## 1 Algebra

Absolute Value Inequalities:

$|f(x)| < a \Rightarrow -a < f(x) < a$

$|f(x)| > a \Rightarrow f(x) > a$ or $f(x) < -a$

## 2 Calculus

Differentiation under the integral sign

$\frac{\mathrm{d}}{\mathrm{d}x}\left(\int_{a(x)}^{b(x)} f(x,t)\mathrm{d}t\right) = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} f_x(x,t)\mathrm{d}t.$

**Concavity in 1 dimension**

If $g: I \to \mathbb{R}$ is twice differentiable in the interval $I$:

concave:
if and only if $g''(x) \leq 0$ for all $x \in I$

strictly concave:
if $g''(x) < 0$ for all $x \in I$

convex:
if and only if $g''(x) \geq 0$ for all $x \in I$

strictly convex if:
$g''(x) > 0$ for all $x \in I$

**Multivariate Calculus**

The Gradient $\nabla$ of a twice differntiable function $f$ is defined as:

$\nabla f : \mathbb{R}^d \to \mathbb{R}^d$

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_d \end{pmatrix} \mapsto \begin{pmatrix} \frac{\partial f}{\partial \theta_1} \\ \frac{\partial f}{\partial \theta_2} \\ \vdots \\ \frac{\partial f}{\partial \theta_d} \end{pmatrix}\Bigg|_\theta$$

**Hessian**

The Hessian of $f$ is a symmetric matrix of second partial derivatives of $f$

$$\mathbf{H}h(\theta) = \nabla^2 h(\theta) = \begin{pmatrix} \frac{\partial^2 h}{\partial\theta_1\partial\theta_1}(\theta) & \cdots & \frac{\partial^2 h}{\partial\theta_1\partial\theta_d}(\theta) \\ & \vdots & \\ \frac{\partial^2 h}{\partial\theta_d\partial\theta_1}(\theta) & \cdots & \frac{\partial^2 h}{\partial\theta_d\partial\theta_d}(\theta) \end{pmatrix} \in \mathbb{R}^{d \times d}$$

A symmetric (real-valued) $d \times d$ matrix $\mathbf{A}$ is:

Positive semi-definite:
$\mathbf{x}^T \mathbf{A}\mathbf{x} \geq 0 \quad$ for all $\mathbf{x} \in \mathbb{R}^d$.

Positive definite:
$\mathbf{x}^T \mathbf{A}\mathbf{x} > 0$ for all non-zero vectors $\mathbf{x} \in \mathbb{R}^d$

Negative semi-definite (resp. negative definite):
$\mathbf{x}^T \mathbf{A}\mathbf{x}$ is negative for all $\mathbf{x} \in \mathbb{R}^d - \{0\}$.

Positive (or negative) definiteness implies positive (or negative) semi-definiteness.

If the Hessian is positive definite then $f$ attains a local minimum at $a$ (convex).

If the Hessian is negative definite at $a$, then f attains a local maximum at $a$ (concave).

If the Hessian has both positive and negative eigenvalues then $a$ is a saddle point for $f$.

## 3 Important probability distributions

**Bernoulli**

Parameter $p \in [0,1]$, discrete

$p_x(k) = \begin{cases} p, & \text{if k = 1} \\ (1-p), & \text{if k = 0} \end{cases}$

$\mathbb{E}[X] = p$
$Var(X) = p(1-p)$

**Binomial**

Parameters $p$ and $n$, discrete. Describes the number of successes in n independent Bernoulli trials.

$p_x(k) = \binom{n}{k} p^k (1-p)^{n-k}, k = 1, \ldots, n$

$\mathbb{E}[X] = np$

$Var(X) = np(1-p)$

**Multinomial**

Parameters $n > 0$ and $p_1, \ldots, p_r$.

$p_x(x) = \frac{n!}{x_1!, \ldots, x_n!} p_1, \ldots, p_r$

$\mathbb{E}[X_i] = n * p_i$
$Var(X_i) = np_i(1-p_i)$

**Poisson**

Parameter $\lambda$. discrete, approximates the binomial PMF when $n$ is large, $p$ is small, and $\lambda = np$.

