Tables for supersymmetry.

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Very sparse references, not always to original papers.
Help welcome at `https://github.com/blefloch/tables-for-supersymmetry`

§1 Lie algebras and groups (dimension $<\infty$)

§1.1 Lie algebras

Complex simple Lie algebras. Infinite series $\mathfrak{a}_{n\geq1}$, $\mathfrak{b}_{n\geq1}$, $\mathfrak{c}_{n\geq1}$, $\mathfrak{d}_{n\geq2}$ with $\mathfrak{a}_1=\mathfrak{b}_1=\mathfrak{c}_1$, $\mathfrak{b}_2=\mathfrak{c}_2$, $\mathfrak{d}_2=\mathfrak{a}_1\oplus\mathfrak{a}_1$, $\mathfrak{d}_3=\mathfrak{a}_3$. Five exceptions with dimensions $\left|\begin{matrix}\mathfrak{e}_6 & \mathfrak{e}_7 & \mathfrak{e}_8 & \mathfrak{f}_4 & \mathfrak{g}_2\\ 78 & 133 & 248 & 52 & 14\end{matrix}\right|$.

Type	Dimension	Lie algebra
$\mathfrak{a}_n$	$n(n+2)$	$\mathfrak{sl}(n+1,\mathbb{C})=\{\text{traceless}\}$
$\mathfrak{b}_n$	$n(2n+1)$	$\mathfrak{so}(2n+1,\mathbb{C})=\{\text{antisymmetric}\}$
$\mathfrak{c}_n$	$n(2n+1)$	$\mathfrak{sp}(2n,\mathbb{C})=\left\{\left(\begin{smallmatrix}0 & \mathbb{1}_n\\ -\mathbb{1}_n & 0\end{smallmatrix}\right)\times\text{symmetric}\right\}$
$\mathfrak{d}_n$	$n(2n-1)$	$\mathfrak{so}(2n,\mathbb{C})=\{\text{antisymmetric}\}$

Roots and Weyl group. The Weyl group has $\prod_i d_i$ elements where d_i are degrees of fundamental invariants. (Below, $\mathbb{1}_i$ denotes the i-th unit vector in $\mathbb{Z}^n$ and $1\leq i\neq j\leq n$.)

$\mathfrak{a}_{n-1}$: (note shifted rank) roots $\mathbb{1}_i-\mathbb{1}_j$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$. The Weyl group S_n permutes the $\mathbb{1}_i$. Fundamental invariants: $x_1^k+\cdots+x_n^k$ for $2\leq k\leq n$.

$\mathfrak{b}_n$: roots $\pm\mathbb{1}_i$ and $\pm\mathbb{1}_i\pm\mathbb{1}_j$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$ and $\mathbb{1}_n$. The Weyl group $\{\pm1\}^n\rtimes S_n$ permutes and changes signs of the $\mathbb{1}_i$. Fundamental invariants: $x_1^{2k}+\cdots+x_n^{2k}$ for $2\leq 2k\leq 2n$.

$\mathfrak{c}_n$: roots $\pm2\mathbb{1}_i$ and $\pm\mathbb{1}_i\pm\mathbb{1}_j$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$ and $2\mathbb{1}_n$. Same Weyl group and invariants as $\mathfrak{b}_n$.

$\mathfrak{d}_n$: roots $\pm\mathbb{1}_i\pm\mathbb{1}_j$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$ and $\mathbb{1}_{n-1}+\mathbb{1}_n$. The Weyl group $\{\pm1\}^{n-1}\rtimes S_n$ permutes the $\mathbb{1}_i$ and changes an even number of signs. Fundamental invariants $x_1\cdots x_n$ and $x_1^{2k}+\cdots+x_n^{2k}$ for $2\leq 2k\leq 2n-2$.

$\mathfrak{e}_8$: $\{\pm\mathbb{1}_i\pm\mathbb{1}_j\}\cup\{\frac12\sum_{k=1}^8\epsilon_k\mathbb{1}_k \mid \epsilon_k=\pm1,\prod_{k=1}^8\epsilon_k=-1\}$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$ and $\frac12(-\mathbb{1}_1-\cdots-\mathbb{1}_5+\mathbb{1}_6+\mathbb{1}_7+\mathbb{1}_8)$. The $2^{14}\,3^5\,5^2\,7=696729600$-element Weyl group is $O_8^+(\mathbb{F}_2)$. Degrees of invariants are $\{d_i\}=\{2,8,12,14,18,20,24,30\}$, with mnemonic $1+$(primes from 7 to 29).

$\mathfrak{e}_7$: roots $\sum_{i=1}^8 a_i\mathbb{1}_i$ of $\mathfrak{e}_8$ with $a_1=\sum_{i=2}^8 a_i$, simple roots are those of $\mathfrak{e}_8$ except $\mathbb{1}_1-\mathbb{1}_2$. The $2^{10}\times3^4\times5\times7=2903040$-element Weyl group is $\mathbb{Z}_2\times\mathrm{PSp}_6(\mathbb{F}_2)$. Degrees of invariants are $\{d_i\}=\{2,6,8,10,12,14,18\}$.

$\mathfrak{e}_6$: roots $\sum_{i=1}^8 a_i\mathbb{1}_i$ of $\mathfrak{e}_8$ with $a_1=a_2$ and $\sum_{i=3}^8 a_i=0$, simple roots are those of $\mathfrak{e}_8$ except $\mathbb{1}_1-\mathbb{1}_2$ and $\mathbb{1}_2-\mathbb{1}_3$. The $2^7\,3^4\,5=51840$-element Weyl group is $\mathrm{Aut}(\mathrm{PSp}_4(\mathbb{F}_3))$. Degrees of invariants are $\{d_i\}=\{2,5,6,8,9,12\}$.

$\mathfrak{f}_4$: roots $\pm\mathbb{1}_i$, $\pm\mathbb{1}_i\pm\mathbb{1}_j$, $\frac12(\pm\mathbb{1}_1\pm\mathbb{1}_2\pm\mathbb{1}_3\pm\mathbb{1}_4)$, simple roots $\mathbb{1}_1-\mathbb{1}_2$, $\mathbb{1}_2-\mathbb{1}_3$, $\mathbb{1}_3$, $-\frac12(\mathbb{1}_1+\mathbb{1}_2+\mathbb{1}_3+\mathbb{1}_4)$. It has an 1152-element Weyl group and $\{d_i\}=\{2,6,8,12\}$.

$\mathfrak{g}_2$: 12 roots $\mathrm{e}^{2\pi\mathrm{i}k/6},\mathrm{e}^{2\pi\mathrm{i}(2k+1)/12}\sqrt3\in\mathbb{C}$ for $0\leq k<6$, simple roots 1 and $\mathrm{e}^{5\pi\mathrm{i}/6}\sqrt3$. The 12-element Weyl group is the dihedral group D_6, and $\{d_i\}=\{2,6\}$.

The Coxeter number $h(\mathfrak{g})=(\dim\mathfrak{g}/\operatorname{rank}\mathfrak{g})-1$ is the largest d_i. A Coxeter element is the product of all simple reflections, in any order. Its eigenvalues $\mathrm{e}^{2\pi\mathrm{i}(d_i-1)/h}$ come in conjugate pairs.

A real simple Lie algebra is a complex algebra (see above) or a real form of it. Let $\mathfrak{sp}(m,n)=\mathfrak{usp}(2m,2n)=\mathfrak{u}(m,n,\mathbb{H})$, $\mathfrak{su}^*(2n)=\mathfrak{sl}(n,\mathbb{H})=\{\operatorname{Re}\operatorname{Tr}M=0\text{ in }\mathfrak{gl}(n,\mathbb{H})\}\simeq\mathfrak{gl}(n,\mathbb{H})/\mathbb{R}$, $\mathfrak{so}^*(2n)=\mathfrak{o}(n,\mathbb{H})$. A Lie algebra is called compact if it exponentiates to a compact Lie group. In $\mathfrak{e}_{r(s)}$, s is the number of (non-compact) $-$ (compact) generators. The maximal compact subalgebra of a complex algebra is its compact real form.

	Real form	Max compact subalgebra	Range
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(n)$	compact	
	$\mathfrak{sl}(n,\mathbb{R})$	$\supset \mathfrak{so}(n)$	
	$\mathfrak{su}(n-p,p)$	$\supset \mathfrak{su}(n-p)\oplus\mathfrak{su}(p)\oplus\mathfrak{u}(1)$	$0<p<n$
	$\mathfrak{su}^*(n)$	$\supset \mathfrak{usp}(n)$	n even
$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{so}(n)$	compact	
	$\mathfrak{so}(p,n-p)$	$\supset \mathfrak{so}(p)\oplus\mathfrak{so}(n-p)$	$0<p<n$
	$\mathfrak{so}^*(n)$	$\supset \mathfrak{u}(n/2)$	n even
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathfrak{usp}(2n)$	compact	
	$\mathfrak{sp}(2n,\mathbb{R})$	$\supset \mathfrak{u}(n)$	
	$\mathfrak{usp}(2n-2p,2p)$	$\supset \mathfrak{usp}(2n-2p)\oplus\mathfrak{usp}(2p)$	$0<p<n$

$\mathfrak{e}_{6(-78)}$	compact	$\mathfrak{e}_{8(-248)}$	compact
$\mathfrak{e}_{6(-26)}$	$\supset \mathfrak{f}_4$	$\mathfrak{e}_{8(-24)}$	$\supset \mathfrak{e}_7\oplus\mathfrak{su}(2)$
$\mathfrak{e}_{6(-14)}$	$\supset \mathfrak{so}(10)\oplus\mathfrak{so}(2)$	$\mathfrak{e}_{8(8)}$	$\supset \mathfrak{so}(16)$
$\mathfrak{e}_{6(2)}$	$\supset \mathfrak{su}(6)\oplus\mathfrak{su}(2)$	$\mathfrak{g}_{2(-14)}$	compact
$\mathfrak{e}_{6(6)}$	$\supset \mathfrak{usp}(8)$	$\mathfrak{g}_{2(2)}$	$\supset \mathfrak{su}(2)\oplus\mathfrak{su}(2)$
$\mathfrak{e}_{7(-133)}$	compact	$\mathfrak{f}_{4(-52)}$	compact
$\mathfrak{e}_{7(-25)}$	$\supset \mathfrak{e}_6\oplus\mathfrak{so}(2)$	$\mathfrak{f}_{4(-20)}$	$\supset \mathfrak{so}(9)$
$\mathfrak{e}_{7(-5)}$	$\supset \mathfrak{so}(12)\oplus\mathfrak{su}(2)$	$\mathfrak{f}_{4(4)}$	$\supset \mathfrak{usp}(6)\oplus\mathfrak{su}(2)$
$\mathfrak{e}_{7(7)}$	$\supset \mathfrak{su}(8)$		

