About Chapter 5

In the last chapter, we saw a proof of the fundamental status of the entropy as a measure of average information content. We defined a data compression scheme using *fixed length block codes*, and proved that as N increases, it is possible to encode N i.i.d. variables $\mathbf{x} = (x_1, \ldots, x_N)$ into a block of $N(H(X)+\epsilon)$ bits with vanishing probability of error, whereas if we attempt to encode X^N into $N(H(X) - \epsilon)$ bits, the probability of error is virtually 1.

We thus verified the *possibility* of data compression, but the block coding defined in the proof did not give a practical algorithm. In this chapter and the next, we study practical data compression algorithms. Whereas the last chapter's compression scheme used large blocks of *fixed* size and was *lossy*, in the next chapter we discuss *variable-length* compression schemes that are practical for small block sizes and that are *not lossy*.

Imagine a rubber glove filled with water. If we compress two fingers of the glove, some other part of the glove has to expand, because the total volume of water is constant. (Water is essentially incompressible.) Similarly, when we shorten the codewords for some outcomes, there must be other codewords that get longer, if the scheme is not lossy. In this chapter we will discover the information-theoretic equivalent of water volume.

Before reading Chapter 5, you should have worked on exercise 2.26 (p.37).

We will use the following notation for intervals:

$$\begin{array}{ll} x \in [1, 2) & \text{means that } x \geq 1 \text{ and } x < 2; \\ x \in (1, 2] & \text{means that } x > 1 \text{ and } x \leq 2. \end{array}$$

5

Symbol Codes

In this chapter, we discuss *variable-length symbol codes*, which encode one source symbol at a time, instead of encoding huge strings of N source symbols. These codes are *lossless:* unlike the last chapter's block codes, they are guaranteed to compress and decompress without any errors; but there is a chance that the codes may sometimes produce encoded strings longer than the original source string.

The idea is that we can achieve compression, on average, by assigning *shorter* encodings to the more probable outcomes and *longer* encodings to the less probable.

The key issues are:

What are the implications if a symbol code is *lossless*? If some codewords are shortened, by how much do other codewords have to be lengthened?

Making compression practical. How can we ensure that a symbol code is easy to decode?

Optimal symbol codes. How should we assign codelengths to achieve the best compression, and what is the best achievable compression?

We again verify the fundamental status of the Shannon information content and the entropy, proving:

Source coding theorem (symbol codes). There exists a variable-length encoding C of an ensemble X such that the average length of an encoded symbol, $L(C, X)$, satisfies $L(C, X) \in [H(X), H(X) + 1)$.

The average length is equal to the entropy $H(X)$ only if the codelength for each outcome is equal to its Shannon information content.

We will also define a constructive procedure, the Huffman coding algorithm, that produces optimal symbol codes.

Notation for alphabets. $\mathcal{A}^N$ denotes the set of ordered N-tuples of elements from the set $\mathcal{A}$, i.e., all strings of length N. The symbol $\mathcal{A}^+$ will denote the set of all strings of finite length composed of elements from the set $\mathcal{A}$.

Example 5.1. $\{0, 1\}^3 = \{000, 001, 010, 011, 100, 101, 110, 111\}$.

Example 5.2. $\{0, 1\}^+ = \{0, 1, 00, 01, 10, 11, 000, 001, \ldots\}$.

▶ 5.1 Symbol codes

A (binary) symbol code C for an ensemble X is a mapping from the range of x, $\mathcal{A}_X = \{a_1, \ldots, a_I\}$, to $\{\mathtt{0},\mathtt{1}\}^+$. $c(x)$ will denote the *codeword* corresponding to x, and $l(x)$ will denote its length, with $l_i = l(a_i)$.

The *extended code* C^+ is a mapping from $\mathcal{A}_X^+$ to $\{\mathtt{0},\mathtt{1}\}^+$ obtained by concatenation, without punctuation, of the corresponding codewords:

$$c^+(x_1 x_2 \ldots x_N) = c(x_1)c(x_2)\ldots c(x_N). \tag{5.1}$$

[The term 'mapping' here is a synonym for 'function'.]

Example 5.3. A symbol code for the ensemble X defined by

$$\begin{array}{rcl} \mathcal{A}_X & = & \{\ \mathtt{a},\ \mathtt{b},\ \mathtt{c},\ \mathtt{d}\ \}, \\ \mathcal{P}_X & = & \{\ 1/2, 1/4, 1/8, 1/8\ \}, \end{array} \tag{5.2}$$

is C_0, shown in the margin.

C_0:

a_i	$c(a_i)$	l_i
a	1000	4
b	0100	4
c	0010	4
d	0001	4

Using the extended code, we may encode `acdbac` as

$$c^+(\mathtt{acdbac}) = \mathtt{100000100001010010000010}. \tag{5.3}$$

There are basic requirements for a useful symbol code. First, any encoded string must have a unique decoding. Second, the symbol code must be easy to decode. And third, the code should achieve as much compression as possible.

Any encoded string must have a unique decoding

A code $C(X)$ is uniquely decodeable if, under the extended code C^+, no two distinct strings have the same encoding, i.e.,

$$\forall\, \mathbf{x}, \mathbf{y} \in \mathcal{A}_X^+,\ \ \mathbf{x} \neq \mathbf{y} \ \Rightarrow\ c^+(\mathbf{x}) \neq c^+(\mathbf{y}). \tag{5.4}$$

The code C_0 defined above is an example of a uniquely decodeable code.

The symbol code must be easy to decode

A symbol code is easiest to decode if it is possible to identify the end of a codeword as soon as it arrives, which means that no codeword can be a *prefix* of another codeword. [A word c is a *prefix* of another word d if there exists a tail string t such that the concatenation ct is identical to d. For example, `1` is a prefix of `101`, and so is `10`.]

We will show later that we don't lose any performance if we constrain our symbol code to be a prefix code.

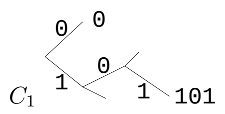
A symbol code is called a prefix code if no codeword is a prefix of any other codeword.

A prefix code is also known as an *instantaneous* or *self-punctuating* code, because an encoded string can be decoded from left to right without looking ahead to subsequent codewords. The end of a codeword is immediately recognizable. A prefix code is uniquely decodeable.

Prefix codes are also known as 'prefix-free codes' or 'prefix condition codes'.

Prefix codes correspond to trees, as illustrated in the margin of the next page.

Example 5.4. The code $C_1 = \{\texttt{0}, \texttt{101}\}$ is a prefix code because `0` is not a prefix of `101`, nor is `101` a prefix of `0`.

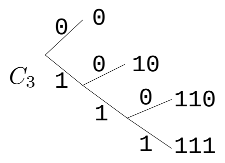


Example 5.5. Let $C_2 = \{\texttt{1}, \texttt{101}\}$. This code is not a prefix code because `1` is a prefix of `101`.

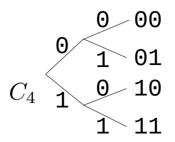
Example 5.6. The code $C_3 = \{\texttt{0}, \texttt{10}, \texttt{110}, \texttt{111}\}$ is a prefix code.

Example 5.7. The code $C_4 = \{\texttt{00}, \texttt{01}, \texttt{10}, \texttt{11}\}$ is a prefix code.

Exercise 5.8.[1, p.104] Is C_2 uniquely decodeable?