$\mathbf{p_x}(k) = exp(-\lambda)\frac{\lambda^k}{k!}$ for $k = 0, 1, \ldots,$

$\mathbb{E}[X] = \lambda$
$Var(X) = \lambda$

**Exponential**

Parameter $\lambda$, continuous

$f_x(x) = \begin{cases} \lambda exp(-\lambda x), & \text{if x >= 0} \\ 0, & \text{o.w.} \end{cases}$

$F_x(x) = \begin{cases} 1 - exp(-\lambda x), & \text{if x >= 0} \\ 0, & \text{o.w.} \end{cases}$

$\mathbb{E}[X] = \frac{1}{\lambda}$

$Var(X) = \frac{1}{\lambda^2}$

**Univariate Gaussians**

Parameters $\mu$ and $\sigma^2 > 0$, continuous

$f(x) = \frac{1}{\sqrt{(2\pi\sigma)}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$

$\mathbb{E}[X] = \mu$
$Var(X) = \sigma^2$

Invariant under affine transformation:

$aX + b \sim N(X + b, a^2\sigma^2)$

Symmetry:

If $X \sim N(0, \sigma^2)$, then $-X \sim N(0, \sigma^2)$

$\mathbb{P}(|X| > x) = 2\mathbb{P}(X > x)$

Standardization:

$Z = \frac{X-\mu}{\sigma} \sim N(0,1)$

$\mathbf{P}(X \leq t) = \mathbf{P}\left(Z \leq \frac{t-\mu}{\sigma}\right)$

Higher moments:

$\mathbb{E}[X^2] = \mu^2 + \sigma^2$
$\mathbb{E}[X^3] = \mu^3 + 3\mu\sigma^2$
$\mathbb{E}[X^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4$

**Multivariate Gaussians**

The distribution of , the -dimensional Gaussian or normal distribution , is completely specified by the vector mean and the covariance matrix If is invertible, then the pdf of is

**Uniform**

Parameters $a$ and $b$, continuous.

$\mathbf{f_x}(x) = \begin{cases} \frac{1}{b-a}, & \text{if a < x <b} \\ 0, & \text{o.w.} \end{cases}$

$\mathbb{E}[X] = \frac{a+b}{2}$

$Var(X) = \frac{(b-a)^2}{12}$

Maximum of n iid uniform r.v.

Minimum of n iid uniform r.v.

**Cauchy**

continuous, parameter $m$,

$f_m(x) = \frac{1}{\pi}\frac{1}{1+(x-m)^2}$

$\mathbb{E}[X] = notdefined!$
$Var(X) = notdefined!$

$\text{med}(X) = P(X > M) = P(X < M)$

$= 1/2 = \int_{1/2}^{\infty} \frac{1}{\pi} \cdot \frac{1}{1+(x-m)^2} dx$

## 4 Random Vectors

A random vector $\mathbf{X} = \left(X^{(1)}, \ldots, X^{(d)}\right)^T$ of dimension $d \times 1$ is a vector-valued function from a probability space $\omega$ to $\mathbb{R}^d$:

$\mathbf{X} : \Omega \longrightarrow \mathbb{R}^d$

$$\omega \longrightarrow \begin{pmatrix} X^{(1)}(\omega) \\ X^{(2)}(\omega) \\ \vdots \\ X^{(d)}(\omega) \end{pmatrix}$$

where each $X^{(k)}$ , is a (scalar) random variable on $\Omega$.

PDF of $\mathbf{X}$: joint distribution of its components $X^{(1)}, \ldots, X^{(d)}$.

CDF of $\mathbf{X}$:

$\mathbb{R}^d \to [0,1]$

$\mathbf{x} \mapsto \mathbf{P}(X^{(1)} \leq x^{(1)}, \ldots, X^{(d)} \leq x^{(d)})$.

The sequence $\mathbf{X}_1, \mathbf{X}_2, \ldots$ converges in probability to $\mathbf{X}$ if and only if each component of the sequence $X_1^{(k)}, X_2^{(k)}, \ldots$ converges in probability to $X^{(k)}$.

