CSE107 Discrete Mathematics and Statistics

1. Number Systems and Proof Techniques

1.1. Types of Numbers

Name	Property

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Natural Numbers	**Closed under addition**
Integer	
Relational Numbers	Can be expressed as $\frac{x}{y}$ where $x, y \in Q$ and $y \neq 0$
Real Numbers	
Prime Numbers *质数*	You all know that.

1.2. Proof Techniques

1.2.1. Proof by contradiction

Example: Use proof by contracdition to prove $\sqrt{2}$ is a irrational number:

- If $\sqrt{2}$ were rational, then it can be written as $\sqrt{2} = \frac{x}{y}$ where $x, y \in Q$ and $x, y \neq 0$
- By repeatedly cancelling common factors, we can make sure that x, y have no common factors so they are not both even.
- Since $2 = \frac{x^2}{y^2}$, so x^2 is even, then x is even.
- Let $x = 2w$, $w \in N$.
- Then $x^2 = 4w^2$ so$4w^2 = 2y^2$, $y^2 = 2w^2$ so y^2 is even then so is y.
- This contradicts the fact that x and y are not both even, so 2 being rational, must have been wrong.

1.2.2. Proof by induction

Normal steps:

1. Prove the first number holds the property.
2. Prove that if $n = m$ holds the property, then so is $n = m + 1$.

Examples:

- For every natural number n, $2^{n+2} + 3^{2n+1}$ is divisible by 7

1.3. Tutorial 1

- Use counter-example for easy question.
- When it comes to rational numbers, use the form $\frac{x}{y}$ to represent it.
- To prove $\sqrt{n}$ is not rational, usually the parity on the sides of the equation is different.

2. Set Theory

2.1. Notation

$B = \{1, 3, 5\} \Rightarrow 1 \in B, 2 \notin B$

For inexplicit set: $S = \{x | P(x)\}$ e.g.

- $S = \{x | x \text{ is an odd positive integer}\}$
- $C = \{n^2 | n \text{ is an integer}\}$

2.2. Important Sets

Name	Notation	Field
Natural Numbers	N	$\{0, 1, 2 \ldots\}$
Integers	Z	$\{\ldots -1, 0, 1 \ldots\}$
Positive Integers	Z^+	$\{1, 2, 3 \ldots\}$
Rationals	Q	$\lbrace\frac{x}{y}
Real numbers	R	$Z + Q$

2.3. Subset

- If **every** element of B is an element of A, then B is the subset of A. Notation: $B \subseteq A$.

2.4. *Equality*

If $B \subseteq A$ and $A \subseteq B$, then $A = B$.

2.5. Operations on sets

First of all, the sets to be operated must have **the same data type**.

Name	Notation	Definition
Union	$A \bigcup B$	$\{x \mid x \in A \bigvee x \in B\}$
Intersection	$A \bigcap B$	$\{x \mid x \in A \bigwedge x \in B\}$
Relative complement	$A - B$	$\{x \mid x \in A \bigwedge x \notin B\}$
Complement	$\sim A$	$\{x \notin A\}$
Symmetric difference	$A \triangle B$	$\{x \mid (x \in A \bigwedge x \notin B) \bigvee (x \notin A \bigwedge x \in B)\}$

2.6. Algebra of sets

Suppose $A, B, C \subseteq U$ in all the situations below.

Name	Content
Commutative laws	$A \bigcup B = B \bigcup A, A \bigcap B = B \bigcap A$
Associative laws 结合律	$A \bigcup (B \bigcup C) = (A \bigcup B) \bigcup C, A \bigcap (B \bigcap C) = (A \bigcap B) \bigcap C$
Identity laws	$A \bigcup \emptyset = A, A \bigcup U = U, A \bigcap U = A, A \bigcap \emptyset = \emptyset$

Name	Content
Distributive laws分配律	$A\bigcap(B\bigcup C)=(A\bigcap B)\bigcup(A\bigcap C), A\bigcup(B\bigcap C)=(A\bigcup B)\bigcap(A\bigcap C)$
Complement laws	$A\bigcup\sim A=U, \sim U=\emptyset, \sim(\sim A)=A, A\bigcap\sim A=\emptyset, \sim\emptyset=U$
De Morgan's laws	$\sim(A\bigcup B)=\sim A\bigcap\sim B, \sim(A\bigcap B)=\sim A\bigcup\sim B$

2.7. Power set

The power set *Pow*(A) is the set of all the subsets of A, denoted by $Pow(A)=\{C|C\subseteq A\}$.

2.7.1. Some laws

- $\forall A, B, Pow(A\bigcap B)=Pow(A)\bigcap Pow(B)$
- s$\exists A, B, Pow(A\bigcup B)\neq Pow(A)\bigcup Pow(B)$

2.8. Cardinality of sets

For a **finite** set S, the cardinality equals the number of the elements in S, denoted by $|S|$.