Example 5.9. Consider exercise 4.1 (p.66) and figure 4.2 (p.69). Any weighing strategy that identifies the odd ball and whether it is heavy or light can be viewed as assigning a *ternary* code to each of the 24 possible states. This code is a prefix code.



Prefix codes can be represented on binary trees. *Complete* prefix codes correspond to binary trees with no unused branches. C_1 is an incomplete code.

The code should achieve as much compression as possible

The expected length $L(C, X)$ of a symbol code C for ensemble X is

$$L(C, X) = \sum_{x \in \mathcal{A}_X} P(x)\, l(x). \tag{5.5}$$

We may also write this quantity as

$$L(C, X) = \sum_{i=1}^{I} p_i l_i \tag{5.6}$$

where $I = |\mathcal{A}_X|$.

Example 5.10. Let

$$\begin{array}{rrcl} & \mathcal{A}_X & = & \{\, \texttt{a},\ \texttt{b},\ \texttt{c},\ \texttt{d}\, \}, \\ \text{and} & \mathcal{P}_X & = & \{\, 1/2, 1/4, 1/8, 1/8\, \}, \end{array} \tag{5.7}$$

and consider the code C_3. The entropy of X is 1.75 bits, and the expected length $L(C_3, X)$ of this code is also 1.75 bits. The sequence of symbols $\mathbf{x} = (\texttt{acdbac})$ is encoded as $c^+(\mathbf{x}) = \texttt{0110111100110}$. C_3 is a prefix code and is therefore uniquely decodeable. Notice that the codeword lengths satisfy $l_i = \log_2(1/p_i)$, or equivalently, $p_i = 2^{-l_i}$.

C_3:

a_i	$c(a_i)$	p_i	$h(p_i)$	l_i
a	0	1/2	1.0	1
b	10	1/4	2.0	2
c	110	1/8	3.0	3
d	111	1/8	3.0	3

Example 5.11. Consider the fixed length code for the same ensemble X, C_4. The expected length $L(C_4, X)$ is 2 bits.

	C_4	C_5
a	00	0
b	01	1
c	10	00
d	11	11

Example 5.12. Consider C_5. The expected length $L(C_5, X)$ is 1.25 bits, which is less than $H(X)$. But the code is not uniquely decodeable. The sequence $\mathbf{x} = (\texttt{acdbac})$ encodes as `000111000`, which can also be decoded as $(\texttt{cabdca})$.

C_6:

a_i	$c(a_i)$	p_i	$h(p_i)$	l_i
a	0	1/2	1.0	1
b	01	1/4	2.0	2
c	011	1/8	3.0	3
d	111	1/8	3.0	3

Example 5.13. Consider the code C_6. The expected length $L(C_6, X)$ of this code is 1.75 bits. The sequence of symbols $\mathbf{x} = (\texttt{acdbac})$ is encoded as $c^+(\mathbf{x}) = \texttt{0011111010011}$.

Is C_6 a prefix code? It is not, because $c(\texttt{a}) = \texttt{0}$ is a prefix of both $c(\texttt{b})$ and $c(\texttt{c})$.

Is C_6 uniquely decodeable? This is not so obvious. If you think that it might *not* be uniquely decodeable, try to prove it so by finding a pair of strings $\mathbf{x}$ and $\mathbf{y}$ that have the same encoding. [The definition of unique decodeability is given in equation (5.4).]

C_6 certainly isn't *easy* to decode. When we receive '`00`', it is possible that $\mathbf{x}$ could start '`aa`', '`ab`' or '`ac`'. Once we have received '`001111`', the second symbol is still ambiguous, as $\mathbf{x}$ could be '`abd`. . . ' or '`acd`. . . '. But eventually a unique decoding crystallizes, once the next `0` appears in the encoded stream.

C_6 *is* in fact uniquely decodeable. Comparing with the prefix code C_3, we see that the codewords of C_6 are the reverse of C_3's. That C_3 is uniquely decodeable proves that C_6 is too, since any string from C_6 is identical to a string from C_3 read backwards.

▶ 5.2 What limit is imposed by unique decodeability?

We now ask, given a list of positive integers $\{l_i\}$, does there exist a uniquely decodeable code with those integers as its codeword lengths? At this stage, we ignore the probabilities of the different symbols; once we understand unique decodeability better, we'll reintroduce the probabilities and discuss how to make an *optimal* uniquely decodeable symbol code.

In the examples above, we have observed that if we take a code such as $\{\texttt{00}, \texttt{01}, \texttt{10}, \texttt{11}\}$, and shorten one of its codewords, for example $\texttt{00} \rightarrow \texttt{0}$, then we can retain unique decodeability only if we lengthen other codewords. Thus there seems to be a constrained budget that we can spend on codewords, with shorter codewords being more expensive.

Let us explore the nature of this budget. If we build a code purely from codewords of length l equal to three, how many codewords can we have and retain unique decodeability? The answer is $2^l = 8$. Once we have chosen all eight of these codewords, is there any way we could add to the code another codeword of some *other* length and retain unique decodeability? It would seem not.

What if we make a code that includes a length-one codeword, '`0`', with the other codewords being of length three? How many length-three codewords can we have? If we restrict attention to prefix codes, then we can have only four codewords of length three, namely $\{\texttt{100}, \texttt{101}, \texttt{110}, \texttt{111}\}$. What about other codes? Is there any other way of choosing codewords of length 3 that can give more codewords? Intuitively, we think this unlikely. A codeword of length 3 appears to have a cost that is 2^2 times smaller than a codeword of length 1.

Let's define a total budget of size 1, which we can spend on codewords. If we set the cost of a codeword whose length is l to 2^{-l}, then we have a pricing system that fits the examples discussed above. Codewords of length 3 cost $^1\!/_8$ each; codewords of length 1 cost $1/2$ each. We can spend our budget on any codewords. If we go over our budget then the code will certainly not be uniquely decodeable. If, on the other hand,

$$\sum_i 2^{-l_i} \leq 1, \tag{5.8}$$

then the code may be uniquely decodeable. This inequality is the Kraft inequality.

Kraft inequality. For any uniquely decodeable code $C(X)$ over the binary alphabet $\{0, 1\}$, the codeword lengths must satisfy:

$$\sum_{i=1}^{I} 2^{-l_i} \leq 1, \tag{5.9}$$

where $I = |\mathcal{A}_X|$.

Completeness. If a uniquely decodeable code satisfies the Kraft inequality with equality then it is called a *complete* code.

We want codes that are uniquely decodeable; prefix codes are uniquely decodeable, and are easy to decode. So life would be simpler for us if we could restrict attention to prefix codes. Fortunately, for any source there *is* an optimal symbol code that is also a prefix code.

Kraft inequality and prefix codes. Given a set of codeword lengths that satisfy the Kraft inequality, there exists a uniquely decodeable prefix code with these codeword lengths.

The Kraft inequality might be more accurately referred to as the Kraft–McMillan inequality: Kraft proved that if the inequality is satisfied, then a prefix code exists with the given lengths. McMillan (1956) proved the converse, that unique decodeability implies that the inequality holds.