## 5 Quantiles of a Distribution

Let $\alpha$ in $(0,1)$. The quantile of order $1-\alpha$ of a random variable $X$ is the number $q_\alpha$ such that:

$q_\alpha = \mathbb{P}(X \leq q_\alpha) = 1 - \alpha$

$\mathbb{P}(X \geq q_\alpha) = \alpha$

$F_X(q_\alpha) = 1 - \alpha$

$F_X^{-1}(1-\alpha) = \alpha$

If $X \sim N(0,1)$:

$\mathbb{P}(|X| > q_\alpha) = \alpha$

## 6 Expectation

$\mathbb{E}[X] = \int_{-inf}^{+inf} x \cdot f_X(x) \quad dx$

$\mathbb{E}[g(X)] = \int_{-inf}^{+inf} g(x) \cdot f_X(x) dx$

$\mathbb{E}[X\big|Y = y] = \int_{-inf}^{+inf} x \cdot f_{X|Y}(x|y) \quad dx$

Integration limits only have to be over the support of the pdf. Discrete r.v. same as continuous but with sums and pmfs.

Total expectation theorem:

$\mathbb{E}[X] = \int_{-inf}^{+inf} f_Y(y) \cdot \mathbb{E}[X\big|Y = y] dy$

Expectation of constant $a$:

$\mathbb{E}[a] = a$

Product of **independent** r.vs $X$ and $Y$ :

$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$

Product of **dependent** r.vs $X$ and $Y$ :

$\mathbb{E}[X \cdot Y] \neq \mathbb{E}[X] \cdot \mathbb{E}[Y]$

$\mathbb{E}[X \cdot Y] = \mathbb{E}[\mathbb{E}[Y \cdot X|Y]] = \mathbb{E}[Y \cdot \mathbb{E}[X|Y]]$

Linearity of Expectation where $a$ and $c$ are given scalars:

$\mathbb{E}[aX + cY] = a\mathbb{E}[X] + c\mathbb{E}[Y]$

If Variance of $X$ is known:

$\mathbb{E}[X^2] = var(X) - \mathbb{E}[X]$

The expectation of a random vector is the elementwise expectation. Let $\mathbf{X}$ be a random vector of dimension $d \times 1$.

$$\mathbb{E}[\mathbf{X}] = \begin{pmatrix} \mathbb{E}[X^{(1)}] \\ \vdots \\ \mathbb{E}[X^{(d)}] \end{pmatrix}.$$

## 7 Variance

Variance is the squared distance from the mean.

$Var(X) = \mathbb{E}[(X - \mathbb{E}(X))^2]$

$Var(X) = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2$

Variance of a product with constant $a$:

$Var(aX) = a^2 Var(X)$

Variance of sum of two **dependent** r.v.:

$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X,Y)$

Variance of sum of two **independent** r.v.:

$Var(X + Y) = Var(X) + Var(Y)$

## 8 Covariance

The Covariance is a measure of how much the values of each of two correlated random variables determine each other

$Cov(X,Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$

$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$Cov(X,Y) = \mathbb{E}[(X)(Y - \mu_Y)]$

Possible notations:

$Cov(X,Y) = \sigma(X,Y) = \sigma_{(X,Y)}$

Covariance is commutative:

$Cov(X,Y) = Cov(Y,X)$

Covariance with of r.v. with itself is variance:

$Cov(X,X) = \mathbb{E}[(X-\mu_X)^2] = Var(X)$

Useful properties:

$Cov(aX+h, bY+c) = abCov(X,Y)$

$Cov(X, X+Y) = Var(X) + cov(X,Y)$

$Cov(aX + bY, Z) = aCov(X,Z) + bCov(Y,Z)$

If $Cov(X,Y) = 0$, we say that X and Y are uncorrelated. If $X$ and $Y$ are independent, their Covariance is zero. The converse is not always true. It is only true if $X$ and $Y$ form a gaussian vector, ie. any linear combination $\alpha X + \beta Y$ is gaussian for all $(\alpha, \beta) \in \mathbb{R}^2$ without $\{0,0\}$.