Accidental isomorphisms.

$$\mathfrak{so}(2)=\mathfrak{u}(1),\quad \mathfrak{so}(1,1)=\mathbb{R} \qquad \mathfrak{so}(4,1)=\mathfrak{usp}(2,2)$$
$$\mathfrak{so}(3)=\mathfrak{su}(2)=\mathfrak{su}^*(2)=\mathfrak{usp}(2) \qquad \mathfrak{so}(3,2)=\mathfrak{sp}(4,\mathbb{R})$$
$$\mathfrak{so}(2,1)=\mathfrak{su}(1,1)=\mathfrak{sl}(2,\mathbb{R})=\mathfrak{sp}(2,\mathbb{R}) \qquad \mathfrak{so}(6)=\mathfrak{su}(4)$$
$$\mathfrak{so}(4)=\mathfrak{su}(2)\oplus\mathfrak{su}(2) \qquad \mathfrak{so}(5,1)=\mathfrak{su}^*(4)$$
$$\mathfrak{so}(3,1)=\mathfrak{sl}(2,\mathbb{C})=\mathfrak{sp}(2,\mathbb{C}) \qquad \mathfrak{so}(4,2)=\mathfrak{su}(2,2)$$
$$\mathfrak{so}(2,2)=\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{sl}(2,\mathbb{R}) \qquad \mathfrak{so}(3,3)=\mathfrak{sl}(4,\mathbb{R})$$
$$\mathfrak{so}^*(4)=\mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{su}(2) \qquad \mathfrak{so}^*(6)=\mathfrak{su}(3,1)$$
$$\mathfrak{so}(5)=\mathfrak{usp}(4) \qquad \mathfrak{so}^*(8)=\mathfrak{so}(6,2)$$

ADE classification of symmetric matrices with eigenvalues in $(-2,2)$ and $\mathbb{Z}_{\geq 0}$ entries (adjacency matrices of ADE diagrams), of simply laced simple Lie algebras, of binary polyhedral groups Γ (discrete subgroups of $\mathrm{SU}(2)$) and du Val singularities $\mathbb{C}^2/\Gamma \simeq$(zeros of Kleinian polynomial), of integers $1\leq p\leq q\leq r$ with $1/p+1/q+1/r>1$, of singularities with no moduli (Arnold) hence of $\mathcal{N}=2$ minimal models ($c<3$), of $\mathcal{N}=0$ unitary minimal models ($c<1$), of quivers of finite type,...

$\mathfrak{g}$	(p,q,r)	Kleinian polynomial
$\mathfrak{a}_k$	$(1,q,1+k-q)$	$w^2+x^2+y^{k+1}$
$\mathfrak{d}_k$	$(2,2,k-2)$	$w^2+x^2y+y^{k-1}$
$\mathfrak{e}_6$	$(2,3,3)$	$w^2+x^3+y^4$
$\mathfrak{e}_7$	$(2,3,4)$	$w^2+x^3+xy^3$
$\mathfrak{e}_8$	$(2,3,5)$	$w^2+x^3+y^5$

§1.2 Lie groups

Basics. The identity component G_0 is a normal subgroup: G/G_0 is the group of components. The maximal compact subgroup K is unique up to conjugation.

Every compact connected Lie group K is a quotient of $\mathrm{U}(1)^n\times\prod_{i=1}^m K_i$ by a finite subgroup Γ of its center, where K_i are simple, compact, simply-connected, connected. Then $\pi_1(K)/\mathbb{Z}^n\simeq\Gamma$ for some embedding $\mathbb{Z}^n\hookrightarrow\pi_1(K)$, and the center of K is $Z(K)=\big(\mathrm{U}(1)^n\times\prod_{i=1}^m Z(K_i)\big)/\Gamma$.

Center of all such K_i: $Z(\mathrm{SU}(n))=\mathbb{Z}_n$, $Z(\mathrm{USp}(2n))=\mathbb{Z}_2$, $Z(\mathrm{Spin}(n\geq 3))=(\mathbb{Z}_2$ for n odd, $\mathbb{Z}_4$ for $n/2$ odd, $\mathbb{Z}_2^2$ otherwise$)$, $Z(\widetilde{\mathrm{E}}_{6(-78)})=\mathbb{Z}_3$, $Z(\widetilde{\mathrm{E}}_{7(-133)})=\mathbb{Z}_2$, while $\mathrm{E}_{8(-248)}$, $\mathrm{F}_{4(-52)}$, $\mathrm{G}_{2(-14)}$ have no center.

Named quotients: $\mathrm{SO}(n)=\mathrm{Spin}(n)/\mathbb{Z}_2$ and $\mathrm{P}G=G/Z(G)$ for $G=\mathrm{SU},\mathrm{USp},\mathrm{SO}$ (also $\mathrm{U},\mathrm{GL},\mathrm{SL}$). The other two quotients $\mathrm{Spin}(4n)/\mathbb{Z}_2$ have no name.

Real connected simple Lie groups are the simply-connected $\widetilde{G}$ (classified by simple Lie algebras) and their quotients by a subgroup $\Gamma\subset Z(\widetilde{G})$ of the center; equivalently, covers of the center-free $G_{\mathrm{cf}}=\widetilde{G}/Z(\widetilde{G})$. One has $\pi_1(\widetilde{G}/\Gamma)=\Gamma$ and $Z(\widetilde{G}/\Gamma)=Z(\widetilde{G})/\Gamma$. The algebraic universal cover $\widetilde{G}_{\mathrm{alg}}$ (largest with a faithful finite-dimensional representation) may be a quotient of $\widetilde{G}$. We define $\pi_1^{\mathrm{alg}}(\widetilde{G}_{\mathrm{alg}}/\Gamma)=\Gamma$. For each real simple Lie algebra $\mathfrak{g}$, we tabulate: G_{cf} as a quotient of $\widetilde{G}_{\mathrm{alg}}$; the (topological) π_1; the real rank r_{Re}; and the maximal compact subgroup $K\subset G_{\mathrm{cf}}$. Below, $\iota(l)=(1$ for l odd, 2 otherwise$)$, $p+q=n$ with $p,q\geq 1$, and $2k=n$ when n is even. For $\mathfrak{sl}(2)$ use $\mathrm{SU}(2)=\mathrm{Sp}(2)$, $\mathrm{SL}(2,\mathbb{R})=\mathrm{Sp}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C})=\mathrm{Sp}(2,\mathbb{C})$.

	$\widetilde{G}_{\mathrm{alg}}/\pi_1^{\mathrm{alg}}(G_{\mathrm{cf}})$	K	π_1	r_{Re}
$\mathfrak{sl}(n\geq 3)$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathbb{Z}_n$	0
	$\mathrm{SL}(n,\mathbb{R})/\mathbb{Z}_{\iota(n)}$	$\mathrm{PSpin}(n)^{\ddagger\S}$	$Z(\mathrm{Spin}(n))^{\ddagger\S}$	$n-1$
	$\mathrm{SU}(p,q)/\mathbb{Z}_{p+q}$	$\frac{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}{\mathbb{Z}_{pq/\gcd(p,q)}}{}^{\P}$	$\mathbb{Z}$	$\min(p,q)$
	$\mathrm{SU}^*(2k)/\mathbb{Z}_2$	$\mathrm{USp}(2k)/\mathbb{Z}_2$	$\mathbb{Z}_2$	$k-1$
	$\mathrm{SL}(n,\mathbb{C})/\mathbb{Z}_n$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathbb{Z}_n$	$n-1$
$\mathfrak{so}(n\geq 3)$	$\mathrm{PSpin}(n)^{\ddagger}$	$\mathrm{PSpin}(n)$	$Z(\mathrm{Spin}(n))^{\ddagger}$	0
	$\mathrm{PSpin}(p,q)^{\ddagger}$	$\frac{\mathrm{SO}(p)\times\mathrm{SO}(q)}{\mathbb{Z}_2\text{ if }p,\ q\text{ even}}$	$\Gamma^{\Vert}$	$\min(p,q)$
	$\mathrm{SO}^*(2k)/\mathbb{Z}_2$	$\mathrm{U}(k)/\mathbb{Z}_2$	$\mathbb{Z}_{\iota(k)}\times\mathbb{Z}$	$\lfloor k/2\rfloor$
	$\mathrm{PSpin}(n,\mathbb{C})$	$\mathrm{PSpin}(n)$	$Z(\mathrm{Spin}(n))^{\ddagger}$	$\lfloor n/2\rfloor$
$\mathfrak{sp}(2n\geq 2)$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	$\mathrm{Sp}(2n,\mathbb{R})/\mathbb{Z}_2$	$\mathrm{U}(n)/\mathbb{Z}_2$	$\mathbb{Z}_{\iota(n)}\times\mathbb{Z}$	n
	$\mathrm{USp}(2p,2q)/\mathbb{Z}_2$	$\frac{\mathrm{USp}(2p)\times\mathrm{USp}(2q)}{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\min(p,q)$
	$\mathrm{Sp}(2n,\mathbb{C})/\mathbb{Z}_2$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathbb{Z}_2$	n

‡ For $r+s\geq 3$, $\mathrm{PSpin}(r,s)=\mathrm{Spin}(r,s)/Z(\mathrm{Spin}(r,s))$ and $Z(\mathrm{Spin}(r,s))=(\mathbb{Z}_2$ if r or s odd, $\mathbb{Z}_4$ if $\frac{r+s}{2}$ odd, else $\mathbb{Z}_2^2)$.

§ Exception: for $n=2$, $K=\mathrm{SO}(2)/\mathbb{Z}_2$ and $\pi_1=\mathbb{Z}$.

