

CSE107 Discrete Mathematics and Statistics

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1. Number Systems and Proof Techniques

1.1. Types of Numbers

Name	Property
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Name	Property
Natural Numbers	Closed under addition
Integer	
Relational Numbers	Can be expressed as $\frac{x}{y}$ where $x, y \in \mathbb{Q}$ and $y \neq 0$
Real Numbers	
Prime Numbers 质数	You all know that.

1.2. Proof Techniques

1.2.1. Proof by contradiction

Example: Use proof by contradiction to prove $\sqrt{2}$ is a irrational number:

- If $\sqrt{2}$ were rational, then it can be written as $\sqrt{2} = \frac{x}{y}$ where $x, y \in \mathbb{Q}$ and $x, y \neq 0$
- By repeatedly cancelling common factors, we can make sure that x, y have no common factors so they are not both even.
- Since $2 = \frac{x^2}{y^2}$, so x^2 is even, then x is even.
- Let $x = 2w, w \in \mathbb{N}$.
- Then $x^2 = 4w^2$ so $4w^2 = 2y^2, y^2 = 2w^2$ so y^2 is even then so is y .
- This contradicts the fact that x and y are not both even, so 2 being rational, must have been wrong.

1.2.2. Proof by induction

Normal steps:

1. Prove the first number holds the property.
2. Prove that if $n = m$ holds the property, then so is $n = m + 1$.

Examples:

- For every natural number n , $2^{n+2} + 3^{2n+1}$ is divisible by 7

1.3. Tutorial 1

- Use counter-example for easy question.
- When it comes to rational numbers, use the form $\frac{x}{y}$ to represent it.
- To prove \sqrt{n} is not rational, usually the parity on the sides of the equation is different.

2. Set Theory

2.1. Notation

$$B = \{1, 3, 5\} \Rightarrow 1 \in B, 2 \notin B$$

For inexplicit set: $S = \{x | P(x)\}$ e.g.

- $S = \{x | x \text{ is an odd positive integer}\}$
- $C = \{n^2 | n \text{ is an integer}\}$

2.2. Important Sets

Name	Notation	Field
Natural Numbers	N	$\{0, 1, 2 \dots\}$
Integers	Z	$\{\dots - 1, 0, 1 \dots\}$
Positive Integers	Z^+	$\{1, 2, 3 \dots\}$
Rationals	Q	$\{\frac{x}{y}\}$
Real numbers	R	$Z + Q$

2.3. Subset

- If **every** element of B is an element of A, then B is the subset of A. Notation: $B \subseteq A$.

2.4. Equality

If $B \subseteq A$ and $A \subseteq B$, then $A = B$.

2.5. Operations on sets

First of all, the sets to be operated must have **the same data type**.

Name	Notation	Definition
Union	$A \cup B$	$\{x x \in A \vee x \in B\}$
Intersection	$A \cap B$	$\{x x \in A \wedge x \in B\}$
Relative complement	$A - B$	$\{x x \in A \wedge x \notin B\}$
Complement	$\sim A$	$\{x \notin A\}$
Symmetric difference	$A \Delta B$	$\{x (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B)\}$

2.6. Algebra of sets

Suppose $A, B, C \subseteq U$ in all the situations below.

Name	Content
Commutative laws	$A \cup B = B \cup A, A \cap B = B \cap A$
Associative laws 结合律	$A \cup (B \cup C) = (A \cup B) \cup C, A \cap (B \cap C) = (A \cap B) \cap C$
Identity laws	$A \cup \emptyset = A, A \cup U = U, A \cap U = A, A \cap \emptyset = \emptyset$

Name	Content
Distributive laws 分配律	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C), A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
<i>Complement laws</i>	$A \cup \sim A = U, \sim U = \emptyset, \sim(\sim A) = A, A \cap \sim A = \emptyset, \sim \emptyset = U$
De Morgan's laws	$\sim(A \cup B) = \sim A \cap \sim B, \sim(A \cap B) = \sim A \cup \sim B$

2.7. Power set

The power set $Pow(A)$ is the set of all the subsets of A, denoted by $Pow(A) = \{C | C \subseteq A\}$.

2.7.1. Some laws

- $\forall A, B, Pow(A \cap B) = Pow(A) \cap Pow(B)$
- $\exists A, B, Pow(A \cup B) \neq Pow(A) \cup Pow(B)$

2.8. Cardinality of sets

For a **finite** set S, the cardinality equals the number of the elements in S, denoted by $|S|$.