## 9 Covariance Matrix

Let $X$ be a random vector of dimension $d \times 1$ with expectation $\mu_X$.
Matrix outer products!

$\Sigma = \mathbb{E}[(X-\mu_X)(X-\mu_X)^T]$
$= \mathbb{E}[XX^T] - \mathbb{E}[X]\mathbb{E}[X]^T$
$= \mathbb{E}[XX^T] - \mu_X \mu_X^T$

## 10 Law of large Numbers and Central Limit theorem univariate

Let $X_1, ..., X_n \overset{iid}{\sim} P_\mu$, where $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$ for all $i = 1, 2, ..., n$ and $\overline{X_n} = \frac{1}{n}\sum_{i=1}^n X_i$.

Law of large numbers:

$\overline{X_n} \xrightarrow[n\to\infty]{P, a.s.} \mu$ .

$\frac{1}{n}\sum_{i=1}^n g(X_i) \xrightarrow[n\to\infty]{P, a.s.} \mathbb{E}[g(X)]$

Central Limit Theorem:

$\sqrt(n)\frac{\overline{X_n}-\mu}{\sqrt(\sigma^2)} \xrightarrow[n\to\infty]{(d)} N(0,1)$

$\sqrt(n)(\overline{X_n} - \mu) \xrightarrow[n\to\infty]{(d)} N(0, \sigma^2)$

Variance of the Mean:

$Var(\overline{X_n}) =$
$(\frac{\sigma^2}{n})^2 Var(X_1 + X_2, ..., X_n) = \frac{\sigma^2}{n}$.

Expectation of the mean:

$E[\overline{X_n}] = \frac{1}{n}E[X_1 + X_2, ..., X_n] = \mu$.

## 11 Law of large Numbers and Central Limit theorem multivariate

## 12 Statistical models

$E, \{P_\theta\}_{\theta \in \Theta}$

$E$ is a sample space for $X$ i.e. a set that contains all possible outcomes of $X$

$\{\mathbb{P}_\theta\}_{\theta\in\Theta}$ is a family of probability distributions on $E$.

$\Theta$ is a parameter set, i.e. a set consisting of some possible values of $\Theta$.

$\theta$ is the true parameter and unknown. In a parametric model we assume that $\Theta \subset \mathbb{R}^d$, for some $d \geq 1$.

Identifiability:

$\theta \neq \theta' \Rightarrow \mathbb{P}_\theta \neq \mathbb{P}_{\theta'}$

$\mathbb{P}_\theta = \mathbb{P}_{\theta'} \Rightarrow \theta = \theta'$

A Model is well specified if:

$\exists \theta \ s.t.\ \mathbb{P} = \mathbb{P}_\theta$

## 13 Estimators

A statistic is any measurable functionof the sample, e.g. $\overline{X_n}, max(X_i)$, etc. An Estimator of $\theta$ is any statistic which does not depend on $\theta$.

An estimator $\hat{\theta}_n$ is weakly consistent if: $\lim_{n\to\infty} \hat{\theta}_n = \theta$ or $\hat{\theta}_n \xrightarrow[n\to\infty]{P} \mathbb{E}[g(X)]$. If the convergence is almost surely it is strongly consistent.

Asymptotic normality of an estimator:

$\sqrt(n)(\hat{\theta}_n - \theta) \xrightarrow[n\to\infty]{(d)} N(0, \sigma^2)$

$\sigma^2$ is called the **Asymptotic Variance** of $\hat{\theta}_n$. In the case of the sample mean it the variance of a single $X_i$. If the estimator is a function of the sample mean the **Delta Method** is needed to compute the Asymptotic Variance. Asymptotic Variance $\neq$ Variance of an estimator.