Proof of the Kraft inequality. Define $S = \sum_i 2^{-l_i}$. Consider the quantity

$$S^N = \left[\sum_i 2^{-l_i}\right]^N = \sum_{i_1=1}^{I}\sum_{i_2=1}^{I}\cdots\sum_{i_N=1}^{I} 2^{-\left(l_{i_1} + l_{i_2} + \cdots l_{i_N}\right)}. \tag{5.10}$$

The quantity in the exponent, $(l_{i_1} + l_{i_2} + \cdots + l_{i_N})$, is the length of the encoding of the string $\mathbf{x} = a_{i_1}a_{i_2}\ldots a_{i_N}$. For every string $\mathbf{x}$ of length N, there is one term in the above sum. Introduce an array A_l that counts how many strings $\mathbf{x}$ have encoded length l. Then, defining $l_{\min} = \min_i l_i$ and $l_{\max} = \max_i l_i$:

$$S^N = \sum_{l=Nl_{\min}}^{Nl_{\max}} 2^{-l} A_l. \tag{5.11}$$

Now assume C is uniquely decodeable, so that for all $\mathbf{x} \neq \mathbf{y}$, $c^+(\mathbf{x}) \neq c^+(\mathbf{y})$. Concentrate on the $\mathbf{x}$ that have encoded length l. There are a total of 2^l distinct bit strings of length l, so it must be the case that $A_l \leq 2^l$. So

$$S^N = \sum_{l=Nl_{\min}}^{Nl_{\max}} 2^{-l} A_l \leq \sum_{l=Nl_{\min}}^{Nl_{\max}} 1 \;\leq\; Nl_{\max}. \tag{5.12}$$

Thus $S^N \leq l_{\max}N$ for all N. Now if S were greater than 1, then as N increases, S^N would be an exponentially growing function, and for large enough N, an exponential always exceeds a polynomial such as $l_{\max}N$. But our result ($S^N \leq l_{\max}N$) is true for *any* N. Therefore $S \leq 1$. □

▷ Exercise 5.14.[*3*, p.104] Prove the result stated above, that for any set of codeword lengths $\{l_i\}$ satisfying the Kraft inequality, there is a prefix code having those lengths.

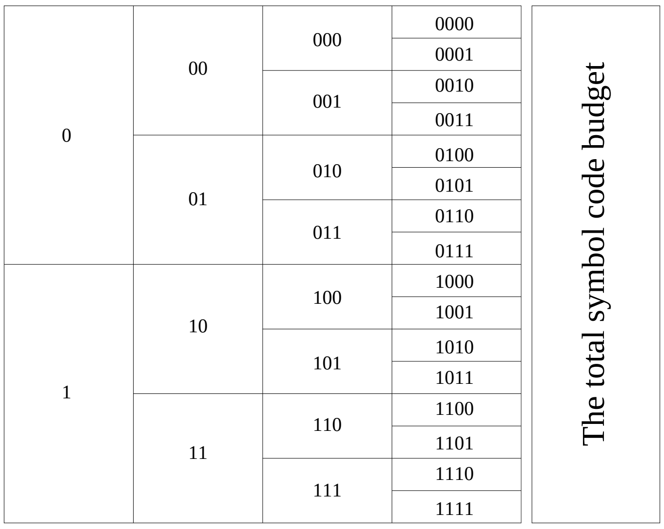


Figure 5.1. The symbol coding budget. The 'cost' 2^{-l} of each codeword (with length l) is indicated by the size of the box it is written in. The total budget available when making a uniquely decodeable code is 1.
You can think of this diagram as showing a *codeword supermarket*, with the codewords arranged in aisles by their length, and the cost of each codeword indicated by the size of its box on the shelf. If the cost of the codewords that you take exceeds the budget then your code will not be uniquely decodeable.

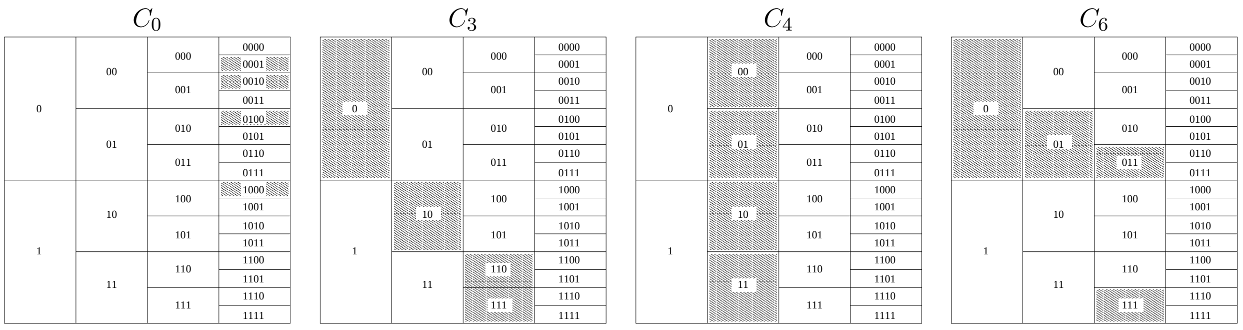


Figure 5.2. Selections of codewords made by codes C_0, C_3, C_4 and C_6 from section 5.1.

A pictorial view of the Kraft inequality may help you solve this exercise. Imagine that we are choosing the codewords to make a symbol code. We can draw the set of all candidate codewords in a supermarket that displays the 'cost' of the codeword by the area of a box (figure 5.1). The total budget available – the '1' on the right-hand side of the Kraft inequality – is shown at one side. Some of the codes discussed in section 5.1 are illustrated in figure 5.2. Notice that the codes that are prefix codes, C_0, C_3, and C_4, have the property that to the right of any selected codeword, there are no other selected codewords – because prefix codes correspond to trees. Notice that a *complete* prefix code corresponds to a *complete* tree having no unused branches.

We are now ready to put back the symbols' probabilities $\{p_i\}$. Given a set of symbol probabilities (the English language probabilities of figure 2.1, for example), how do we make the best symbol code – one with the smallest possible expected length $L(C,X)$? And what is that smallest possible expected length? It's not obvious how to assign the codeword lengths. If we give short codewords to the more probable symbols then the expected length might be reduced; on the other hand, shortening some codewords necessarily causes others to lengthen, by the Kraft inequality.

▶ 5.3 What's the most compression that we can hope for?

We wish to minimize the expected length of a code,

$$L(C,X) \;=\; \sum_i p_i l_i. \tag{5.13}$$

As you might have guessed, the entropy appears as the lower bound on the expected length of a code.

Lower bound on expected length. The expected length $L(C,X)$ of a uniquely decodeable code is bounded below by $H(X)$.

Proof. We define the *implicit probabilities* $q_i \equiv 2^{-l_i}/z$, where $z = \sum_{i'} 2^{-l_{i'}}$, so that $l_i = \log 1/q_i - \log z$. We then use Gibbs' inequality, $\sum_i p_i \log 1/q_i \geq \sum_i p_i \log 1/p_i$, with equality if $q_i = p_i$, and the Kraft inequality $z \leq 1$:

$$\begin{aligned} L(C,X) &= \sum_i p_i l_i = \sum_i p_i \log 1/q_i - \log z & (5.14)\\ &\geq \sum_i p_i \log 1/p_i - \log z & (5.15)\\ &\geq H(X). & (5.16)\end{aligned}$$

The equality $L(C,X) = H(X)$ is achieved only if the Kraft equality $z = 1$ is satisfied, and if the codelengths satisfy $l_i = \log(1/p_i)$. □

This is an important result so let's say it again:

Optimal source codelengths. The expected length is minimized and is equal to $H(X)$ only if the codelengths are equal to the *Shannon information contents*:

$$l_i = \log_2(1/p_i). \tag{5.17}$$

Implicit probabilities defined by codelengths. Conversely, any choice of codelengths $\{l_i\}$ *implicitly* defines a probability distribution $\{q_i\}$,

$$q_i \equiv 2^{-l_i}/z, \tag{5.18}$$

for which those codelengths would be the optimal codelengths. If the code is complete then $z = 1$ and the implicit probabilities are given by $q_i = 2^{-l_i}$.