¶ $K\ni\overline{(A,B,\lambda)}\mapsto\begin{pmatrix}\lambda^{q/(p+q)}A & 0\\ 0 & \lambda^{-p/(p+q)}B\end{pmatrix}\in\mathrm{PSU}(p,q)$.

$^{\Vert}$ $\Gamma=\pi_1(\mathrm{SO}(p))\times\pi_1(\mathrm{SO}(q))$ for p or q odd (each factor is $\mathbb{Z}_2$ except $\pi_1(\mathrm{SO}(1))=0$ and $\pi_1(\mathrm{SO}(2))=\mathbb{Z}$); otherwise $\Gamma\subset\pi_1(\mathrm{SO}(p)/\mathbb{Z}_2)\times\pi_1(\mathrm{SO}(q)/\mathbb{Z}_2)$ consists of (γ_p,γ_q) such that both or neither γ is in the corresponding $\pi_1(\mathrm{SO})\subset\pi_1(\mathrm{SO}/\mathbb{Z}_2)$.

Discrete groups in this table should not be trusted.

$\widetilde{G}_{\text{alg}}/\pi_1^{\text{alg}}(G_{\text{cf}})$	K	π_1	r_{Re}
$\widetilde{\mathrm{E}}_{6(-78)}/\mathbb{Z}_3$	$=\mathrm{E}_{6(-78)}$	$\mathbb{Z}_3$	0
$\mathrm{E}_{6(-26)}$	$\mathrm{F}_{4(-52)}$	1	2
$\widetilde{\mathrm{E}}_{6(-14)}/\mathbb{Z}$	$\mathrm{Spin}(10)\times \mathrm{U}(1)/?$	$\mathbb{Z}$	2
$\widetilde{\mathrm{E}}_{6(2)}/\mathbb{Z}_6$	$(\mathrm{SU}(6)/\mathbb{Z}_6)\times \mathrm{SU}(2)$	$\mathbb{Z}_6$	4
$\widetilde{\mathrm{E}}_{6(6)}/\mathbb{Z}_2$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathbb{Z}_2$	6
$\widetilde{\mathrm{E}}_6^{\mathbb{C}}/\mathbb{Z}_3$	$\mathrm{E}_{6(-78)}$	$\mathbb{Z}_3$	6
$\widetilde{\mathrm{E}}_{7(-133)}/\mathbb{Z}_2$	$=\mathrm{E}_{7(-133)}$	$\mathbb{Z}_2$	0
$\widetilde{\mathrm{E}}_{7(-25)}/\mathbb{Z}$	$\mathrm{E}_{6(-78)}\times \mathrm{U}(1)/?$	$\mathbb{Z}$	3
$\widetilde{\mathrm{E}}_{7(-5)}/\mathbb{Z}_2^2$	$\mathrm{Spin}(12)\times \mathrm{SU}(2)/\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	4
$\widetilde{\mathrm{E}}_{7(7)}/\mathbb{Z}_4$	$\mathrm{SU}(8)/\mathbb{Z}_4$	$\mathbb{Z}_4$	7
$\widetilde{\mathrm{E}}_7^{\mathbb{C}}/\mathbb{Z}_2$	$\mathrm{E}_{7(-133)}$	$\mathbb{Z}_2$	7
$\mathrm{E}_{8(-248)}$	$\mathrm{E}_{8(-248)}$	1	0
$\widetilde{\mathrm{E}}_{8(-24)}/\mathbb{Z}_2$	$\widetilde{\mathrm{E}}_{7(-133)}\times \mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\widetilde{\mathrm{E}}_{8(8)}/\mathbb{Z}_2$	$\mathrm{SO}(16)/\mathbb{Z}_2$	$\mathbb{Z}_2$	8
$\mathrm{E}_8^{\mathbb{C}}$	$\mathrm{E}_{8(-248)}$	1	8
$\mathrm{F}_{4(-52)}$	$\mathrm{F}_{4(-52)}$	1	0
$\widetilde{\mathrm{F}}_{4(-20)}/\mathbb{Z}_2$	$\mathrm{Spin}(9)/\mathbb{Z}_2$	$\mathbb{Z}_2$	1
$\widetilde{\mathrm{F}}_{4(4)}$	$\mathrm{USp}(6)\times \mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\mathrm{F}_4^{\mathbb{C}}$	$\mathrm{F}_{4(-52)}$	1	4
$\mathrm{G}_{2(-14)}$	$\mathrm{G}_{2(-14)}$	1	0
$\widetilde{\mathrm{G}}_{2(2)}/\mathbb{Z}_2$	$\mathrm{SU}(2)\times \mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\mathrm{G}_2^{\mathbb{C}}$	$\mathrm{G}_{2(-14)}$	1	4

Spin and Pin groups. $\mathrm{SO}(n)$ has a double cover $\mathrm{Spin}(n)$. Since $\pi_0(\mathrm{O}(n))=\mathbb{Z}_2$ there are two double covers: $\mathrm{Pin}_+(n)$ in which a reflection R obeys $R^2=1$, and $\mathrm{Pin}_-(n)$ in which $R^2=(-1)^F$. For $p,q\geq 1$, $\pi_0(\mathrm{O}(p,q))=\pi_0(\mathrm{O}(p))\times\pi_0(\mathrm{O}(q))=\mathbb{Z}_2^2$; the identity component $\mathrm{SO}_+(p,q)$ has a double cover $\mathrm{Spin}(p,q)$. The eight double covers of $\mathrm{O}(p,q)$ differ in whether R^2, T^2 and $(RT)^2$ are $+1$ or $(-1)^F$.

Accidental isomorphisms (real reductive Lie groups) $\mathbb{R}/\mathbb{Z}=\mathrm{U}(1)$; $\mathrm{SU}(2)=\mathrm{Spin}(3)\twoheadrightarrow\mathrm{SO}(3)$; …

Homotopy. Any connected Lie group is homeomorphic to its maximal compact subgroup K times a Euclidean space $\mathbb{R}^p$. All $\pi_{j\geq 1}(K)$ are abelian and finitely generated, $\pi_2(K)=0$, $\pi_3(K)=\mathbb{Z}^m$ where m counts simple factors in a finite cover $\mathrm{U}(1)^n\times\prod_{i=1}^m K_i\twoheadrightarrow K$, and $\pi_j(K)=\prod_{i=1}^m\pi_j(K_i)$ for $j\geq 2$.

For any G there exists $\prod_{i=1}^{\operatorname{rank} G}S^{2d_i-1}\to G$ which induces isomorphisms of rational (i.e., torsion-free part of) homotopy/cohomology groups where d_i are the degrees of fundamental invariants. For compact simple K,

Group	$(2d_i-1)$
A_n	$3,5,\ldots,2n+1$
$\mathrm{B}_n,\mathrm{C}_n$	$3,7,\ldots,4n-1$
D_n	$3,7,\ldots,4n-5,2n-1$

E_6	$3,9,11,15,17,23$
E_7	$3,11,15,19,23,27,35$
E_8	$3,15,23,27,35,39,47,59$
F_4	$3,11,15,23$
G_2	$3,11$

$\pi_{j\geq 2}(G)$ has a factor $\mathbb{Z}$ for each S^j above, and some torsion. Explicitly, $\pi_j(\mathrm{SU}(n))$ is $\mathbb{Z}$ for odd $j<2n$, 0 for even $j<2n$, and is pure torsion for $j\geq 2n$. Similarly, $\pi_{j<4n+2}(\mathrm{USp}(2n))$ is $\mathbb{Z}$ for $j\equiv 3,7 \bmod 8$, $\mathbb{Z}_2$ for $j\equiv 4,5 \bmod 8$, and 0 otherwise.

§1.3 Simple Lie superalgebras

Classical Lie superalgebras: the bosonic algebra acts on the fermionic generators in a completely reducible representation. This excludes Cartan-type superalgebras $\mathfrak{w}(n)$, $\mathfrak{s}(n)$, $\tilde{\mathfrak{s}}(n)$ and $\mathfrak{h}(n)$. In this table, $m,n\geq 1$ and we do not list purely bosonic Lie algebras. The factor $\mathbb{C}$ of $\mathfrak{sl}(m|n)$ must be removed if $m=n$.

	Bosonic algebra	Fermionic repr.
$\mathfrak{sl}(m\vert n)$	$\mathfrak{sl}(m,\mathbb{C})\oplus\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{C}$	$(m,\overline{n})\oplus(\overline{m},n)$
$\mathfrak{osp}(m\vert 2n)$	$\mathfrak{so}(m,\mathbb{C})\oplus\mathfrak{sp}(2n,\mathbb{C})$	$(m,2n)$
$\mathfrak{d}(2,1,\alpha)$	$\mathfrak{sl}(2,\mathbb{C})^3$	$(2,2,2)$
$\mathfrak{f}(4)$	$\mathfrak{so}(7,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	$(8,2)$
$\mathfrak{g}(3)$	$\mathfrak{g}_2\oplus\mathfrak{sl}(2,\mathbb{C})$	$(7,2)$
$\mathfrak{p}(m)$	$\mathfrak{sl}(m+1,\mathbb{C})$	sym $\oplus$ (antisym)*
$\mathfrak{q}(m)$	$\mathfrak{sl}(m+1,\mathbb{C})$	adjoint

Real forms of Lie superalgebras, starting from their compact form ($p=q=0$). $\mathfrak{p}(m)$ has no compact form. Here, $m,n\geq 1$, $0\leq p\leq m/2$, $0\leq q\leq n/2$. The forms $\mathfrak{su}^*$, $\mathfrak{osp}^*$, $\mathfrak{q}^*$ only exist for even rank; $\mathfrak{sl}'$ only if $m=n$.

