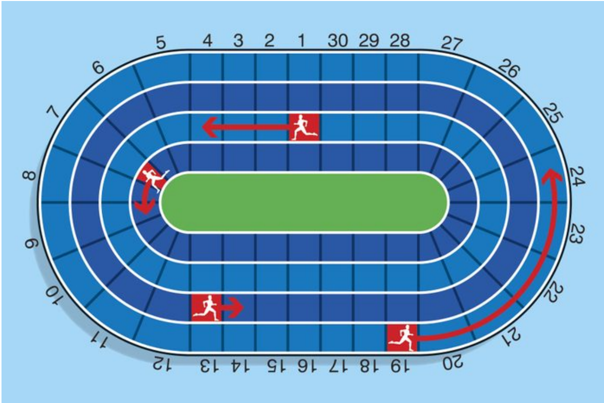
Here are the full, or partial solutions.

Year 8 and below

Four runners are at different positions around an athletics track. The track is divided into thirty sectors as indicated by the numbers round the periphery. Each runner moves forward the number of sectors indicated by the red arrow in front of them, in the same time. When they are next all in the same sector, which sector number will it be?



Solution

Let's call the runner in sector 1 Runner 1, the runner in sector 7 Runner 2, the person in sector 13 Runner 3 and the athlete in sector 19 Runner 4.
We don't know how long it takes Runner 1 to move from Sector 1 to Sector 4, but in that same time period, Runner 2 moves forward 2 sectors, Runner 3 moves on 1 sector and Runner 4 moves 5 sectors.
We can write an expression for the sector number reached after t periods.

	Sector after t periods	0	1	2	3
Runner 1	$1+3t$	1	4	7	10
Runner 2	$7+2t$	7	9	11	13
Runner 3	$13+t$	13	14	15	16
Runner 4	$19+5t$	19	24	29	4

Notice that Runner 4's sector number 'wrapped round' to 4. $29+5$ is 34 but they reached sector 4 on the track. This is arithmetic modulo 30, you have met the idea before with hours of the day. The hour after the 24^{th} hour, isn't the 25^{th} hour, it's the 1^{st} hour (of the next day).
We could just keep adding columns to our table until all the Runners' sector numbers became the same. This is a valid way of solving the problem, sometimes called a trial and error, or 'brute force' approach. The downside is that we often may have little idea of how many repetitions would be needed. Let's try to be a bit more analytical: We need to find the first value of t for which all runners are in the same sector.
We need to find t for which the following is true:

$$1+3t=7+2t=13+t=19+5t$$

we can subtract t throughout,

$$1+2t=7+t=13=19+4t$$

and we can subtract 1 throughout,

$$2t=6+t=12=18+4t$$

Taking equations pairwise from the last line, we have $2t = 6+t \implies t = 6$ and $6+t = 12 \implies t = 6$ and $12 = 18+4t$. The first two give an answer of six time periods. Let's try that in our table:

	Sector after t periods	0	1	2	3	4	5	6
Runner 1	$1+3t$	1	4	7	10	13	16	19
Runner 2	$7+2t$	7	9	11	13	15	17	19
Runner 3	$13+t$	13	14	15	16	17	18	19
Runner 4	$19+5t$	19	24	29	4	9	14	19

So we can see that $t=6$ is a solution and the runners will be all together on sector 19 of the track after six time periods.
But what about that last equation $12 = 18+4t$ above? If we simplify: $-6 = 4t$ and then substitute $t = 6$ into it we get $-6 = 24$, which seems strange until we consider that we are counting modulo 30, six back from thirty is 24.
So Sector 19 is the answer.

Year 9 and above

You have a four-digit, positive integer. Now you remove one of the four digits. The three digits that are left, in their original order from the four digit number, make a three-digit number. The sum of the four-digit number and the three-digit number is 6031. What is the four-digit number?

Solution

We'll begin by describing the four-digit number as $abcd$. The three digit number could be bcd, removing the a, acd, removing the b, abd, or abc. The additions for each of these cases look like:

Case 1: a removed

$$\begin{array}{rcccc} & a & b & c & d \\ + & & b & c & d \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

Case 2: b removed

$$\begin{array}{rcccc} & a & b & c & d \\ + & & a & c & d \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

Case 3: c removed

$$\begin{array}{rcccc} & a & b & c & d \\ + & & a & b & d \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

Case 4: d removed

$$\begin{array}{rcccc} & a & b & c & d \\ + & & a & b & c \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

But notice that in the first three cases, the answer has a 1 in the units place, each obtained from $d + d$. This is not possible, $2d$ must be even, so the three-digit number must be abc as in the fourth case.

Now, let's think about the value of a. It could be 6, (no carry from the hundreds column), or it could be 5, (carry 1 from the hundreds colum.

Case 1: $a = 6$

$$\begin{array}{rcccc} & 6 & b & c & d \\ + & & 6 & b & c \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

Case 2: $a = 5$

$$\begin{array}{rcccc} & 1 & & & \\ & 5 & b & c & d \\ + & & 5 & b & c \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

We can see that $a = 6$ will not work; if there is no carry, what value of b plus 6 can give zero in the hundreds place? None. So the case $a = 5$ must be correct. So now looking at Case 2, we must find the value of b such that $b + 5 = 10$, giving a 0 in the hundreds' answer column. So now we have two cases again, $b = 5$ (with no carry from the tens column), or $b = 4$ with 1 carried from the tens column:

Case 1: $b = 5$

$$\begin{array}{rcccc} & 1 & & & \\ & 5 & 5 & c & d \\ + & & 5 & 5 & c \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

Case 2: $b = 4$

$$\begin{array}{rcccc} & 1 & 1 & & \\ & 5 & 4 & c & d \\ + & & 5 & 4 & c \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

In Case 1, in the tens column we must have $c + 5 = 3$, and yet there can be no carry. This is not possible so Case 1 is wrong, Case 2 is correct. So $b = 4$, and we must find $c + 4 = 3$. So c could be 9 with no carry from the ones column or c could be 8 with a carry from the ones column:

Case 1: $c = 9$

$$\begin{array}{rcccc} & 1 & 1 & & \\ & 5 & 4 & 9 & d \\ + & & 5 & 4 & 9 \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

Case 2: $c = 8$

$$\begin{array}{rcccc} & 1 & 1 & 1 & \\ & 5 & 4 & 8 & d \\ + & & 5 & 4 & 8 \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

Nearly there! If $c = 9$ then in the ones column we would have $d + 9 = 1$ which means that $d = 2$ and there must be a carry, but Case 1 has no carry to the tens place so Case 1 is impossible. Therefore Case 2 with $c = 8$ must be true. But if $c = 8$ and we do have a carry to the tens column, then d has to be 3.

$$\begin{array}{rcccc} & 1 & 1 & 1 & \\ & 5 & 4 & 8 & 3 \\ + & & 5 & 4 & 8 \\ \hline & 6 & 0 & 3 & 1 \end{array}$$

The four digit number is 5483.