Bias of an estimator:

$Bias(\hat{\theta}_n = \mathbb{E}[\hat{\theta_n}] - \theta$

Quadratic risk of an estimator:

$R(\hat{\theta}_n) = \mathbb{E}[(\hat{\theta}_n - \theta)^2] = Bias^2 + Variance$

## 14 Confidence intervals

Let $(E, (\mathbb{P}_\theta)_{\theta\in\Theta})$ be a statistical model based on observations $X_1, \dots X_n$ and assume $\Theta \subseteq \mathbb{R}$. Let $\alpha \in (0,1)$.

Confidence interval of level $1-\alpha$ for $\theta$:

Any random interval $\mathcal{I}$, depending on the sample $X_1, \dots X_n$ but not at $\theta$ and such that:

$\mathbb{P}_\theta[\mathcal{I} \ni \theta] \geq 1 - \alpha, \ \ \forall \theta \in \Theta$

Confidence interval of **asymptotic level** $1-\alpha$ for $\theta$:

Any random interval $\mathcal{I}$ whose boundaries do not depend on $\theta$ and such that:

$\lim_{n\to\infty} \mathbb{P}_\theta[\mathcal{I} \ni \theta] \geq 1 - \alpha, \ \ \forall \theta \in \Theta$

**Two-sided asymptotic CI**

Let $X_1, \dots, X_n = \tilde{X}$ and $\tilde{X} \overset{iid}{\sim} P_\theta$. A two-sided CI is a function depending on $\tilde{X}$ giving an upper and lower bound in which the estimated parameter lies $\mathcal{I} = [l(\tilde{X}), u(\tilde{X})]$ with a certain probability $\mathbb{P}(\theta \in \mathcal{I}) \geq 1 - q_\alpha$ and conversely $\mathbb{P}(\theta \notin \mathcal{I}) \leq \alpha$

Since the estimator is a r.v. depending on $\tilde{X}$ it has a variance $Var(\hat{\theta}_n$ and a mean $\mathbb{E}[\hat{theta}_n]$. After finding those it is possible to standardize the estimator using the CLT. This yields an asymptotic CI:

$\mathcal{I} = \hat{\theta}_n + [\frac{-q_{\alpha/2}\sqrt{Var(\theta_n)}}{\sqrt{n}}, \frac{q_{\alpha/2}\sqrt{Var(\theta_n)}}{\sqrt{n}}]$

Since this expression depends on the real variance $Var(X_i)$ of the r.vs, the variance has to be estimated. Three possible methods: plugin (use sample mean), solve (solve quadratic inequality), conservative (use the maximum of the variance).

**Delta Method**

$\sqrt{n}(g(\widehat{m}_1) - g(m_1(\theta))) \xrightarrow[n\to\infty]{(d)} \mathcal{N}(0, g'(m_1(\theta))^2\sigma^2)$

## 15 Hypothesis tests

**Onesided**

**Twosided**

**P-Value**

## 16 Distance between distributions

**Total variation**

The total variation distance TV between the propability measures $P$ and $Q$ with a sample space $E$ is defined as:

$\text{TV}(\mathbf{P}, \mathbf{Q}) = \max_{A\subset E} |\mathbf{P}(A) - \mathbf{Q}(A)|,$

Calculation with $f$ and $g$:

$$\text{TV}(\mathbf{P}, \mathbf{Q}) = \begin{cases} \frac{1}{2}\sum_{x\in E} |f(x) - g(x)|, & \text{discr} \\ \frac{1}{2}\int_{x\in E} |f(x) - g(x)| dx, & \text{cont} \end{cases}$$

Symmetry:
$d(\mathbf{P}, \mathbf{Q}) = d(\mathbf{Q}, \mathbf{P})$
nonnegative:
$d(\mathbf{P}, \mathbf{Q}) \geq 0$
definite:
$d(\mathbf{P}, \mathbf{Q}) = 0 \iff \mathbf{P} = \mathbf{Q}$
triangle inequality:
$d(\mathbf{P}, \mathbf{V}) \leq d(\mathbf{P}, \mathbf{Q}) + d(\mathbf{Q}, \mathbf{V})$
If the support of $\mathbf{P}$ and $\mathbf{Q}$ is disjoint:
$d(\mathbf{P}, \mathbf{V}) = 1$
TV between continuous and discrete r.v:
$d(\mathbf{P}, \mathbf{V}) = 1$