▶ 5.4 How much can we compress?

So, we can't compress below the entropy. How close can we expect to get to the entropy?

Theorem 5.1 Source coding theorem for symbol codes. *For an ensemble X there exists a prefix code C with expected length satisfying*

$$H(X) \leq L(C, X) < H(X) + 1. \tag{5.19}$$

Proof. We set the codelengths to integers slightly larger than the optimum lengths:

$$l_i = \lceil \log_2(1/p_i) \rceil \tag{5.20}$$

where $\lceil l^* \rceil$ denotes the smallest integer greater than or equal to l^*. [We are not asserting that the *optimal* code necessarily uses these lengths, we are simply choosing these lengths because we can use them to prove the theorem.]

We check that there *is* a prefix code with these lengths by confirming that the Kraft inequality is satisfied.

$$\sum_i 2^{-l_i} = \sum_i 2^{-\lceil \log_2(1/p_i) \rceil} \leq \sum_i 2^{-\log_2(1/p_i)} = \sum_i p_i = 1. \tag{5.21}$$

Then we confirm

$$L(C, X) = \sum_i p_i \lceil \log(1/p_i) \rceil < \sum_i p_i (\log(1/p_i) + 1) = H(X) + 1. \tag{5.22}$$

□

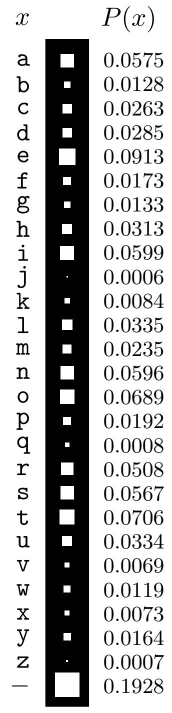
The cost of using the wrong codelengths

If we use a code whose lengths are not equal to the optimal codelengths, the average message length will be larger than the entropy.

If the true probabilities are $\{p_i\}$ and we use a complete code with lengths l_i, we can view those lengths as defining implicit probabilities $q_i = 2^{-l_i}$. Continuing from equation (5.14), the average length is

$$L(C, X) = H(X) + \sum_i p_i \log p_i / q_i, \tag{5.23}$$

i.e., it exceeds the entropy by the relative entropy $D_{\mathrm{KL}}(\mathbf{p}||\mathbf{q})$ (as defined on p.34).



x	$P(x)$
a	0.0575
b	0.0128
c	0.0263
d	0.0285
e	0.0913
f	0.0173
g	0.0133
h	0.0313
i	0.0599
j	0.0006
k	0.0084
l	0.0335
m	0.0235
n	0.0596
o	0.0689
p	0.0192
q	0.0008
r	0.0508
s	0.0567
t	0.0706
u	0.0334
v	0.0069
w	0.0119
x	0.0073
y	0.0164
z	0.0007
–	0.1928

Figure 5.3. An ensemble in need of a symbol code.

▶ 5.5 Optimal source coding with symbol codes: Huffman coding

Given a set of probabilities $\mathcal{P}$, how can we design an optimal prefix code? For example, what is the best symbol code for the English language ensemble shown in figure 5.3? When we say 'optimal', let's assume our aim is to minimize the expected length $L(C, X)$.

How not to do it

One might try to roughly split the set $\mathcal{A}_X$ in two, and continue bisecting the subsets so as to define a binary tree from the root. This construction has the right spirit, as in the weighing problem, but it is not necessarily optimal; it achieves $L(C, X) \leq H(X) + 2$.

The Huffman coding algorithm

We now present a beautifully simple algorithm for finding an optimal prefix code. The trick is to construct the code *backwards* starting from the tails of the codewords; *we build the binary tree from its leaves.*

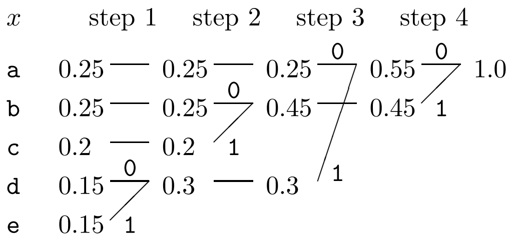
1. Take the two least probable symbols in the alphabet. These two symbols will be given the longest codewords, which will have equal length, and differ only in the last digit.

2. Combine these two symbols into a single symbol, and repeat.

Algorithm 5.4. Huffman coding algorithm.

Since each step reduces the size of the alphabet by one, this algorithm will have assigned strings to all the symbols after $|\mathcal{A}_X| - 1$ steps.

Example 5.15. Let $\mathcal{A}_X = \{$ `a`, `b`, `c`, `d`, `e` $\}$
and $\mathcal{P}_X = \{ 0.25, 0.25, 0.2, 0.15, 0.15 \}$.



a_i	p_i	$h(p_i)$	l_i	$c(a_i)$
`a`	0.25	2.0	2	`00`
`b`	0.25	2.0	2	`10`
`c`	0.2	2.3	2	`11`
`d`	0.15	2.7	3	`010`
`e`	0.15	2.7	3	`011`

Table 5.5. Code created by the Huffman algorithm.

The codewords are then obtained by concatenating the binary digits in reverse order: $C = \{$`00`, `10`, `11`, `010`, `011`$\}$. The codelengths selected by the Huffman algorithm (column 4 of table 5.5) are in some cases longer and in some cases shorter than the ideal codelengths, the Shannon information contents $\log_2 {}^1\!/\!_{p_i}$ (column 3). The expected length of the code is $L = 2.30$ bits, whereas the entropy is $H = 2.2855$ bits. □

If at any point there is more than one way of selecting the two least probable symbols then the choice may be made in any manner – the expected length of the code will not depend on the choice.

Exercise 5.16.[*3*, p.105] Prove that there is no better symbol code for a source than the Huffman code.

Example 5.17. We can make a Huffman code for the probability distribution over the alphabet introduced in figure 2.1. The result is shown in figure 5.6. This code has an expected length of 4.15 bits; the entropy of the ensemble is 4.11 bits. Observe the disparities between the assigned codelengths and the ideal codelengths $\log_2 {}^1\!/\!_{p_i}$.