Real form	Bosonic algebra
$\mathfrak{su}(m-p,p\vert n-q,q)$	$\mathfrak{su}(m-p,p)\oplus\mathfrak{su}(n-q,q)\oplus\mathfrak{u}(1)^\ddagger$
$\mathfrak{sl}(m\vert n)$	$\mathfrak{sl}(m,\mathbb{R})\oplus\mathfrak{sl}(n,\mathbb{R})\oplus\mathfrak{so}(1,1)^\ddagger$
$\mathfrak{sl}'(n\vert n)$ $(m=n)$	$\mathfrak{sl}(n,\mathbb{C})$
$\mathfrak{su}^*(m\vert n)$ $(m,n$ even$)$	$\mathfrak{su}^*(m)\oplus\mathfrak{su}^*(n)\oplus\mathfrak{so}(1,1)^\ddagger$
$\mathfrak{osp}(m-p,p\vert 2n)$	$\mathfrak{so}(m-p,p)\oplus\mathfrak{sp}(2n,\mathbb{R})$
$\mathfrak{osp}^*(m\vert 2n-2q,2q)$ $(m$ even$)$	$\mathfrak{so}^*(m)\oplus\mathfrak{usp}(2n-2q,2q)^\P$
$\mathfrak{d}^p(2,1,\alpha)$ §	$\mathfrak{so}(4-p,p)\oplus\mathfrak{sl}(2,\mathbb{R})$ $(p=0,1,2)$
$\mathfrak{f}^p(4)$ for $p=0,3$	$\mathfrak{so}(7-p,p)\oplus\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{f}^p(4)$ for $p=1,2$	$\mathfrak{so}(7-p,p)\oplus\mathfrak{su}(2)$
$\mathfrak{g}_s(3)$ for $s=-14,2$	$\mathfrak{g}_{2(s)}\oplus\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{p}(m)$	$\mathfrak{sl}(m+1,\mathbb{R})$
$\mathfrak{uq}(m-p,p)$	$\mathfrak{su}(m+1-p,p)$
$\mathfrak{q}(m)$	$\mathfrak{sl}(m+1,\mathbb{R})$
$\mathfrak{q}^*(m)$ $(m$ odd$)$	$\mathfrak{su}^*(m+1)$

‡ For $m=n$, $\mathfrak{u}(1)$ and $\mathfrak{so}(1,1)$ factors are absent. Additionally, one can project down to a single bosonic factor.

¶ ⚠ A real form of $\mathfrak{osp}(2|2,\mathbb{C})=\mathfrak{sl}(2|1,\mathbb{C})$ is missing.

§ The three $\mathfrak{sl}(2)$ bosonic factors of $\mathfrak{d}(2,1,\alpha)$ appear with weights 1, α and $-1-\alpha$ in fermion anticommutators. For $\mathfrak{d}^0$ and $\mathfrak{d}^2$, α is real. For $\mathfrak{d}^1$, $\alpha=1+ia$ with a real.

Some isomorphisms: $\mathfrak{su}(1,1|1)=\mathfrak{sl}(2|1)=\mathfrak{osp}(2|2)$ and $\mathfrak{su}(2|1)=\mathfrak{osp}^*(2|2,0)$ and $\mathfrak{d}^p(2,1,\alpha=1)=\mathfrak{osp}(4-p,p|2)$ and $\mathfrak{osp}(6,2|4)=\mathfrak{osp}^*(8|4)$.

§1.4 Lie supergroups

§1.5 Representations

Complex dimensions of smalest irreps

E_6	$1,27,27,78,351,351,351,351,650,1728,1728,\ldots$
E_7	$1,56,133,912,1463,1539,6480,7371,8645,24320,\ldots$
E_8	$1,248,3875,27000,30380,147250,779247,1763125,\ldots$
F_4	$1,26,52,273,324,1053,1053,1274,2652,4096,\ldots$
G_2	$1,7,14,27,64,77,77,182,189,273,286,378,448,\ldots$

§2 Quantum field theory

§2.1 Generalities

Yang–Mills term. A gauge group is a compact reductive Lie group G such as $(\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1))/\mathbb{Z}_6$. The gauge kinetic term is $\mathcal{L}_{\mathrm{SYM}} = g^{-2}\operatorname{Tr} F\wedge\star F$, with one real gauge coupling g per simple factor.

Theta term in even dimension: $\theta\operatorname{Tr}F^{\wedge(d/2)}$ with θ periodic. In 4d, θ and g combine to $\tau=\theta/(2\pi)+4\pi i/g^2$.

Chern–Simons term in 3d: $k\operatorname{Tr}\bigl(A\wedge \mathrm{d}A+\frac{2}{3}A\wedge A\wedge A\bigr)$ with k quantized (normalization missing).

Boundaries and gauge redundancies. On a non-compact spacetime one can consider the group of gauge redundancies with various boundary conditions. Let $H\subset F$ be the constant transformations included as gauge redundancies (including constant gauge transformations by $H\cap G$). The Higgs branch flavour symmetry is then $\{x\in F\mid xG=Gx, xH=Hx\}/H$.

Loops. A Polyakov loop is just a Wilson loop in the fundamental representation and wrapping the time direction.

A redundant operator is $\mathcal{O}$ such that $\langle\mathcal{O}(p)\mathcal{O}(-p)\rangle$ is polynomial in the momentum p; equivalently, $\langle\mathcal{O}(x)\mathcal{O}(y)\rangle$ vanishes at separated points.

An abelian q-form symmetry [1412.5148] (for $q=0$ this is a standard flavor symmetry and can be nonabelian) is a $(q+1)$-form current J such that $\mathrm{d}{\star J}=0$; it couples as $\int B\wedge\star J$ to a background $(q+1)$-form gauge field B with gauge transformation $B\to B+\mathrm{d}\Lambda$.

The conserved charge $Q(\Sigma)=\int_\Sigma\star J$ is measured along a $(d-1-q)$-cycle Σ. Charged operators are supported on q-cycles linking Σ; charged excitations are dynamical q-branes intersecting Σ transversally. The symmetry is U(1) rather than $\mathbb{R}$ if all $Q(\Sigma)\in\mathbb{Z}$ so all $\int_{\Sigma'}\mathrm{d}\Lambda$ and $\int_{\Sigma''}\mathrm{d}B\in 2\pi\mathbb{Z}$.

A q-form symmetry may emerge in the IR. The partition function $Z[B]$ may suffer shifts under $B\to B+\mathrm{d}\Lambda$: such a 't Hooft anomaly prevents gauging (making B dynamical); it is an RG invariant hence matches in UV and IR.

Coleman–Mermin–Wagner (see [1802.07747] for p-forms) Spontaneous symmetry breaking of continuous p-form symmetry can only happen for $D>p+2$ ($p+1$ for discrete symmetries). The IR theory has massless p-form Nambu–Goldstone bosons.

§2.2 Anomalies

Continuous anomalies in $d=2n$, in $(n+1)$-point functions of currents: $(n+1)$-gon fermion loop, summed over fermions. Forbid simultaneous nontrivial backgrounds for all $n+1$ symmetries. Anomaly with $(n+1)$ gauge currents $\implies$ theory is sick. Anomaly with n gauge, one flavour $\implies$ classical flavour symmetry fails at one-loop, $D_\mu J^\mu\sim\operatorname{Tr}(\mathrm{d}A_1\wedge\cdots\wedge\mathrm{d}A_n)+O(A^{n+1})$.

Fermion effective action $\Gamma[A]$ defined by $\exp(-\Gamma[A])=\int \mathrm{D}\bar\chi\,\mathrm{D}\psi\exp(-\int \mathrm{d}^4x\,\bar\chi i\not{D}\gamma_-\psi)$ always has gauge-invariant and diffeomorphism-invariant real part but varies by the imaginary $D_\mu(\delta\Gamma/\delta A_\mu)=D_\mu\langle J^\mu\rangle$ and $D_\mu(\delta\Gamma/\delta g_{\mu\nu})=\frac{1}{2}D_\mu\langle T^{\mu\nu}\rangle$, nonzero in case of anomaly.

Anomaly polynomial: formal $(d+2)$-form built from field strengths F and Riemann R two-forms, traced.

Continuous gravitational anomaly. $\mathrm{Spin}(1,d-1)$ plus CPT only has complex representations for $d=4k+2$: for other even d, CPT exchanges chirality so we cannot have a single Weyl spinor. Focus on $d=1+(4k+1)$, Weyl fermions of spin $\frac{1}{2}$ and $\frac{3}{2}$ and self-dual $2k+1$ form. In 10d, unique theory with anomaly cancellation between fields of different spins: IIB supergravity. Above 10d, only same spins can cancel.

Discrete gravitational anomaly. In $d=8k$ and $d=8k+1$, single Majorana spinors cannot be given mass (but pairs can). It turns out that coupling an odd number of spin $\frac{1}{2}$ Majorana fermions to gravity is inconsistent.

Mixed gauge-gravity anomaly (or flavour-gravity) corresponds to $(\frac{1}{2}d+1)$-gons fermion loops with an even number of stress-tensors and some currents.

§2.3 Supersymmetric theories

Vector. Yang–Mills, theta, Chern–Simons terms have supersymmetric completion. Additionally, FI parameter, real for 4 supercharges, triplet for 8 supercharges. Dimensionless in 2d, the FI parameter combines with the theta angle.

Matter. For 16 supercharges, none. For 8 supercharges, symplectic representation $V\simeq\mathbb{H}^n$ namely $G\to F=\mathrm{USp}(2n)$. For 4 supercharges, unitary representation $V\simeq\mathbb{C}^n$ namely $G\to F=\mathrm{U}(n)$. Canonical kinetic term for bosons: $D_\mu\phi_i D^\mu\phi_i$.

Superpotential term. For 4 supercharges, $\int \mathrm{d}^2\theta\, W$ gives a potential for scalars and Yukawa-type interactions. W is holomorphic in chiral fields and in couplings seen as background fields. Example: the kinetic term $\operatorname{Im}\int \mathrm{d}^2\theta[\tau W_\alpha^2]$ of an abelian gauge field: W_α^2 is a chiral field so τ is the background value of a chiral field.

An accidental symmetry is a flavour symmetry of the IR but not of the UV.

R-symmetry. In 2d and higher the IR R-symmetry is part of the superconformal algebra, rather than an outer automorphism of it. The manifest (UV) R-symmetry can be a mixture of the IR R-symmetry and of a flavour symmetry: $R_{\mathrm{UV}}\subset R_{\mathrm{IR}}\times F$. For nonabelian R-symmetry that flavour symmetry must be accidental as it does not commute with R_{UV}. For abelian R-symmetry the mixing is continuous; assuming no accidental flavour symmetries it is fixed in 4d $\mathcal{N}=1$ by a-extremization, in 3d $\mathcal{N}=2$ by Z_{S^3}-extremization, in 2d $\mathcal{N}=(0,2)$ by c-extremization.