2.8.1. Computing the cardinality of sets

If A, B, C are sets then:

$$\begin{cases} |A \cup B| = |A| + |B| - |A \cap B| \\ |A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \end{cases}$$

Those above are special cases of **the principle of inclusion and exclusion**.

2.9. Ordered pairs

Introduced **cartesian product** $A \times B$, the set of consisting all pairs (a,b) with $a \in A \wedge b \in B$.

2.9.1. Catesian plane

Use a cartesian coordinate system to represent pairs of numbers.

2.10. Bit strings of lengs n

Used to represent the **characteristic vector** of subsets of a set.

e.g. $S = \{1, 2, 3, 4, 5\}, A = \{1, 3, 5\}, B = \{3, 4\}$

- the CV(characteristic vector) of A is {1,0,1,0,1}
- the CV of B is {0,0,1,1,0}

2.11. Tutorial 2

- The cartesian product holds the **distributive laws** as well.

3. Relations

3.1. Definition (Binary relation 二元关系)

3.2. Representation

Type	Example
Directed Graph(Digraph)	<pre> graph TD 1((1)) --> 2((2)) 1((1)) --> 4((4)) 1((1)) --> 6((6)) 3((3)) --> 4((4)) 3((3)) --> 6((6)) 5((5)) --> 6((6)) 7((7)) </pre>
Martix	$M = \begin{bmatrix} T & T & F \\ F & F & F \\ T & T & T \end{bmatrix}$

3.3. Unary relation 单元关系

3.4. Infix noatation for binary relation

- Suppose R is a binary relation, then it can be write as xRy where $(x, y) \in R$.

3.5. Properties of binary relations

Suppose a binary relation R on a set A.

Name	Meaning
reflexive	$\forall x \in A, xRx$
symmetric	$\forall x, y \in A, xRy \rightarrow yRx$
antisymmetric	$\forall x, y \in A, xRy \wedge yRx \rightarrow x = y$
transitive	$\forall x, y, z \in A, xRy \wedge yRz \rightarrow xRz$

- It can be reflected by digraph as well.

3.6. Transitive closure 传递闭包

- Example: Let $A = \{1, 2, 3\}$. Find the transitive closure of $R = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 1)\}$.

$$(3, 1), (1, 2) \rightarrow (3, 2)$$

$$(2, 3), (3, 1) \rightarrow (2, 1)$$

$$(2, 1), (1, 2) \rightarrow (2, 2)$$

$$(3, 1), (1, 3) \rightarrow (3, 3)$$

$$\text{so } t(R) = \{(1, 1), (1, 2), (1, 3), (2, 3), (3, 1), (3, 2), (2, 1), (2, 2), (3, 3)\}$$

- Obviously, digraphs are more intuitive: check the graph of R , start at each node, find its end point, **if there is no edge from that node to that end point, add that edge**, and will get the relationship of $t(R)$.

3.7. Equivalence Relations

- A binary relation being reflexive, transitive, and symmetric is called a equivalence relation.
- Equivalence class can be denoted by $E_x = \{y \mid yRx\}$ where $\forall x \in A$.
 - e.g. $E_0 = Z$ is the equivalence class of 0.

3.8. Partition of a set

Just remeber it has something to with equivalence relations.

3.9. Partial orders

- A binary relation being s reflexive,transitive and antisymmetric is called a **paritial order**.
 - e.g. the relation \subseteq on $Pow(A)$

3.9.1. Predecessors (我也不知道翻译是啥)(好了我知道了，是前元，感谢Ramos)

- x is a **predecessor** of y when R is a partial order on a set A and $xRy, x \neq y$
- x is an **immediate predecessor** of y when x is a predecessor of y and there is no $z \notin \{x, y\}$ where xRz, zRy .

3.10. Hasse diagram

- The Hasse Diagram of a partial order is a digraph.

3.11. Total Order

- A binary relation R on a set A is a total order if it is a partial order such that $\forall x, y \in A, xRy \vee yRx$.
- The Hasse diagram of a total order is a **chain**.

3.12. Tutorial 3

- For matrices, figure out the right way to read it. The first number is the horizontal coordinate (up to down); the second is the vertical coordinate(left to right).
- A relation cannot be proved reflexive by combining transitive and symmetric, because there may not be all the elements in the relation compared with the set.⁹