Constructing a binary tree top-down is suboptimal

In previous chapters we studied weighing problems in which we built ternary or binary trees. We noticed that balanced trees – ones in which, at every step, the two possible outcomes were as close as possible to equiprobable – appeared to describe the most efficient experiments. This gave an intuitive motivation for entropy as a measure of information content.

a_i	p_i	$\log_2 \frac{1}{p_i}$	l_i	$c(a_i)$
a	0.0575	4.1	4	0000
b	0.0128	6.3	6	001000
c	0.0263	5.2	5	00101
d	0.0285	5.1	5	10000
e	0.0913	3.5	4	1100
f	0.0173	5.9	6	111000
g	0.0133	6.2	6	001001
h	0.0313	5.0	5	10001
i	0.0599	4.1	4	1001
j	0.0006	10.7	10	1101000000
k	0.0084	6.9	7	1010000
l	0.0335	4.9	5	11101
m	0.0235	5.4	6	110101
n	0.0596	4.1	4	0001
o	0.0689	3.9	4	1011
p	0.0192	5.7	6	111001
q	0.0008	10.3	9	110100001
r	0.0508	4.3	5	11011
s	0.0567	4.1	4	0011
t	0.0706	3.8	4	1111
u	0.0334	4.9	5	10101
v	0.0069	7.2	8	11010001
w	0.0119	6.4	7	1101001
x	0.0073	7.1	7	1010001
y	0.0164	5.9	6	101001
z	0.0007	10.4	10	1101000001
–	0.1928	2.4	2	01

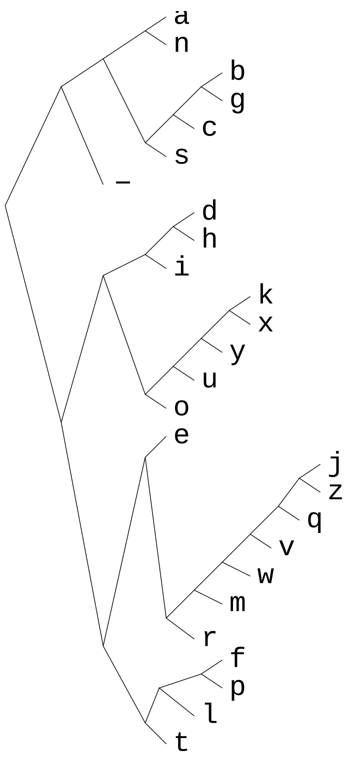


Figure 5.6. Huffman code for the English language ensemble (monogram statistics).

It is not the case, however, that optimal codes can *always* be constructed by a greedy top-down method in which the alphabet is successively divided into subsets that are as near as possible to equiprobable.

Example 5.18. Find the optimal binary symbol code for the ensemble:

$$\begin{array}{rl} \mathcal{A}_X = \{ & \texttt{a}, \quad \texttt{b}, \quad \texttt{c}, \quad \texttt{d}, \quad \texttt{e}, \quad \texttt{f}, \quad \texttt{g} \; \} \\ \mathcal{P}_X = \{ & 0.01, 0.24, 0.05, 0.20, 0.47, 0.01, \; 0.02 \, \} \end{array} . \tag{5.24}$$

Notice that a greedy top-down method can split this set into two subsets $\{\texttt{a},\texttt{b},\texttt{c},\texttt{d}\}$ and $\{\texttt{e},\texttt{f},\texttt{g}\}$ which both have probability 1/2, and that $\{\texttt{a},\texttt{b},\texttt{c},\texttt{d}\}$ can be divided into subsets $\{\texttt{a},\texttt{b}\}$ and $\{\texttt{c},\texttt{d}\}$, which have probability 1/4; so a greedy top-down method gives the code shown in the third column of table 5.7, which has expected length 2.53. The Huffman coding algorithm yields the code shown in the fourth column, which has expected length 1.97. □

a_i	p_i	Greedy	Huffman
a	.01	000	000000
b	.24	001	01
c	.05	010	0001
d	.20	011	001
e	.47	10	1
f	.01	110	000001
g	.02	111	00001

Table 5.7. A greedily-constructed code compared with the Huffman code.

▶ 5.6 Disadvantages of the Huffman code

The Huffman algorithm produces an optimal symbol code for an ensemble, but this is not the end of the story. Both the word 'ensemble' and the phrase 'symbol code' need careful attention.

Changing ensemble

If we wish to communicate a sequence of outcomes from one unchanging ensemble, then a Huffman code may be convenient. But often the appropriate ensemble changes. If for example we are compressing text, then the symbol frequencies will vary with context: in English the letter `u` is much more probable after a `q` than after an `e` (figure 2.3). And furthermore, our knowledge of these context-dependent symbol frequencies will also change as we learn the statistical properties of the text source.

Huffman codes do not handle changing ensemble probabilities with any elegance. One brute-force approach would be to recompute the Huffman code every time the probability over symbols changes. Another attitude is to deny the option of adaptation, and instead run through the entire file in advance and compute a good probability distribution, which will then remain fixed throughout transmission. The code itself must also be communicated in this scenario. Such a technique is not only cumbersome and restrictive, it is also suboptimal, since the initial message specifying the code and the document itself are partially redundant. This technique therefore wastes bits.

The extra bit

An equally serious problem with Huffman codes is the innocuous-looking 'extra bit' relative to the ideal average length of $H(X)$ – a Huffman code achieves a length that satisfies $H(X) \leq L(C, X) < H(X)+1$, as proved in theorem 5.1. A Huffman code thus incurs an overhead of between 0 and 1 bits per symbol. If $H(X)$ were large, then this overhead would be an unimportant fractional increase. But for many applications, the entropy may be as low as one bit per symbol, or even smaller, so the overhead $L(C, X) - H(X)$ may dominate the encoded file length. Consider English text: in some contexts, long strings of characters may be highly predictable. For example, in the context '`strings_of_ch`', one might predict the next nine symbols to be '`aracters_`' with a probability of 0.99 each. A traditional Huffman code would be obliged to use at least one bit per character, making a total cost of nine bits where virtually no information is being conveyed (0.13 bits in total, to be precise). The entropy of English, given a good model, is about one bit per character (Shannon, 1948), so a Huffman code is likely to be highly inefficient.

A traditional patch-up of Huffman codes uses them to compress *blocks* of symbols, for example the 'extended sources' X^N we discussed in Chapter 4. The overhead per block is at most 1 bit so the overhead per symbol is at most $1/N$ bits. For sufficiently large blocks, the problem of the extra bit may be removed – but only at the expenses of (a) losing the elegant instantaneous decodeability of simple Huffman coding; and (b) having to compute the probabilities of all relevant strings and build the associated Huffman tree. One will end up explicitly computing the probabilities and codes for a huge number of strings, most of which will never actually occur. (See exercise 5.29 (p.103).)

Beyond symbol codes

Huffman codes, therefore, although widely trumpeted as 'optimal', have many defects for practical purposes. They *are* optimal *symbol* codes, but for practical purposes *we don't want a symbol code.*

The defects of Huffman codes are rectified by *arithmetic coding*, which dispenses with the restriction that each symbol must translate into an integer number of bits. Arithmetic coding is the main topic of the next chapter.

▶ 5.7 Summary

Kraft inequality. If a code is *uniquely decodeable* its lengths must satisfy

$$\sum_i 2^{-l_i} \leq 1. \tag{5.25}$$

For any lengths satisfying the Kraft inequality, there exists a prefix code with those lengths.

Optimal source codelengths for an ensemble are equal to the Shannon information contents

$$l_i = \log_2 \frac{1}{p_i}, \tag{5.26}$$

and conversely, any choice of codelengths defines *implicit probabilities*

$$q_i = \frac{2^{-l_i}}{z}. \tag{5.27}$$

The relative entropy $D_{\mathrm{KL}}(\mathbf{p}||\mathbf{q})$ measures how many bits per symbol are wasted by using a code whose implicit probabilities are $\mathbf{q}$, when the ensemble's true probability distribution is $\mathbf{p}$.

Source coding theorem for symbol codes. For an ensemble X, there exists a prefix code whose expected length satisfies

$$H(X) \leq L(C, X) < H(X) + 1. \tag{5.28}$$

The Huffman coding algorithm generates an optimal symbol code iteratively. At each iteration, the two least probable symbols are combined.