2.8.1. Computing the cardinality of sets

If A, B, C are sets then:

> $|A\bigcup B|=|A|+|B|-|A\bigcap B|$
>
> $|A\bigcup B\bigcup C|=|A|+|B|+|C|-|A\bigcap B|-|A\bigcap C|-|B\bigcap C|+|A\bigcap B+\bigcap C|$

Those above are special cases of **the principle of inclusion and exclusion**.

2.9. Ordered pairs

Introduced **cartesian product** $A\times B$, the set of consisting all pairs (a,b) with $a\in A\bigwedge b\in B$.

2.9.1. *Catesian plane*

Use a cartesian coordinate system to represent pairs of numbers.

2.10. Bit strings of lengs n

Used to represent the **characteristic vector** of subsets of a set.

e.g. $S=\{1,2,3,4,5\}, A=\{1,3,5\}, B=\{3,4\}$

- the CV(characteristic vector) of A is {1,0,1,0,1}
- the CV of B is {0,0,1,1,0}

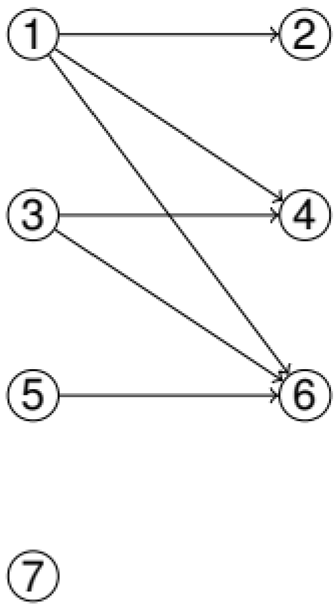
2.11. Tutorial 2

- The catesian product holds the **distrubutive laws** as well.

3. Relations

3.1. Definition (Binary relation 二元关系)

3.2. Representation

Type	Example
Directed Graph(Digraph)	
Martix	$M = \begin{bmatrix} T & T & F \\ F & F & F \\ T & T & T \end{bmatrix}$

3.3. *Unary relation 单元关系*

3.4. Infix noatation for binary relation

- Suppose R is a binary relation, then it can be write as xRy where $(x, y) \in R$.

3.5. Properties of binary relations

> Suppose a binary relation R on a set A.

Name	Meaning
reflexive	$\forall x \in A, xRx$
symmetric	$\forall x, y \in A, xRy \rightarrow yRx$
antisymmetric	$\forall x, y \in A, xRy \bigwedge yRx \rightarrow x = y$
transitive	$\forall x, y, z \in A, xRy \bigwedge yRz \rightarrow xRz$

- It can be reflected by digraph as well.

3.6. **Transitive closure** *传递闭包*

- Example: Let A = {1, 2, 3}. Find the transitive closure of R = {(1, 1),(1, 2),(1, 3),(2, 3),(3, 1)}.

> $(3,1),(1,2) \rightarrow (3,2)$
>
> $(2,3),(3,1) \rightarrow (2,1)$
>
> $(2,1),(1,2) \rightarrow (2,2)$
>
> $(3,1),(1,3) \rightarrow (3,3)$
>
> so $t(R) = \{(1,1),(1,2),(1,3),(2,3),(3,1),(3,2),(2,1),(2,2),(3,3)\}$

- Obiviously, digraphs are more intuitive: check the graph of R, start at each node, find its end point, i**f there is no edge from that node to that end point, add that edge,** and will get the relationship of t(R).

3.7. Equivalence Relations

- A binary relation being reflexive, transitive, and symmetric is called a equivalence relation.
- Equivalence class can be denoted by $E_x = \{y \mid yRx\}$ where $\forall x \in A$.
  - e.g. $E_0 = Z$ is the equivalence class of 0.

3.8. *Partition of a set*

Just remeber it has something to with equivalence relations.

3.9. **Partial orders**

- A binary relation being s reflexive,transitive and antisymmetric is called a **paritial order**.
  - e.g. the relation $\subseteq$ on $Pow(A)$

3.9.1. Predecessors (我也不知道翻译是啥)(好了我知道了，是前元，感谢Ramos)

- x is a **predecessor** of y when R is a partial order on a set A and $xRy, x \neq y$
- x is an **immediate predecessor** of y when x is a predecessor of y and there is no $z \notin \{x,y\}$ where xRz, zRy.

3.10. Hasse diagram

- The Hasse Diagram of a partial order is a digraph.

3.11. Total Order

- A binary relation R on a set A is a total order if it is a partial order such that $\forall x, y \in A, xRy \bigvee yRx$.

- The Hasse diagram of a total order is a **chain**.

Tutorial

- For matrices, figure out the right way to read it. The first number is the horizontal coordinate (up to down); the second is the vertical coordinate(left to right).
- A relation cannot be proved reflexive by combining transistive and symmetric, because there may not be all the elements in the relation compared with the set.9