Vector multiplet scalars and gauge field are $U(1)_R$ neutral.
For chirals, $\Delta\geq\frac{1}{2}(d-1)|R|$ at the fixed point.

Classical vacua: Coulomb, Higgs and mixed branches. Coulomb branch ($\mathfrak{g}$ modulo conjugation by G) parametrized by vector multiplet scalars, larger in 3d due to monopoles, can be lifted by quantum effects. For 4 supercharges, Higgs branch parametrized by chiral multiplet scalars: Kähler quotient $R//G$. For 8 supercharges, Higgs branch parametrized by hypermultiplet scalars: hyper-Kähler quotient $\widetilde{R}///G$. The Higgs branch has flavour symmetry $\{x\in F\mid xG=Gx\}/G$ normalizer of G in $F=\mathrm{U}(R)$ or $F=\mathrm{USp}(\widetilde{R})$ modulo G. Background vector multiplet scalars (real/twisted masses) reduce the Higgs and mixed branches to fixed points of corresponding flavour symmetries.

§2.4 Spinors (e.g. [hep-th/9910030])

Clifford algebra. Let h_{ab} be diagonal with s '+1' and t '−1', and $d = s + t$. The Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2h_{ab}$ has real dimension 2^d and is isomorphic to a matrix algebra $M_{2^\#}(\bullet)$ with

$s - t$ mod 8	0	1	2	3	4	5	6	7
$\bullet$ is	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$

Charge conjugation. $(-\eta)\Gamma_a^T = \mathcal{C}\Gamma_a\mathcal{C}^{-1}$ are conjugate for $\eta = \pm 1$ because they obey the same algebra. Get $\mathcal{C}^T = -\varepsilon\mathcal{C}$ with $\varepsilon = \pm 1$ by transposing twice. Let $\Gamma^{(n)} = \Gamma_{a_1\dots a_n}$. Using $\left(\mathcal{C}\Gamma^{(n)}\right)^T = -\epsilon(-)^{n(n-1)/2}(-\eta)^n\mathcal{C}\Gamma^{(n)}$ find which n mod 4 give symmetric $\mathcal{C}\Gamma^{(n)}$. The sum of $\binom{d}{n}$ must be $2^{\lfloor d/2\rfloor}(2^{\lfloor d/2\rfloor}+1)/2$. This fixes ϵ, η. Odd d require $\eta = (-1)^{d(d+1)/2}$ to preserve $\Gamma^{(d)}$. Even d allow two choices of signs: consult the rows $d \pm 1$.

d mod 8	n	ϵ	η
0 — 1	0, 1	−1	−1
2 ⟨ 3	1, 2	+1	+1
4 ⟨ 5	2, 3	+1	−1
6 ⟨ 0 — 7	0, 3	−1	+1

Reduced spinors. $M_{ab} \in \mathfrak{so}(s,t)$ acts as $\gamma_a\gamma_b$ on representations of the Clifford algebra. But the $2^{\lceil d/2\rceil}$-dimensional representation is not irreducible as a representation of $\mathfrak{so}(s,t)$.

In even d, Weyl (or chiral) spinors $\Gamma^{(d)}\lambda = \pm\lambda$ have $2^{d/2-1}$ real components. Let B be defined by $\Gamma_a^* = -\eta(-1)^t B\Gamma_a B^{-1}$. Majorana spinors $\lambda^* = B\lambda$ exist for $s - t \equiv 0, \pm 1, \pm 2$ mod 8; the case $s-t \equiv \pm 2$ requires $\eta = \mp(-1)^{d/2}$. When $s-t \equiv 3,4,5$, a set of $2n$ spinors can be symplectic Majorana: $(\lambda^I)^* = B\Omega_{IJ}\lambda^J$ for $\Omega = ((0, \mathbb{1}_n); (-\mathbb{1}_n, 0))$. (Symplectic) Majorana–Weyl spinors exist for $s - t \equiv 0, 4$ mod 8. The table also includes the real dimension of the minimal spinor.

d		$t \equiv 0$		1		2		3 mod 4	
1	(D 2)	M	1	M	1				
2	(W 2)	M$^-$	2	MW	1	M$^+$	2		
3	(D 4)	s	4	M	2	M	2	s	4
4	(W 4)	sW	4	M$^+$	4	MW	2	M$^-$	4
5	(D 8)	s	8	s	8	M	4	M	4
6	(W 8)	M$^+$	8	sW	8	M$^-$	8	MW	4
7	(D 16)	M	8	s	16	s	16	M	8
8	(W16)	MW	8	M$^-$	16	sW	16	M$^+$	16
9	(D 32)	M	16	M	16	s	32	s	32
10	(W32)	M$^-$	32	MW	16	M$^+$	32	sW	32
11	(D 64)	s	64	M	32	M	32	s	64
12	(W64)	sW	64	M$^+$	64	MW	32	M$^-$	64

Flavour symmetries of N minimal spinors. This is also the R-symmetry of the N-extended superalgebra. For (symplectic) Majorana Weyl spinors, specify $N = (N_L, N_R)$ left/right-handed.

$$\text{M} \begin{cases} \mathfrak{u}(N) & \text{if } d \text{ even} \\ \mathfrak{so}(N) & \text{if } d \text{ odd} \end{cases}$$

MW: $\mathfrak{so}(N_L) \times \mathfrak{so}(N_R)$
s : $\mathfrak{usp}(2N)$
sW : $\mathfrak{usp}(2N_L) \times \mathfrak{usp}(2N_R)$

E.g., Lorentzian 6d $(2,0)$ has $\mathfrak{usp}(4) \times \mathfrak{usp}(0)$ R-symmetry.

Products of spinor representations. For odd $d = 2m+1$, let $\mathcal{S}$ be a spinor representation of complex dimension 2^m. The symmetric product $S^2\mathcal{S}$ consists of k-forms with $k \equiv m$ mod 4. Since k-forms and $(d-k)$-forms are the same representation, other descriptions can be given. For the antisymmetric product $\Lambda^2\mathcal{S}$, take $k \equiv m - 1$ mod 4. See the list of forms in the table.

d	1	3	5	7	9	11
$\dim_{\mathbb{C}}\mathcal{S}$	1	2	4	8	16	32
$S^2\mathcal{S}$	0	1	2	0, 3	0, 1, 4	1, 2, 5
$\Lambda^2\mathcal{S}$	.	0	0, 1	1, 2	2, 3	0, 3, 4

For even $d = 2m$, let $\mathcal{S}_\pm$ be the Weyl spinor representations of complex dimension 2^{m-1}. Then

$$S^2(\mathcal{S}_+ \oplus \mathcal{S}_-) = S^2\mathcal{S}_+ \oplus (\mathcal{S}_+ \otimes \mathcal{S}_-) \oplus S^2\mathcal{S}_-$$
$$\Lambda^2(\mathcal{S}_+ \oplus \mathcal{S}_-) = \Lambda^2\mathcal{S}_+ \oplus (\mathcal{S}_+ \otimes \mathcal{S}_-) \oplus \Lambda^2\mathcal{S}_-$$

The tensor product $\mathcal{S}_+ \otimes \mathcal{S}_-$ consists of $(m-1-2j)$-forms for $0 \le j \le (m-1)/2$. The symmetric products $S^2\mathcal{S}_\pm$ decompose into the (anti-)self-dual m-forms (denoted $m^\dagger$) and $(m-4j)$-forms for $0 < j \le m/4$. The antisymmetric products $\Lambda^2\mathcal{S}_\pm$ decompose into $(m-2-4j)$-forms for $0 \le j \le (m-2)/4$.

d	2	4	6	8	10	12
$\dim_{\mathbb{C}}\mathcal{S}_\pm$	1	2	4	8	16	32
$S^2\mathcal{S}_\pm$	$1^\dagger$	$2^\dagger$	$3^\dagger$	$0, 4^\dagger$	$1, 5^\dagger$	$2, 6^\dagger$
$\Lambda^2\mathcal{S}_\pm$	.	0	1	2	3	0, 4
$\mathcal{S}_+ \otimes \mathcal{S}_-$	0	1	0, 2	1, 3	0, 2, 4	1, 3, 5

§3 Supersymmetry

§3.1 Generalities

The Poincaré algebra is $\mathbb{R}^{s,t} \rtimes \mathfrak{so}(s,t)$, the semi-direct product of translations by rotations. Namely, $[P_a, P_b] = 0$, $[M_{ab}, P_c] = 2ih_{c[a}P_{b]}$, and $[M_{ab}, M^{cd}] = 4ih^{[c}_{[a}M^{d]}_{b]}$.

Super-Poincaré algebra. Add supercharges in some spinor representation Q of the Poincaré algebra (so $[P_a, Q] = 0$). Their anticommutator transforms in the representation S^2Q and should include the one-form P. Depending on s, t they can include other k-forms Z, called central charges because $[P, Z] = [Z, Z] = 0$. The super-Poincaré algebra is $((\mathbb{R}^{s,t} \times Z).Q) \rtimes (\mathfrak{so}(s,t) \times R)$, where the R-symmetry acts on Q. This Lie superalgebra is graded: $\mathrm{gr}(\mathbb{R}^{s,t} \times Z) = -2$, $\mathrm{gr}(Q) = -1$, and $\mathrm{gr}(\mathfrak{so}(s,t) \times R) = 0$. The supertranslations consist of $(\mathbb{R}^{s,t} \times Z).Q$.

Example: M-theory algebra. $d = 10+1$ super-Poincaré algebra with $Q =$ Majorana. Since S^2Q has 1, 2, and 5-forms, there are 2-form and 5-form central charges $Z_{(2)}$ and $Z_{(5)}$ (under which M2 and M5 branes are charged):

$$\{Q_\alpha, Q_\beta\} = (\gamma^M C)_{\alpha\beta}P_M + \frac{1}{2}(\gamma_{MN}C)_{\alpha\beta}Z^{MN}_{(2)} + \frac{1}{5!}(\gamma_{MNPQR}C)_{\alpha\beta}Z^{MNPQR}_{(5)}$$

Altogether the M-theory algebra is $\mathfrak{osp}(1|32)$.