▶ 5.8 Exercises

▷ Exercise 5.19.[*2*] Is the code $\{\texttt{00}, \texttt{11}, \texttt{0101}, \texttt{111}, \texttt{1010}, \texttt{100100}, \texttt{0110}\}$ uniquely decodeable?

▷ Exercise 5.20.[*2*] Is the ternary code $\{\texttt{00}, \texttt{012}, \texttt{0110}, \texttt{0112}, \texttt{100}, \texttt{201}, \texttt{212}, \texttt{22}\}$ uniquely decodeable?

Exercise 5.21.[*3*, p.106] Make Huffman codes for X^2, X^3 and X^4 where $\mathcal{A}_X = \{0, 1\}$ and $\mathcal{P}_X = \{0.9, 0.1\}$. Compute their expected lengths and compare them with the entropies $H(X^2)$, $H(X^3)$ and $H(X^4)$.

Repeat this exercise for X^2 and X^4 where $\mathcal{P}_X = \{0.6, 0.4\}$.

Exercise 5.22.[*2*, p.106] Find a probability distribution $\{p_1, p_2, p_3, p_4\}$ such that there are *two* optimal codes that assign different lengths $\{l_i\}$ to the four symbols.

Exercise 5.23.[*3*] (Continuation of exercise 5.22.) Assume that the four probabilities $\{p_1, p_2, p_3, p_4\}$ are ordered such that $p_1 \geq p_2 \geq p_3 \geq p_4 \geq 0$. Let $\mathcal{Q}$ be the set of all probability vectors $\mathbf{p}$ such that there are *two* optimal codes with different lengths. Give a complete description of $\mathcal{Q}$. Find three probability vectors $\mathbf{q}^{(1)}$, $\mathbf{q}^{(2)}$, $\mathbf{q}^{(3)}$, which are the convex hull of $\mathcal{Q}$, i.e., such that any $\mathbf{p} \in \mathcal{Q}$ can be written as

$$\mathbf{p} = \mu_1 \mathbf{q}^{(1)} + \mu_2 \mathbf{q}^{(2)} + \mu_3 \mathbf{q}^{(3)}, \tag{5.29}$$

where $\{\mu_i\}$ are positive.

▷ Exercise 5.24.[*1*] Write a short essay discussing how to play the game of twenty questions optimally. [In twenty questions, one player thinks of an object, and the other player has to guess the object using as few binary questions as possible, preferably fewer than twenty.]

▷ Exercise 5.25.[*2*] Show that, if each probability p_i is equal to an integer power of 2 then there exists a source code whose expected length equals the entropy.

▷ Exercise 5.26.[*2*, p.106] Make ensembles for which the difference between the entropy and the expected length of the Huffman code is as big as possible.

▷ Exercise 5.27.[*2*, p.106] A source X has an alphabet of eleven characters

$$\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}, \mathtt{e}, \mathtt{f}, \mathtt{g}, \mathtt{h}, \mathtt{i}, \mathtt{j}, \mathtt{k}\},$$

all of which have equal probability, 1/11.

Find an optimal uniquely decodeable symbol code for this source. How much greater is the expected length of this optimal code than the entropy of X?

▷ Exercise 5.28.[*2*] Consider the optimal symbol code for an ensemble X with alphabet size I from which all symbols have identical probability $p = 1/I$. I is not a power of 2.

Show that the fraction f^+ of the I symbols that are assigned codelengths equal to

$$l^+ \equiv \lceil \log_2 I \rceil \tag{5.30}$$

satisfies

$$f^+ = 2 - \frac{2^{l^+}}{I} \tag{5.31}$$

and that the expected length of the optimal symbol code is

$$L = l^+ - 1 + f^+. \tag{5.32}$$

By differentiating the excess length $\Delta L \equiv L - H(X)$ with respect to I, show that the excess length is bounded by

$$\Delta L \leq 1 - \frac{\ln(\ln 2)}{\ln 2} - \frac{1}{\ln 2} = 0.086. \tag{5.33}$$

Exercise 5.29.[*2*] Consider a sparse binary source with $\mathcal{P}_X = \{0.99, 0.01\}$. Discuss how Huffman codes could be used to compress this source *efficiently*. Estimate how many codewords your proposed solutions require.

▷ Exercise 5.30.[*2*] *Scientific American* carried the following puzzle in 1975.

The poisoned glass. 'Mathematicians are curious birds', the police commissioner said to his wife. 'You see, we had all those partly filled glasses lined up in rows on a table in the hotel kitchen. Only one contained poison, and we wanted to know which one before searching that glass for fingerprints. Our lab could test the liquid in each glass, but the tests take time and money, so we wanted to make as few of them as possible by simultaneously testing mixtures of small samples from groups of glasses. The university sent over a mathematics professor to help us. He counted the glasses, smiled and said:

‘ “Pick any glass you want, Commissioner. We’ll test it first.”

‘ “But won’t that waste a test?” I asked.

‘ “No,” he said, “it’s part of the best procedure. We can test one glass first. It doesn’t matter which one.” ’

‘How many glasses were there to start with?’ the commissioner’s wife asked.

‘I don’t remember. Somewhere between 100 and 200.’

What was the exact number of glasses?

Solve this puzzle and then explain why the professor was in fact wrong and the commissioner was right. What is in fact the optimal procedure for identifying the one poisoned glass? What is the expected waste relative to this optimum if one followed the professor’s strategy? Explain the relationship to symbol coding.

Exercise 5.31.[*2*, p.106] Assume that a sequence of symbols from the ensemble X introduced at the beginning of this chapter is compressed using the code C_3. Imagine picking one bit at random from the binary encoded sequence $\mathbf{c} = c(x_1)c(x_2)c(x_3)\ldots$. What is the probability that this bit is a 1?

C_3:

a_i	$c(a_i)$	p_i	$h(p_i)$	l_i
`a`	`0`	$1/2$	1.0	1
`b`	`10`	$1/4$	2.0	2
`c`	`110`	$1/8$	3.0	3
`d`	`111`	$1/8$	3.0	3

▷ Exercise 5.32.[*2*, p.107] How should the binary Huffman encoding scheme be modified to make optimal symbol codes in an encoding alphabet with q symbols? (Also known as ‘radix q’.)

Mixture codes

It is a tempting idea to construct a ‘metacode’ from several symbol codes that assign different-length codewords to the alternative symbols, then switch from one code to another, choosing whichever assigns the shortest codeword to the current symbol. Clearly we cannot do this for free. If one wishes to choose between two codes, then it is necessary to lengthen the message in a way that indicates which of the two codes is being used. If we indicate this choice by a single leading bit, it will be found that the resulting code is suboptimal because it is incomplete (that is, it fails the Kraft equality).

Exercise 5.33.[*3*, p.108] Prove that this metacode is incomplete, and explain why this combined code is suboptimal.

▶ 5.9 Solutions

Solution to exercise 5.8 (p.93). Yes, $C_2 = \{\texttt{1}, \texttt{101}\}$ is uniquely decodeable, even though it is not a prefix code, because no two different strings can map onto the same string; only the codeword $c(a_2) = \texttt{101}$ contains the symbol `0`.