**KL divergence**

the KL divergence (also known as relative entropy) KL between between the propability measures $P$ and $Q$ with the common sample space $E$ and pmf/pdf functions $f$ and $g$ is defined as:

$$\text{KL}(\mathbf{P}, \mathbf{Q}) = \begin{cases} \sum_{x\in E} p(x) \ln\left(\frac{p(x)}{q(x)}\right), & \text{discr} \\ \int_{x\in E} p(x) \ln\left(\frac{p(x)}{q(x)}\right) dx, & \text{cont} \end{cases}$$

Not a distance!
Sum over support of $P$!
Asymetric in general:
$\text{KL}(\mathbf{P}, \mathbf{Q}) \neq \text{KL}(\mathbf{Q}, \mathbf{P})$
Nonnegative:
$\text{KL}(\mathbf{P}, \mathbf{Q}) \geq 0$
Definite:
if $\mathbf{P} = \mathbf{Q}$ then $\text{KL}(\mathbf{P}, \mathbf{Q}) = 0$
Does not satisfy triangle inequality in general:
$KL(\mathbf{P}, \mathbf{V}) \nleq KL(\mathbf{P}, \mathbf{Q}) + KL(\mathbf{Q}, \mathbf{V})$

Estimator of KL divergence:

$\text{KL}(\mathbf{P}_{\theta^*}, \mathbf{P}_\theta) = \mathbb{E}_{\theta^*}\left[\ln\left(\frac{p_{\theta^*}(X)}{p_\theta(X)}\right)\right],$

$\widehat{KL}(\mathbf{P}_{\theta_*}, \mathbf{P}_\theta) = const - \frac{1}{n}\sum_{i=1}^n log(p_\theta(X_i))$

## 17 Likelihood

Let $(E, \{P_\theta\}_{\theta\in\Theta})$ denote a discrete or continuous statistical model. Let $p_\theta$ denote the pmf or pdf of $P_\theta$. Let $X_1, \dots, X_n \overset{iid}{\sim} P_{\theta^*}$ where the parameter $\theta^*$ is unknown. Then the likelihood is the function

$L_n : E^n \times \Theta$
$L_n(x_1, \dots, x_n, \theta) = \prod_{i=1}^n P_\theta[X_i = x_i]$

Loglikelihood:
$\ell_n(\theta) = \ln(L(x_1, \dots, x_n\theta)) =$
$= ln(\prod_{i=1}^n f_\theta(x_i)) =$
$= \sum_{i=1}^n ln(f_\theta(x_i))$

**Bernoulli**

Likelihood 1 trial:
$L_1(p) = p^x(1-p)^{1-x}$

Loglikelihood 1 trial:
$\ell_1(p) = xlog(p) + (1-x)log(1-p)$

Likelihood n trials:

$L_n(x_1, \dots, x_n, p) =$
$= p^{\sum_{i=1}^n x_i}(1-p)^{n - \sum_{i=1}^n x_i}$

Loglikelihood n trials:

$\ell_n(p) =$
$= \sum_{i=1}^n x_i \ln(p) + \left(n - \sum_{i=1}^n x_i\right)\ln(1-p)$

**Binomial**

Likelihood:

$L_n(x_1, \dots, x_n, p, n) =$
$= nC_x\, p^x(1-p)^{n-x} = p^{x_i}(1-p)^{1-x_i}$

Loglikelihood:

$\ell_n(p, n) =$
$= \ln(nC_x) + x\ln(p) + (n-x)\ln(1-p)$

$C$ is a constant from n choose k, disappears after differentiating.

**Multinomial**

Parameters $n > 0$ and $p_1, \dots, p_r$. Sample space= $E = 1, 2, 3, \dots, j$

Likelihood:

$p_x(x) = \prod_{j=1}^n p_j{}^{T_j}$, where $T^j = \mathbb{1}(X_i = j)$ is the count how often an outcome is seen in trials.

