TECHNICAL REPORT: SUPPORTING SOFTWARE FOR ARTICLE "QUASI-OPTIMAL DISCONTINUOUS GALERKIN DISCRETISATIONS OF THE p-DIRICHLET PROBLEM"

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Abstract. We discuss an implementation of the smoothing operator from [2, 1].

Key words. smoothing operator, quasi-optimal, DG, p-Laplace

1. Notation. We consider the action of operator $E_p = A_p - B_p(\mathrm{Id} - A_p)$ from [2, 1] for the lowest-order case $p = 1$ on triangles:

$$\begin{aligned} E_1 v &= A_1 v + B_1(v - A_1 v) \\ &= \sum_{z\in\mathcal{L}_1^{\mathrm{int}}} v|_{K_z}(z)\,\varphi_z^1 + \sum_{F\in\mathcal{F}^{\mathrm{int}}} \left(\int_F \{\!\{v\}\!\} - \sum_{z\in\mathcal{L}_1^F} v|_{K_z}(z) \int_F \varphi_z^1 \right) \hat\varphi_F, \end{aligned}$$

where $\mathcal{L}_1^{\mathrm{int}}$ are the interior nodes of the first-order Lagrange space (i.e., evalutation at internal vertices), K_z is an arbitrary mesh element containing z, φ_z^1 denote the first-order Lagrange basis function such that $\varphi_z^1(z') = \delta_{zz'}$ for mesh vertices z, z', $\mathcal{F}^{\mathrm{int}}$ is the set of internal facets, $\{\!\{\cdot\}\!\}$ stands for facet average, $\mathcal{L}_1^F$ denotes the first-order Lagrange nodes on facet F (i.e., facet vertices), and $\hat\varphi_F$ is the facet bubble normalized as $\int_{F'} \hat\varphi_F = \delta_{FF'}$.

The facet bubble can be represented as

$$\hat\varphi_F = \frac{6}{|F|}\varphi^1_{z_F^1}\varphi^1_{z_F^2} = \frac{3}{2|F|}\varphi_F, \tag{1}$$

where $\varphi^1_{z_F^j}$, $j = 1, 2$, are the first-order Lagrange functions associated with facet vertices z_F^1, z_F^2, and φ_F is the nodal facet bubble, i.e., $\varphi_F = 1$ at the midpoint of F.

The above-defined E_1 has in its range only functions vanishing on the boundary and hence, in this formulation, it only applies to homogeneous Dirichlet problems. The same holds for this report and the reference implementation, which can be obtained at https://github.com/blechta/quasioptimal-dg-p-laplace.

2. Lowest-order Crouzeix–Raviart case. A linear functional f assembled against smoothed Crouzeix–Raviart basis $E_1\varphi_F^{\mathrm{CR}}$, $F \in \mathcal{F}^{\mathrm{int}}$, can be expressed as a linear combination of $\langle f, \varphi_z^1\rangle$, $z \in \mathcal{L}_1^{\mathrm{int}}$, and $\langle f, \varphi_F\rangle$, $F \in \mathcal{F}^{\mathrm{int}}$,

$$\begin{aligned} &\langle f, E_1\varphi_F^{\mathrm{CR}}\rangle \\ &= \sum_{z\in\mathcal{L}_1^{\mathrm{int}}} \varphi_F^{\mathrm{CR}}|_{K_z}(z)\,\langle f, \varphi_z^1\rangle + \sum_{F'\in\mathcal{F}^{\mathrm{int}}} \left(\int_{F'} \{\!\{\varphi_F^{\mathrm{CR}}\}\!\} - \sum_{z\in\mathcal{L}_1^{F'}} \varphi_F^{\mathrm{CR}}|_{K_z}(z) \int_{F'} \varphi_z^1 \right) \langle f, \hat\varphi_{F'}\rangle. \end{aligned} \tag{2}$$