Solution to exercise 5.14 (p.95). We wish to prove that for any set of codeword lengths $\{l_i\}$ satisfying the Kraft inequality, there is a prefix code having those lengths. This is readily proved by thinking of the codewords illustrated in figure 5.8 as being in a ‘codeword supermarket’, with size indicating cost. We imagine purchasing codewords one at a time, starting from the shortest codewords (i.e., the biggest purchases), using the budget shown at the right of figure 5.8. We start at one side of the codeword supermarket, say the

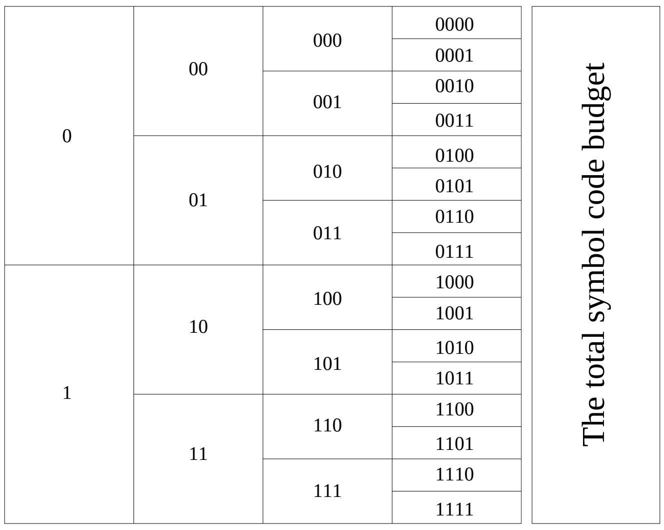


Figure 5.8. The codeword supermarket and the symbol coding budget. The 'cost' 2^{-l} of each codeword (with length l) is indicated by the size of the box it is written in. The total budget available when making a uniquely decodeable code is 1.

symbol	probability	Huffman codewords	Rival code's codewords	Modified rival code
a	p_a □	$\boxed{c_{\rm H}(a)}$	$\boxed{c_{\rm R}(a)}$	$\boxed{c_{\rm R}(c)}$
b	p_b □	$\boxed{c_{\rm H}(b)}$	$\boxed{c_{\rm R}(b)}$	$\boxed{c_{\rm R}(b)}$
c	p_c ▭	$\boxed{c_{\rm H}(c)}$	$\boxed{c_{\rm R}(c)}$	$\boxed{c_{\rm R}(a)}$

Figure 5.9. Proof that Huffman coding makes an optimal symbol code. We assume that the rival code, which is said to be optimal, assigns *unequal* length codewords to the two symbols with smallest probability, a and b. By interchanging codewords a and c of the rival code, where c is a symbol with rival codelength as long as b's, we can make a code better than the rival code. This shows that the rival code was not optimal.

top, and purchase the first codeword of the required length. We advance down the supermarket a distance 2^{-l}, and purchase the next codeword of the next required length, and so forth. Because the codeword lengths are getting longer, and the corresponding intervals are getting shorter, we can always buy an adjacent codeword to the latest purchase, so there is no wasting of the budget. Thus at the Ith codeword we have advanced a distance $\sum_{i=1}^{I} 2^{-l_i}$ down the supermarket; if $\sum 2^{-l_i} \leq 1$, we will have purchased all the codewords without running out of budget.

Solution to exercise 5.16 (p.99). The proof that Huffman coding is optimal depends on proving that the key step in the algorithm – the decision to give the two symbols with smallest probability equal encoded lengths – cannot lead to a larger expected length than any other code. We can prove this by contradiction.

Assume that the two symbols with smallest probability, called a and b, to which the Huffman algorithm would assign equal length codewords, do *not* have equal lengths in *any* optimal symbol code. The optimal symbol code is some other rival code in which these two codewords have unequal lengths l_a and l_b with $l_a < l_b$. Without loss of generality we can assume that this other code is a complete prefix code, because any codelengths of a uniquely decodeable code can be realized by a prefix code.

In this rival code, there must be some other symbol c whose probability p_c is greater than p_a and whose length in the rival code is greater than or equal to l_b, because the code for b must have an adjacent codeword of equal or greater length – a complete prefix code never has a solo codeword of the maximum length.

Consider exchanging the codewords of a and c (figure 5.9), so that a is encoded with the longer codeword that was c's, and c, which is more probable than a, gets the shorter codeword. Clearly this reduces the expected length of the code. The change in expected length is $(p_a - p_c)(l_c - l_a)$. Thus we have contradicted the assumption that the rival code is optimal. Therefore it is valid to give the two symbols with smallest probability equal encoded lengths. Huffman coding produces optimal symbol codes. □

Solution to exercise 5.21 (p.102). A Huffman code for X^2 where $\mathcal{A}_X = \{0, 1\}$ and $\mathcal{P}_X = \{0.9, 0.1\}$ is $\{00, 01, 10, 11\} \rightarrow \{1, 01, 000, 001\}$. This code has $L(C, X^2) = 1.29$, whereas the entropy $H(X^2)$ is 0.938.

A Huffman code for X^3 is

$$\{000, 100, 010, 001, 101, 011, 110, 111\} \rightarrow \\ \{1, 011, 010, 001, 00000, 00001, 00010, 00011\}.$$

This has expected length $L(C, X^3) = 1.598$ whereas the entropy $H(X^3)$ is 1.4069.

A Huffman code for X^4 maps the sixteen source strings to the following codelengths:

$$\{0000, 1000, 0100, 0010, 0001, 1100, 0110, 0011, 0101, 1010, 1001, 1110, 1101, \\ 1011, 0111, 1111\} \rightarrow \{1, 3, 3, 3, 4, 6, 7, 7, 7, 7, 7, 9, 9, 9, 10, 10\}.$$

This has expected length $L(C, X^4) = 1.9702$ whereas the entropy $H(X^4)$ is 1.876.

When $\mathcal{P}_X = \{0.6, 0.4\}$, the Huffman code for X^2 has lengths $\{2, 2, 2, 2\}$; the expected length is 2 bits, and the entropy is 1.94 bits. A Huffman code for X^4 is shown in table 5.10. The expected length is 3.92 bits, and the entropy is 3.88 bits.

a_i	p_i	l_i	$c(a_i)$
0000	0.1296	3	000
0001	0.0864	4	0100
0010	0.0864	4	0110
0100	0.0864	4	0111
1000	0.0864	3	100
1100	0.0576	4	1010
1010	0.0576	4	1100
1001	0.0576	4	1101
0110	0.0576	4	1110
0101	0.0576	4	1111
0011	0.0576	4	0010
1110	0.0384	5	00110
1101	0.0384	5	01010
1011	0.0384	5	01011
0111	0.0384	4	1011
1111	0.0256	5	00111

Table 5.10. Huffman code for X^4 when $p_0 = 0.6$. Column 3 shows the assigned codelengths and column 4 the codewords. Some strings whose probabilities are identical, e.g., the fourth and fifth, receive different codelengths.

Solution to exercise 5.22 (p.102). The set of probabilities $\{p_1, p_2, p_3, p_4\} = \{1/6, 1/6, 1/3, 1/3\}$ gives rise to two different optimal sets of codelengths, because at the second step of the Huffman coding algorithm we can choose any of the three possible pairings. We may either put them in a constant length code $\{00, 01, 10, 11\}$ or the code $\{000, 001, 01, 1\}$. Both codes have expected length 2.

Another solution is $\{p_1, p_2, p_3, p_4\} = \{1/5, 1/5, 1/5, 2/5\}$.

