π's is a useless idea, because it also disintegrates into 3 π's!

Therefore we have a *new idea*: we do not have to know what things are made of inside; we cannot and need not identify, inside a particle, which of the energy is rest energy of the parts into which it is going to disintegrate. It is not convenient and often not possible to separate the total mc^2 energy of an object into rest energy of the inside pieces, kinetic energy of the pieces, and potential energy of the pieces; instead, we simply speak of the *total energy* of the particle. We "shift the origin" of energy by adding a constant m_0c^2 to everything, and say that the total energy of a particle is the mass in motion times c^2, and when the object is standing still, the energy is the mass at rest times c^2.

Finally, we find that the velocity v, momentum P, and total energy E are related in a rather simple way. That the mass in motion at speed v is the mass m_0 at rest divided by $\sqrt{1-v^2/c^2}$, surprisingly enough, is rarely used. Instead, the following relations are easily proved, and turn out to be very useful:

$$E^2 - P^2c^2 = m_0^2c^4 \tag{16.13}$$

and

$$Pc = Ev/c. \tag{16.14}$$

Note : For a given speed, the direction which gives the maximum range of the projectile on an inclined plane, bisects the angle between the incline and the vertical, for upward or downward projection.

Standard results for projectile motion on an incline plane

	Up the incline	**Down the incline**
Range	$\frac{2u^2\cos\theta\sin(\theta-\alpha)}{g\cos^2\alpha}$	$\frac{2u^2\cos\theta\sin(\theta+\alpha)}{g\cos^2\alpha}$
Time of flight	$\frac{2u\sin(\theta-\alpha)}{g\cos\alpha}$	$\frac{2u\sin(\theta+\alpha)}{g\cos\alpha}=\frac{2u_y}{a_y}$
Maximum Range	$\frac{u^2}{g\,[1+\sin\alpha]}$	$\frac{u^2}{g\,[1-\sin\alpha]}$
Angle of projection for maximum range (from inclined surface)	$\frac{\pi-2\alpha}{4}$	$\frac{\pi+2\alpha}{4}$