Lorentzian superconformal algebras are the same as super AdS_{d+1}. The bosonic part is $\mathfrak{so}(d,2)$ and R-symmetries. As a supermatrix: $\begin{pmatrix} \mathfrak{so}(d,2) & Q+S \\ Q-S & R \end{pmatrix}$ or $\begin{pmatrix} R & Q+S \\ Q-S & \mathfrak{so}(d,2) \end{pmatrix}$. Note that $\{Q, S\}$ contains R. For $d = 2$, the finite conformal algebra is $\mathfrak{so}(2,2) = \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$, sum of two $d = 1$ algebras, so the superalgebra is sum of two $d = 1$ superalgebras.

d	Superalgebra	R-symm (compact)	#Q+#S
1	$\mathfrak{osp}(N\|2)$	$\mathfrak{o}(N)$	$2N$
	$\mathfrak{su}(N\|1,1)$	$\mathfrak{su}(N)\oplus\mathfrak{u}(1)$ for $N\neq 2$	$4N$
	$\mathfrak{su}(2\|1,1)$	$\mathfrak{su}(2)$	8
	$\mathfrak{osp}(4^*\|2N)$	$\mathfrak{su}(2)\oplus\mathfrak{usp}(2N)$	$8N$
	$\mathfrak{g}_{-14}(3)$	$\mathfrak{g}_{2(-14)}$	14
	$\mathfrak{f}^0(4)$	$\mathfrak{so}(7)$	16
	$\mathfrak{d}^0(2,1,\alpha)$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	8
3	$\mathfrak{osp}(N\|4)$	$\mathfrak{so}(N)$	$4N$
4	$\mathfrak{su}(2,2\|N)$	$\mathfrak{su}(N)\oplus\mathfrak{u}(1)$ for $N\neq 4$	$8N$
	$\mathfrak{su}(2,2\|4)$	$\mathfrak{su}(4)$	32
5	$\mathfrak{f}^2(4)$	$\mathfrak{su}(2)$	16
6	$\mathfrak{osp}(8^*\|N)$	$\mathfrak{usp}(N)$ (N even)	$8N$

Dimensional reduction of Lorentzian supersymmetry algebras. The 1d column gives the number of real supercharges.

10d	6d	5d	4d	3d	2d	1d
$\mathcal{N}=(1,0)$	$(1,1)$	2	4	8	$(8,8)$	16
	$(1,0)$	1	2	4	$(4,4)$	8
			1	2	$(2,2)$	4

Supersymmetry on symmetric curved spaces 4d $\mathcal{N}=2$ supersymmetry on S^4 is $\mathfrak{osp}(2|4)$. 2d $\mathcal{N}=(2,2)$ supersymmetry on S^2 is $\mathfrak{osp}(2|2)$.

§3.2 Explicit supersymmetry algebras

4d $\mathcal{N}=2$. $\{Q^A_\alpha,\overline{Q}^B_{\dot\alpha}\}=\epsilon^{AB}P_{\alpha\dot\alpha}$; $0=\{Q^A_\alpha,Q^B_\beta\}=\{\overline{Q}^A_{\dot\alpha},\overline{Q}^B_{\dot\beta}\}$.

3d $\mathcal{N}=2$. $\{Q_\alpha,\overline{Q}_\beta\}=2\sigma^\mu_{\alpha\beta}P_\mu+2i\epsilon_{\alpha\beta}Z$ with $Z=P_3$ a central charge; $0=\{Q_\alpha,Q_\beta\}=\{\overline{Q}_\alpha,\overline{Q}_\beta\}$.

§3.3 Spin ≤ 1 supermultiplets

For 16 **supercharges**, there is only the vector multiplet.

For 8 **supercharges**, vector multiplet and (half-)hypermultiplet; in 3d and lower also twisted vector multiplet and twisted hypermultiplet.

For 4 **supercharges**, vector ($V=V^\dagger$) and chiral ($\overline{D}_{\dot\alpha}X=0$) multiplets; in 3d $\mathcal{N}=2$ also linear multiplets ($\epsilon^{\alpha\beta}D_\alpha D_\beta\Sigma=0=\epsilon^{\alpha\beta}\overline{D}_\alpha\overline{D}_\beta\Sigma$); in 2d $\mathcal{N}=(2,2)$ also twisted vector, twisted chirals, semichirals, …

For 2 **supercharges**, vector, chiral, linear, Fermi, …

§3.4 Other supermultiplets

6d $\mathcal{N}=(2,0)$ tensor multiplet with self-dual two-form gauge field B (namely $\mathrm{d}B=\star\mathrm{d}B$), four spinors, five scalars.

6d $\mathcal{N}=(1,0)$ tensor multiplet (contains one scalar), reduces to 4d $\mathcal{N}=2$ vector.

6d $\mathcal{N}=(1,0)$ supergravity multiplet, reduces to 4d $\mathcal{N}=2$ supergravity multiplet and two vectors.

4d $\mathcal{N}=1$ supercurrent multiplet contains stress tensor and/or R-symmetry current; is a source for supergravity. Ferrara–Zumino supercurrent $\overline{D}^{\dot\alpha}J_{\alpha\dot\alpha}=D_\alpha X$ with $\overline{D}_{\dot\alpha}X=0$ contains stress tensor; sources old minimal supergravity. R-symmetry multiplet $\overline{D}^{\dot\alpha}R_{\alpha\dot\alpha}=\chi_\alpha$, $\overline{D}_{\dot\alpha}\chi_\alpha=0$, $D^\alpha\chi_\alpha=\overline{D}_{\dot\alpha}\overline{\chi}^{\dot\alpha}$ contains (conserved) R-symmetry current; sources new minimal supergravity. Komargodski–Seiberg multiplet [1002.2228] $\overline{D}^{\dot\alpha}S_{\alpha\dot\alpha}=\chi_\alpha+D_\alpha X$, with χ_α and X as above, contains both stress tensor and R-symmetry current and sources 16/16 supergravity.

§4 Supersymmetric (gauge) theories

§4.1 Maximal super Yang–Mills

Data: gauge group.

Lorentzian 10d $\mathcal{N}=1$ SYM is anomalous unless the gauge group is abelian. Its dimensional reductions are anomaly-free and have one gauge field, $10-d$ scalars and $\mathcal{N}$ (symplectic or Majorana, and Weyl or not) spinors. The Lagrangian's R-symmetry Spin$(10-d)$ is contained in the automorphism group of the superalgebra (they coincide for $d\geq 5$).

dim.	$\mathcal{N}$ spinors	autom. $\supset$ R-sym.
10d	$(1,0)$ MW	
9d	1 M	
8d	1 M	$\mathrm{U}(1)=\mathrm{Spin}(2)$
7d	1 s	$\mathrm{USp}(2)=\mathrm{Spin}(3)$
6d	$(1,1)$ sW	$\mathrm{USp}(2)^2=\mathrm{Spin}(4)$
5d	2 s	$\mathrm{USp}(4)=\mathrm{Spin}(5)$
4d	4 M	$\mathrm{U}(4)\supset\mathrm{Spin}(6)$
3d	8 M	$\mathrm{Spin}(8)\supset\mathrm{Spin}(7)$
2d	$(8,8)$ MW	$\mathrm{Spin}(8)^2\supset\mathrm{Spin}(8)$
1d	16 M	$\mathrm{Spin}(16)\supset\mathrm{Spin}(9)$

4d $\mathcal{N}=4$ has exactly marginal parameter $\tau=\theta/(2\pi)+4\pi i/g^2$. Lagrangian theories are characterized by G but non-Lagrangian theories are not ruled out.

3d $\mathcal{N}=8$ [0806.1218] Bagger-Lambert, ABJM

§4.2 Theories with 9 to 12 supercharges

4d $\mathcal{N}=3$ theories exist, always non-Lagrangian.

3d $\mathcal{N}=5,6$ [0806.1218, 0807.4924] ABJM, ABJ

§4.3 Theories with 8 supercharges

6d $\mathcal{N}=(1,0)$ UV-complete Lagrangian gauge theories classified in [1502.05405, 1502.06594].

5d $\mathcal{N}=1$ SCFTs built from 5-brane diagrams or UV fixed point of gauge theory. [hep-th/9608111]

SU$(2N)$ SYM with $N_f\leq 7$ fundamental hypermultiplets has $\mathrm{SO}(2N_f)\times\mathrm{U}(1)_T\subset\mathrm{E}_{N_f+1}$ flavour symmetry enhancement. For SU(2) and $N_f=0$, non-trivial "θ" in $\pi_4(\mathrm{SU}(2))=\mathbb{Z}_2$ gives the $\widetilde{E}_1$ theory with $\mathrm{U}(1)_T$ symmetry only.

4d $\mathcal{N}=2$ gauge theories classified in [1309.5160]: $\mathrm{SU}(2)^n$ gauge group with trifundamental hypermultiplets; quiver in the shape of a (possibly single-node) Dynkin or affine Dynkin diagram; finitely many exceptions.

4d $\mathcal{N}=2$ generalities

There can be no continuous flavour symmetry enhancement.

The theory on $\mathbb{R}^4_{\epsilon_1,0}$ (Nekrasov–Shatashvili limit) $\leftrightarrow$ quantum integrable system with Planck constant ϵ_1.

Coulomb moduli $\leftrightarrow$ action variables.

Supersymmetric vacua $\leftrightarrow$ eigenstates.

Lift to $\mathbb{R}^4\times S^1$ gives K-theoretic Nekrasov partition function. The 5d theory $\leftrightarrow$ relativistic version of the integrable system.

4d $\mathcal{N}=2$ (G,G') Argyres–Douglas theories (with G and G' among A_k, D_k, $\mathrm{E}_{6,7,8}$) are engineered as IIB strings on three-fold singularity $f_G(x_1,x_2)+f_{G'}(x_3,x_4)=0$ where $f_{\mathrm{A}_k}(x,y)=x^2+y^{k+1}$ etc. (see page 2).