And a third is $\{p_1, p_2, p_3, p_4\} = \{1/3, 1/3, 1/3, 0\}$.

Solution to exercise 5.26 (p.103). Let $p_{\max}$ be the largest probability in $p_1, p_2, \ldots, p_I$. The difference between the expected length L and the entropy H can be no bigger than $\max(p_{\max}, 0.086)$ (Gallager, 1978).

See exercises 5.27–5.28 to understand where the curious 0.086 comes from.

Solution to exercise 5.27 (p.103). Length − entropy = 0.086.

Solution to exercise 5.31 (p.104). There are two ways to answer this problem correctly, and one popular way to answer it incorrectly. Let's give the incorrect answer first:

Erroneous answer. "We can pick a random bit by first picking a random source symbol x_i with probability p_i, then picking a random bit from $c(x_i)$. If we define f_i to be the fraction of the bits of $c(x_i)$ that are 1s, we find

$$P(\text{bit is 1}) = \sum_i p_i f_i \tag{5.34}$$

$$= 1/2 \times 0 + 1/4 \times 1/2 + 1/8 \times 2/3 + 1/8 \times 1 = 1/3.\text{"} \tag{5.35}$$

C_3:

a_i	$c(a_i)$	p_i	l_i
a	0	1/2	1
b	10	1/4	2
c	110	1/8	3
d	111	1/8	3

This answer is wrong because it falls for the bus-stop fallacy, which was introduced in exercise 2.35 (p.38): if buses arrive at random, and we are interested in 'the average time from one bus until the next', we must distinguish two possible averages: (a) the average time from a randomly chosen bus until the next; (b) the average time between the bus you just missed and the next bus. The second 'average' is twice as big as the first because, by waiting for a bus at a random time, you bias your selection of a bus in favour of buses that follow a large gap. You're unlikely to catch a bus that comes 10 seconds after a preceding bus! Similarly, the symbols `c` and `d` get encoded into longer-length binary strings than `a`, so when we pick a bit from the compressed string at random, we are more likely to land in a bit belonging to a `c` or a `d` than would be given by the probabilities p_i in the expectation (5.34). All the probabilities need to be scaled up by l_i, and renormalized.

Correct answer in the same style. Every time symbol x_i is encoded, l_i bits are added to the binary string, of which $f_i l_i$ are `1`s. The expected number of `1`s added per symbol is

$$\sum_i p_i f_i l_i; \tag{5.36}$$

and the expected total number of bits added per symbol is

$$\sum_i p_i l_i. \tag{5.37}$$

So the fraction of `1`s in the transmitted string is

$$\begin{aligned} P(\text{bit is } \mathtt{1}) &= \frac{\sum_i p_i f_i l_i}{\sum_i p_i l_i} & (5.38) \\ &= \frac{1/2 \times 0 + 1/4 \times 1 + 1/8 \times 2 + 1/8 \times 3}{7/4} = \frac{7/8}{7/4} = 1/2. \end{aligned}$$

For a general symbol code and a general ensemble, the expectation (5.38) is the correct answer. But in this case, we can use a more powerful argument.

Information-theoretic answer. The encoded string $\mathbf{c}$ is the output of an optimal compressor that compresses samples from X down to an expected length of $H(X)$ bits. We can't expect to compress this data any further. But if the probability $P(\text{bit is } \mathtt{1})$ were not equal to $1/2$ then it *would* be possible to compress the binary string further (using a block compression code, say). Therefore $P(\text{bit is } \mathtt{1})$ must be equal to $1/2$; indeed the probability of any sequence of l bits in the compressed stream taking on any particular value must be 2^{-l}. The output of a perfect compressor is always perfectly random bits.

To put it another way, if the probability $P(\text{bit is } \mathtt{1})$ were not equal to $1/2$, then the information content per bit of the compressed string would be at most $H_2(P(\mathtt{1}))$, which would be less than 1; but this contradicts the fact that we can recover the original data from $\mathbf{c}$, so the information content per bit of the compressed string must be $H(X)/L(C,X) = 1$.

Solution to exercise 5.32 (p.104). The general Huffman coding algorithm for an encoding alphabet with q symbols has one difference from the binary case. The process of combining q symbols into 1 symbol reduces the number of symbols by $q-1$. So if we start with A symbols, we'll only end up with a complete q-ary tree if $A \bmod (q-1)$ is equal to 1. Otherwise, we know that whatever prefix code we make, it must be an incomplete tree with a number of missing leaves equal, modulo $(q-1)$, to $A \bmod (q-1) - 1$. For example, if a ternary tree is built for eight symbols, then there will unavoidably be one missing leaf in the tree.

The optimal q-ary code is made by putting these extra leaves in the longest branch of the tree. This can be achieved by adding the appropriate number of symbols to the original source symbol set, all of these extra symbols having probability zero. The total number of leaves is then equal to $r(q-1) + 1$, for some integer r. The symbols are then repeatedly combined by taking the q symbols with smallest probability and replacing them by a single symbol, as in the binary Huffman coding algorithm.

Solution to exercise 5.33 (p.104). We wish to show that a greedy metacode, which picks the code which gives the shortest encoding, is actually suboptimal, because it violates the Kraft inequality.

We'll assume that each symbol x is assigned lengths $l_k(x)$ by each of the candidate codes C_k. Let us assume there are K alternative codes and that we can encode which code is being used with a header of length $\log K$ bits. Then the metacode assigns lengths $l'(x)$ that are given by

$$l'(x) = \log_2 K + \min_k l_k(x). \tag{5.39}$$

We compute the Kraft sum:

$$S = \sum_x 2^{-l'(x)} = \frac{1}{K} \sum_x 2^{-\min_k l_k(x)}. \tag{5.40}$$

Let's divide the set $\mathcal{A}_X$ into non-overlapping subsets $\{\mathcal{A}_k\}_{k=1}^K$ such that subset $\mathcal{A}_k$ contains all the symbols x that the metacode sends via code k. Then

$$S = \frac{1}{K} \sum_k \sum_{x \in \mathcal{A}_k} 2^{-l_k(x)}. \tag{5.41}$$

Now if one sub-code k satisfies the Kraft equality $\sum_{x \in \mathcal{A}_X} 2^{-l_k(x)} = 1$, then it must be the case that

$$\sum_{x \in \mathcal{A}_k} 2^{-l_k(x)} \leq 1, \tag{5.42}$$

with equality only if all the symbols x are in $\mathcal{A}_k$, which would mean that we are only using one of the K codes. So

$$S \leq \frac{1}{K} \sum_{k=1}^K 1 = 1, \tag{5.43}$$

with equality only if equation (5.42) is an equality for all codes k. But it's impossible for all the symbols to be in *all* the non-overlapping subsets $\{\mathcal{A}_k\}_{k=1}^K$, so we can't have equality (5.42) holding for *all* k. So $S < 1$.

Another way of seeing that a mixture code is suboptimal is to consider the binary tree that it defines. Think of the special case of two codes. The first bit we send identifies which code we are using. Now, in a complete code, any subsequent binary string is a valid string. But once we know that we are using, say, code A, we know that what follows can only be a codeword corresponding to a symbol x whose encoding is shorter under code A than code B. So some strings are invalid continuations, and the mixture code is incomplete and suboptimal.

For further discussion of this issue and its relationship to probabilistic modelling read about 'bits back coding' in section 28.3 and in Frey (1998).