Loglikelihood:
$\ell_n = \sum_{j=2}^n T_j \ln\left(p_j\right)$

**Poisson**

Likelihood:
$L_n(x_1, \dots, x_n, \lambda) = \prod_{i=1}^n \frac{\lambda^{\sum_{i=1}^n x_i}}{\prod_{i=1}^n x_i!} e^{n\lambda}$

Loglikelihood:
$\ell_n(\lambda) =$
$= -n\lambda + log(\lambda)(\sum_{i=1}^n x_i)) - log(\prod_{i=1}^n x_i!)$

**Gaussian**

Likelihood:

$L(x_1 \dots x_n; \mu, \sigma^2) =$
$= \frac{1}{\left(\sigma\sqrt{2\pi}\right)^n} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2\right)$

Loglikelihood:

$\ell_n(\mu, \sigma^2) =$
$= -nlog(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i - \mu)^2$

**Exponential**

Likelihood:
$L(x_1 \dots x_n; \lambda) = \lambda^n \exp\left(-\lambda\sum_{i=1}^n x_i\right)$
Loglikelihood:

**Uniform**

Likelihood:
$L(x_1 \dots x_n; b) = \frac{\mathbb{1}(\max_i(x_i \leq b))}{b^n}$

Loglikelihood:

**Maximum likelihood estimation**

Cookbook: take the log of the likelihood function. Take the partial derivative of the loglikelihood function with respect to the parameter. Set the partial derivative to zero and solve for the parameter.
If an indicator function on the pdf/pmf does not depend on the parameter, it can be ignored. If it depends on the parameter it can't be ignored because there is an discontinuity in the loglikelihood function. The maximum/minimum of the $X_i$ is then the maximum likelihood estimator.
Maximum likelihood estimator:

Let $\left\{E, (\mathbf{P}_\theta)_{\theta \in \Theta}\right\}$ be a statistical model associated with a sample of i.i.d. random variables $X_1, X_2, \ldots, X_n$. Assume that there exists $\theta^* \in \Theta$ such that $X_i \sim \mathbf{P}_{\theta^*}$.
The maximum likelihood estimator is the (unique) $\theta$ that minimizes $\widehat{\text{KL}}(\mathbf{P}_{\theta^*}, \mathbf{P}_\theta)$ over the parameter space. (The minimizer of the KL divergence is unique due to it being strictly convex in the space of distributions once is fixed.)

$\widehat{\theta}_n^{MLE} =$

$\text{argmin}_{\theta \in \Theta} \widehat{\text{KL}}_n(\mathbf{P}_{\theta^*}, \mathbf{P}_\theta) =$

$\text{argmax}_{\theta \in \Theta} \sum_{i=1}^n \ln p_\theta(X_i) =$

$\text{argmax}_{\theta \in \Theta} \ln\left(\prod_{i=1}^n p_\theta(X_i)\right)$

Gaussian Maximum-loglikelihood estimators:

MLE estimator for $\sigma^2 = \tau$:
$\hat{\tau}_n^{MLE} = \frac{1}{n}\sum_{i=1}^n X_i^2$

MLE estimators:

$\hat{\mu}_n^{MLE} = \frac{1}{n}\sum_{i=1}(x_i)$

### 17.1 Fisher Information

The Fisher information, captures the negative of the expected curvature of the loglikelihood function.
Let $(\mathbb{R}, \{\mathbf{P}_\theta\}_{\theta \in \mathbb{R}})$ denote a continuous statistical model. Let $f_\theta(x)$ denote the pdf (probability density function) of the continuous distribution $\mathbf{P}_\theta$. Assume that $f_\theta(x)$ is twice-differentiable as a function of the parameter $\theta$.

Formula for the calculation of Fisher Information of $X$:

$$\mathcal{I}(\theta) = \int_{-\infty}^{\infty} \frac{\left(\frac{\partial f_\theta(x)}{\partial \theta}\right)^2}{f_\theta(x)}\,dx$$

Models with one parameter (ie. Bernulli):

$\mathcal{I}(\theta) = \mathsf{Var}(\ell'(\theta))$

$\mathcal{I}(\theta) = -\mathbf{E}(\ell''(\theta))$

Models with multiple parameters (ie. Gaussians):

$\mathcal{I}(\theta) = -\mathbb{E}[\mathbf{H}\ell(\theta)]$
Cookbook:

Better to use 2nd derivative.