3d $\mathcal{N}=4$ has $\mathrm{SU}(2)_C\times\mathrm{SU}(2)_H$ R-symmetry acting on the Coulomb and Higgs branch. Both branches are hyper-Kähler and the SU(2) rotates their $\mathbb{CP}^1$ worth of complex structures. Denote $\mathrm{T}\subset\mathrm{G}$ the Cartan torus. The Coulomb branch is a holomorphic Lagrangian fibration $\mathcal{M}_C\to\mathfrak{t}_{\mathbb{C}}/\mathrm{Weyl}$ with generic fiber $\mathrm{T}^\vee_{\mathbb{C}}\simeq(\mathbb{C}^*)^{\operatorname{rank}\mathrm{G}}$. Its classical $\mathrm{Hom}(\pi_1(\mathrm{G}),\mathrm{U}(1))$ topological flavour symmetry can be enhanced quantum mechanically.

Example: $\boxed{\mathrm{SU}(N)}$—$(\mathrm{U}(N-1))$—$\cdots$—$(\mathrm{U}(2))$—$(\mathrm{U}(1))$ (edges denote bifundamental hypermultiplets) is the $T[\mathrm{SU}(N)]$ theory. Its manifest $\mathfrak{su}(N)\times\mathfrak{u}(1)^{N-1}$ flavour symmetry enhances to $\mathfrak{su}(N)^2$. The $T[G]$ theory has flavour symmetry $G\times{}^LG$ acting on Higgs and Coulomb branch respectively. Mirror symmetry exchanges $G\leftrightarrow{}^LG$. Gauging G and ${}^LG$ with two 4d vector multiplets realizes the S-duality domain wall of 4d $\mathcal{N}=4$ SYM [0804.2902].

2d $\mathcal{N}=(4,4)$ gauge theories. Typically get in the IR a direct sum of 2d $\mathcal{N}=(4,4)$ SCFTs (from the Coulomb and Higgs branches) whose central charges are different. Their $\mathrm{SU}(2)\times\mathrm{SU}(2)$ left/right-moving R-symmetries are different.

§4.4 Theories with 4 supercharges

Superspace $\mathbb{R}^{4|4}\ni(x^m|\theta^\alpha,\overline{\theta}^{\dot\alpha})$ in $(2,2)\oplus(2,1)\oplus(1,2)$ of $\mathfrak{so}(1,3)$ or $\mathfrak{so}(4)=\mathfrak{su}(2)^2$. Supercharges $Q_\alpha=\partial_{\theta^\alpha}-i\sigma^m_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial_{x^m}$ and $\overline{Q}_{\dot\alpha}=-\partial_{\overline{\theta}^{\dot\alpha}}+i\sigma^m_{\alpha\dot\alpha}\theta^\alpha\partial_{x^m}$ obey $\{Q_\alpha,\overline{Q}_{\dot\alpha}\}=2i\sigma^m_{\alpha\dot\alpha}\partial_{x^m}$. They commute with superderivatives D_α and $\overline{D}_{\dot\alpha}$ obtained by $\sigma\leftrightarrow-\sigma$. Note $\overline{Q}_{\dot\alpha}=\mathrm{e}^{-A}\overline{D}_{\dot\alpha}\mathrm{e}^A$ for $A=2i\theta\sigma^m\overline{\theta}\partial_{x^m}$; likewise D_α is conjugate to Q_α.

4d $\mathcal{N}=1$ pure SYM classically has $\mathrm{U}(1)_R$ symmetry, broken by instantons to $\mathbb{Z}_{2h}$ with $h=C_2(\mathrm{adj})$. It confines, is mass-gapped, and has $C_2(A)$ vacua associated to breaking $\mathbb{Z}_{2h}$ to $\mathbb{Z}_2$ by gaugino condensation $\langle\lambda\lambda\rangle$. Witten index $\mathrm{Tr}(-1)^F=h$.

$W_\alpha=-\frac14\overline{D}\overline{D}\mathrm{e}^{-V}D_\alpha\mathrm{e}^V$ and $\overline{W}_\alpha=-\frac14 DD\mathrm{e}^{-V}\overline{D}_\alpha\mathrm{e}^V$ field strength.

Wess-Zumino model: chiral multiplet ϕ with superpotential $W=m\phi^2+g\phi^3$.

3d $\mathcal{N}=2$ [hep-th/9703110]

In Abelian theories or (only approximately) deep in the Coulomb branch, the dual photon γ (a periodic real scalar) is defined by $\mathrm{d}\gamma=J_T=\star F$ where J_T is the topological current. Chiral superfield $\Phi=\phi+i\gamma$ where ϕ is the vector multiplet scalar.

Field strength $\Sigma=\epsilon^{\alpha\beta}\overline{D}_\alpha D_\beta V$.

$Z_{S^3}=\int_{\mathfrak{t}}\mathrm{d}u\,\frac{\prod_{\alpha\text{ root}}(2\sinh(\alpha u/2))^2}{\prod_{w\in\mathcal{R}}\cosh(wu/2)}\mathrm{e}^{ik\operatorname{Tr}u^2/(4\pi)}$

2d $\mathcal{N}=(2,2)$. Classical $\mathrm{U}(1)\times\mathrm{U}(1)$ R-symmetry. The axial U(1) R-symmetry has an anomaly with U(1) gauge symmetry proportional to the total charge under that gauge symmetry.

The gauge field strength is a twisted chiral multiplet Σ.

Integrating out massive chirals gives a twisted superpotential $-\operatorname{Tr}_R\big(\Sigma\log(\Sigma/\mu)-\Sigma\big)$ where Σ combines gauge field strength and twisted masses. FI parameters (twisted superpotentials linear in Σ) thus run as $\log(\mu)$ times the sum of charges.

Twisted chiral ring relations: $\partial W/\partial\Sigma_j\in2\pi\mathrm{i}\mathbb{Z}$.

1d $\mathcal{N}=4$

Data: gauge group G, representation V of G for chiral multiplets. Gauge couplings, FI parameters, superpotential W. Flavour Wilson line, twisted and real masses $v,m_1+im_2,m_3\in\mathfrak{g}_F$ that commute.

R-symmetry: SU(2), times U(1) if W has charge 2.

§4.5 Theories with 2 supercharges

3d $\mathcal{N}=1$

2d $\mathcal{N}=(0,2)$

2d $\mathcal{N}=(1,1)$

1d $\mathcal{N}=2$

Discrete data: gauge group G, chiral multiplets in a representation V of G, Wilson line in a unitary representation $M=M_0\oplus M_1$ of $\mathfrak{g}$, flavour symmetry group $G_F\subseteq\mathrm{U}(V)\times\mathrm{U}(M_0)\times\mathrm{U}(M_1)$ commuting with G. Gauge anomaly cancellation: $M\otimes\det^{1/2}V$ must be a representation of G.

Continuous data: gauge couplings, FI parameters, flavour Wilson line and real mass $v,\sigma\in\mathfrak{g}_F$ that commute, $\mathfrak{g}$-equivariant holomorphic odd map $Q\colon V\to\operatorname{End}M$ with $Q^2=0$ describing how supercharges act on M.

Special case: Fermi multiplets in representation V_{f} of G with G-equivariant holomorphic maps $E\colon V\to V_{\mathrm{f}}$ and $J\colon V\to V_{\mathrm{f}}^\vee$ obeying $J\cdot E=0$ are equivalent to Wilson line in $M=\wedge V_{\mathrm{f}}\otimes\det^{-1/2}V_{\mathrm{f}}$ with $Q=E\wedge+J\lrcorner$.

R-symmetry: U(1) if $Q\colon V\to\operatorname{End}M$ has charge 1. Mixing with flavour symmetries not fixed by superconformal algebra.

NLSM Chiral multiplet: scalar ϕ in a Kähler target X and fermion in holomorphic bundle ϕ^*T_X. Wilson line depends on a complex of vector bundles $\mathcal{F}$. Fermi multiplet takes values in a holomorphic vector bundle $\mathcal{E}$ with hermitian metric, equivalent to Wilson line with $\mathcal{F}=\det^{-1/2}\mathcal{E}\otimes\wedge\mathcal{E}$. Anomaly cancellation: $\sqrt{K_X}\otimes\wedge T_X\otimes\det^{-1/2}\mathcal{E}\otimes\wedge\mathcal{E}\otimes\mathcal{F}$ is a well-defined vector bundle on X.

§5 Other theories

Chern–Simons $(2m-1)$-form $m\operatorname{Tr}\big(A\int_0^1\mathrm{d}t(t\mathrm{d}A+t^2A^2)^{m-1}\big)$.

§5.1 2d conformal field theories

Virasoro algebra Vir_c, $c\in\mathbb{R}$: generators L_m, $m\in\mathbb{Z}$ obey $[L_m,L_n]=(m-n)L_{m+n}+\frac{c}{12}(m^3-m)\delta_{m+n,0}$ and $L_n^\dagger=L_{-n}$.

$\mathcal{N}=1$ super-Virasoro algebra additionally $[L_m,G_r]=(m/2-r)G_{m+r}$ and $\{G_r,G_s\}=2L_{r+s}+\frac{c}{3}(r^2-1/4)\delta_{r+s,0}$ where either $r\in\mathbb{Z}$ (Ramond algebra) or $r\in\mathbb{Z}+1/2$ (Neveu–Schwarz algebra). Adjoint $G_r^\dagger=G_{-r}$.

$\mathcal{N}=2$ super-Virasoro algebra $[L_m,J_n]=-nJ_{m+n}$, $[J_m,J_n]=\frac{c}{3}m\delta_{m+n,0}$, $[L_m,G_r^\pm]=(m/2-r)G_{m+r}^\pm$, $[J_m,G_r^\pm]=\pm G_{m+r}^\pm$, $\{G_r^+,G_s^+\}=\{G_r^-,G_s^-\}=0$, $\{G_r^+,G_s^-\}=L_{r+s}+\frac12(r-s)J_{r+s}+\frac{c}{6}(r^2-1/4)\delta_{r+s,0}$. Adjoint $L_m^\dagger=L_{-m}$, $J_m^\dagger=J_{-m}$, $(G_r^\pm)^\dagger=G_{-r}^\mp$, $c^\dagger=c$. The algebras with $r\in\mathbb{Z}$ (Ramond) or $r\in\mathbb{Z}+1/2$ (Neveu–Schwarz) are isomorphic under spectral shift $\alpha_{\pm1/2}$ where $\alpha_\eta(L_n)=L_n+\eta J_n+\frac{c}{6}\eta^2\delta_{n,0}$, $\alpha_\eta(J_n)=J_n+\frac{c}{3}\eta\delta_n$, $\alpha_\eta(G_r^\pm)=G_{r\pm\eta}^\pm$. Another automorphism is $G_r^+\leftrightarrow G_r^-$, $J_m\mapsto-J_m-\frac{c}{3}\delta_{m,0}$. We get a $\mathbb{Z}\rtimes\mathbb{Z}_2$ automorphism group.