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We have

$$\int_{F'} \{\!\{\varphi_F^{\mathrm{CR}}\}\!\} = \delta_{FF'}|F|, \tag{3a}$$

$$\varphi_F^{\mathrm{CR}}|_{K_z}(z) = \begin{cases} 1 & \text{if } F \subset \overline{K_z} \text{ and } z \in \overline{F}, \\ -1 & \text{if } F \subset \overline{K_z} \text{ and } z \text{ is a vertex adjacent to } F, \\ 0 & \text{otherwise,} \end{cases} \tag{3b}$$

and

$$\int_{F'} \varphi_z^1 = \tfrac{1}{2}|F'| \quad \text{if } z \in \overline{F}'. \tag{3c}$$

The first term in (2) results in the index sets and coefficients `coeffs1` computed in `SmoothingOpVeeserZanottiCR.coeffs()`. The second term gives `coeffs2`, where the coefficients `3/2`, `±3/4` result from

$$\begin{aligned} \int_{F'} \{\!\{\varphi_F^{\mathrm{CR}}\}\!\} \langle f, \hat\varphi_{F'}\rangle &= \frac{3}{2}\langle f, \varphi_{F'}\rangle \qquad \text{if } F' = F \in \mathcal{F}^{\mathrm{int}}, \\ \int_{F'} \varphi_z^1 \langle f, \hat\varphi_{F'}\rangle &= \frac{3}{4}\langle f, \varphi_{F'}\rangle \qquad \text{if } F' \in \mathcal{F}^{\mathrm{int}} \text{ and } z \in \mathcal{L}_1^{F'} \cap \mathcal{L}_1^{\mathrm{int}}, \end{aligned}$$

where we have used (1), (3a), and (3c). Recall that Firedrake uses nodal bubbles φ_F normalized to one at the facet midpoint.

3. Lowest-order DG case. Consider DG basis members $\varphi_{K,z}$, for mesh element K, and $z \in \mathcal{L}_1^{\mathrm{int}}$,

$$\varphi_{K,z}|_{K'}(z') = \begin{cases} 1 & \text{if } K = K',\ z = z', \text{ and } z \in \mathcal{L}_1^{\mathrm{int}}, \\ 0 & \text{otherwise,} \end{cases}$$

which in particular implies that

$$\varphi_{K,z}|_{K_{z'}}(z') = \begin{cases} 1 & \text{if } z = z',\ z \in \mathcal{L}_1^{\mathrm{int}}, \text{ and } K_{z'} = K, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Recall that, for $z \in \mathcal{L}_1$, K_z is a uniquely given mesh element containing z (chosen at random or of smallest index, etc.).

Similarly to (2) we have

$$\begin{aligned} \langle f, E_1\varphi_{K,z}\rangle &= \sum_{z' \in \mathcal{L}_1^{\mathrm{int}}} \varphi_{K,z}|_{K_{z'}}(z') \langle f, \varphi_{z'}^1\rangle \\ &\quad + \sum_{F' \in \mathcal{F}^{\mathrm{int}}} \Big(\int_{F'} \{\!\{\varphi_{K,z}\}\!\} - \sum_{z' \in \mathcal{L}_1^{F'}} \varphi_{K,z}|_{K_{z'}}(z') \int_{F'} \varphi_{z'}^1 \Big) \langle f, \hat\varphi_{F'}\rangle. \end{aligned} \tag{5}$$

The first term in (5) with (4) explains `coeffs1` in `SmoothingOpVeeserZanottiDG.coeffs()`. It is

$$\int_{F'} \{\!\{\varphi_{K,z}\}\!\} = \frac{1}{4}|F'| \qquad \text{if } F' \subset K \text{ and } z \in \mathcal{L}_1^{F'} \cap \mathcal{L}_1^{\mathrm{int}},$$

which explains together with (3c) and (1) the values `3/8` and `-3/4` in `coeffs2`, which correspond to the second and third term in (5), respectively.

REFERENCES

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