- Find loglikelihood
- Take second derivative (=Hessian if multivariate)
- Massage second derivative or Hessian to use with $-\mathbf{E}(\ell''(\theta))$ or $-\mathbb{E}[\mathbf{H}\ell(\theta)]$

**Asymptotic normality of the maximum likelihood estimator**

Under certain conditions (see slides) the MLE is asymptotically normal. This applies even if the MLE is not the sample average.
The asymptotic variance of the MLE is the inverse of the fisher information.

$\sqrt(n)(\widehat{\theta}_n^{\text{MLE}} - \theta^*) \xrightarrow[n\to\infty]{(d)} N_d(0, \mathcal{I}(\theta^*)^{-1})$

## 18 Method of Moments

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathbf{P}_{\theta^*}$ associated with model $(\mathbb{E}, \{\mathbf{P}_\theta\}_{\theta \in \Theta})$, with $\mathbb{E} \subseteq \mathbb{R}$ and $\Theta \subseteq \mathbb{R}$, for some d $\geq 1$
Population moments:

$m_k(\theta) = \mathbb{E}_\theta[X_1^k], 1 \leq k \leq d$

Empirical moments:

$\widehat{m_k}(\theta) = \overline{X_n^k} = \frac{1}{n}\sum_{i=1}^n X_i^k$
Convergence of empirical moments:

$\widehat{m_k} \xrightarrow[n\to\infty]{P,a.s.} m_k$

$(\widehat{m_1}, \ldots, \widehat{m_d}) \xrightarrow[n\to\infty]{P,a.s.} (m_1, \ldots, m_d)$
MOM Estimator $M$ is a map from the parameters of a model to the moments of its distribution. This map is invertible, (ie. it results into a system of equations that can be solved for the true parameter vector $\theta^*$). Find the moments (as many as parameters), set up system of equations, solve for parameters, use empirical moments to estimate.
$\psi : \Theta \to \mathbb{R}^d$

$\theta \mapsto (m_1(\theta), m_2(\theta), \ldots, m_d(\theta))$

$M^{-1}(m_1(\theta^*), m_2(\theta^*), \ldots, m_d(\theta^*))$

The MOM estimator uses the empirical moments:

$M^{-1}\left(\frac{1}{n}\sum_{i=1}^n X_i, \frac{1}{n}\sum_{i=1}^n X_i^2, \ldots, \frac{1}{n}\sum_{i=1}^n X_i^d\right)$

Assuming $M^{-1}$ is continuously differentiable at $M(0)$, the asymptotical variance of the MOM estimator is:

$\sqrt(n)(\widehat{\theta_n^{MM}} - \theta) \xrightarrow[n\to\infty]{(d)} N(0, \Gamma)$

where,

$\Gamma(\theta) = \left[\frac{\partial M^{-1}}{\partial \theta}(M(\theta))\right]^T \Sigma(\theta)\left[\frac{\partial M^{-1}}{\partial \theta}(M(\theta))\right]$

$\Gamma(\theta) = \nabla_\theta(M^{-1})^T \Sigma \nabla_\theta(M^{-1})$

$\Sigma_\theta$ is the covariance matrix of the random vector of the moments $(X_1^1, X_1^2 \ldots, X_1^d)$.

## 19 M-estimation

Generalization of maximum likelihood estimation. No statistical model needs to be assumed to perform M-estimation.

Median

## 20 Hubert loss

$h_\delta(x) = \begin{cases} \frac{x^2}{2} & \text{if } |x| < \delta \\ \delta(|x| - \delta/2) & \text{if } |x| > \delta \end{cases}$.

the derivative of Huber's loss is the clip function :

$\text{clip}_\delta(x) := \frac{d}{dx}h_\delta(x) =$
$\begin{cases} \delta & \text{if } x > \delta \\ x & \text{if } -\delta \leq x \leq \delta \end{cases}$