$SW(3/2,2)$ **super-Virasoro algebra** has L, G, W, U

bc **system**

$\beta\gamma$ **system**

Liouville CFT has $c=1+6(b+1/b)^2$ and primary operators with $h(\alpha)=\alpha(b+1/b-\alpha)$ for "momentum" $\alpha\in\frac12(b+1/b)+i\mathbb{R}$.

Minimal model $\mathcal{M}_{p,q}$ for $p > q$ coprime is a quotient of $b = i\sqrt{p/q}$ Liouville CFT. It has $c = 1 - \frac{6(p-q)^2}{pq}$ and primary operators with $h_{r,s} = \frac{(ps-qr)^2-(p-q)^2}{4pq}$ for $0 < r < p$ and $0 < s < q$; no degeneracy besides $h_{r,s} = h_{p-r,q-s}$. Example: Ising model $\mathcal{M}_{4,3}$, tricritical Ising model $\mathcal{M}_{5,4}$, Yang–Lee singularity $\mathcal{M}_{5,2}$.

Unitary minimal model $\mathcal{M}_{k+2,k+1}$ is coset $\frac{\hat{\mathfrak{su}}(2)_{k-1}\times\hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_k}$

$\mathcal{N} = 2$ **minimal models** have an ADE classification; the A_k has $c/3 = 1-2/(k+2)$ and is the IR limit of a Landau–Ginzburg model with $W = \Phi^{k+2}$ superpotential.

§5.2 3d gauge theories

Chern–Simons term $S_{\mathrm{CS}} = \frac{1}{2\pi}\int_M \mathrm{d}^3x \operatorname{Tr}\bigl(A\mathrm{d}A + 2A^3/3\bigr)$ for a trivial G-bundle; otherwise realize A as boundary of an A_{4d} on some 4d manifold X with $M = \partial X$ and use $S_{\mathrm{CS}} = \frac{1}{8\pi}\int_X \mathrm{d}^4x \operatorname{Tr}(F\wedge F)$. This gives a TQFT (topological quantum field theory).

Magnetic global symmetry $G_{\mathrm{mag}} = \operatorname{Hom}(\pi_1(G), \mathrm{U}(1))$. Gauging discrete $\Gamma \subset G_{\mathrm{mag}}$ enlarges gauge group G to a covering with $\tilde{G}/\Gamma = G$. Explicit "topological" current $J = \star\mathrm{d}A/(2\pi)$ for $G = \mathrm{U}(1)$.

Contact term in $\langle JJ\rangle$ where J is a conserved current in 3d: $\langle J_\mu(x)J_\nu(y)\rangle = \cdots + \frac{w}{2\pi}\epsilon_{\mu\nu\rho}\partial_{x^\rho}\delta^3(x-y)$ for some w.

Action of SL$(2,\mathbb{Z})$ [hep-th/0307041] on 3d CFTs with U(1) flavour symmetry (current J). $S = \left(\begin{smallmatrix}0&1\\-1&0\end{smallmatrix}\right)$ gauges the U(1) but gives no kinetic energy to A; $T^k = \left(\begin{smallmatrix}1&k\\0&1\end{smallmatrix}\right)$ shifts background Chern–Simons level by k, equivalently adds contact term to $\langle JJ\rangle$, where $k \in 2\mathbb{Z}$ if the manifold has no spin structure. Relations: $S^2\colon J \to -J$ simply, while $(ST)^3$ multiplies path integral by theory-indepedent topological invariant of M.

§5.3 Supergravity and strings

String actions Polyakov action $L_{\mathrm{P}} = \lambda^{mn}[(\partial_m X)(\partial_n X) - g_{mn}] + \frac{1}{\alpha'}\sqrt{-g}$. Using equations of motion get Nambu–Goto action $L_{\mathrm{NG}} = \frac{1}{\alpha'}\sqrt{-\det[(\partial_m X)(\partial_n X)]}$ or [inspire:109550] action $L_{\mathrm{BdVHDZ}} = \frac{1}{2\alpha'}\sqrt{-g}[g^{mn}(\partial_m X)(\partial_n X) - (d-2)]$ with $d = 2$ the world-sheet dimension.

Pure supergravities in $4 \le d \le 11$. Gravity is topological in $d = 3$. The maximum number of supercharges $Q = 32$ forbids $d > 11$. A priori, all $Q = 4k$ are possible. Focus on $32, 16, 8, 4$.

d	$Q = 32$	16	8	4
11	✓			
10	$\overset{IIB}{(2,0)}$ $\overset{IIA}{(1,1)}$	$\overset{I}{(1,0)}$‡		
9	✓	✓		
8	✓	✓		
7	✓	✓		
6	$(2,2)$	$(2,0)$ $(1,1)$	$(1,0)$	
5	✓	✓	✓	
4	$N = 8$	$N = 4$	$N = 2$	$N = 1$

‡ 10d $(1,0)$ supergravity ("type I") has a gravitational anomaly [inspire:192309]. (Perhaps 6d $(2,0)$ or $(1,0)$ supergravity too?)

M-theory has as its low-energy limit 11d supergravity, which has two $\frac{1}{2}$-BPS membrane solutions (with 16 Killing spinors): M2-brane $\mathrm{d}s^2 = \Lambda^4\,\mathrm{d}x^2 + \frac{\mathrm{d}y^2}{\Lambda^2}$ with $\Lambda = (1 + \frac{c_2 N_2 l^6}{|y|^6})^{-1/6}$, and M5-brane $\mathrm{d}s^2 = \Lambda\,\mathrm{d}x^2 + \mathrm{d}y^2/\Lambda^2$ with $\Lambda = (1 + \frac{c_5 N_5 l^3}{|y|^3})^{-1/3}$, where $x \in \mathbb{R}^{p,1}$ and $y \in \mathbb{R}^{10-p}$. In the near horizon $y \to 0$ these become $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ with 32 Killing spinors.

Branes IIA strings: D0, F1 (strings), D2, D4, O4$^\pm$, $\widetilde{\mathrm{O4}}^+$, NS5, D6, D8 (wall), O8 (wall), etc.. IIB strings: D(-1), F1 (strings), D1, D3, (p,q) 5-branes (includes D5 and NS5), O5$^\pm$, $\widetilde{\mathrm{O5}}^+$, D7, O7$^\pm$, ON^0, etc.. M-theory: M2, M5, OM5, M9.

Flat space brane configurations Flat space preserves 32 supercharges. One stack of parallel branes breaks half; two stacks break all unless: Dp and Dq have 0, 4 or 8 directions that one brane spans and not the other; Dp branes have 1 or 3 directions transverse to any NS5; any pair of NS5 branes has 2, 4, 6 common directions. Kappa-projection: for Dp is $\Gamma_{01\dots p}\epsilon_L = \epsilon_R$, for NS5 is $\Gamma_{01\dots 5}\epsilon_L = \epsilon_L$. Then at least $32/2^{\#\text{stacks}}$ supercharges preserved.

S-rule, brane creation Let a Dp have 3 directions transverse to an NS5. Zero or one D$(p-2)$ can stretch between the two (spanning common directions and directions where neither Dp nor NS5 stretch). Moving Dp through NS5 toggles between zero and one.

Little string theory (LST) Decoupled $g_s \to 0$, fixed α description of k coincident NS5 branes on transverse T^5 gives $(1,1)$ LST for IIB and $(2,0)$ LST for IIA. Has AdS/CFT dual with linear dilaton background.

§5.4 Integrable models

Sine-Gordon model in 2d: $\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + m^2(\cos\phi - 1)$ has eom $\Box\phi + m^2\sin\phi = 0$.

Relativistic quantum Toda chain. $H = \sum_{n=1}^N \bigl(\cos(2\eta\hat{p}_n) + g^2\cos(\eta\hat{p}_n + \eta\hat{p}_{n+1})\mathrm{e}^{x_{n+1}-x_n}\bigr)$. Its non-relativistic limit is $\eta \to 0$ imaginary with $g/(i\eta\sqrt{2}) = c$ fixed.

§6 Dualities

§6.1 4d $\mathcal{N} = 1$ dualities

IR duality: the "electric" theory A has the same IR limit as the "magnetic" theory B: TQFT, SCFT, free theory, infinite flow, or sum of products thereof.

Gauge-invariant chirals $\mathcal{O}$ of trial R-charge $\le 2/3$ are free and of true IR R-charge 2/3 thanks to mixing with accidental $\mathfrak{u}(1)$ acting on $\mathcal{O}$. Other view: given a term δW, trial R-charge obeys $R(\delta W) = 2$ but true R-charge maybe $R(\delta W) > 2$; then δW is irrelevant and leaves some chiral free.

Typical evidence: global symmetries, 't Hooft anomaly matching, moduli space of vacua, chiral rings, behaviour of these under F-term deformations, superconformal index ($S^3_b \times S^1$ partition function).

Seiberg duality [hep-th/9411149] $\boxed{\mathrm{SU}(F)} \xrightarrow{Q} \mathrm{SU}(C) \xrightarrow{\tilde{Q}} \boxed{\mathrm{SU}(F)}$ dual to $\begin{tikzcd} \boxed{\mathrm{SU}(F)} \arrow[rr, bend right, "M"'] & \mathrm{SU}(C') \arrow[l, "\tilde{q}"'] & \boxed{\mathrm{SU}(F)} \arrow[l, "q"'] \end{tikzcd}$ with $W = \operatorname{Tr}(M\tilde{q}q)$ and $C' = F - C \ge 0$ (here $\mathrm{SU}(1) = \mathrm{SU}(0) = \{1\}$).

Non-anomalous global symmetries $\mathfrak{su}(F)^2 \times \mathfrak{u}(1)_B \times \mathfrak{u}(1)_R$ coincide, where $(Q, \tilde{Q})$, M, $(\tilde{q}, q)$ have $\mathfrak{u}(1)_B$ charges $\pm\frac{1}{C}$, 0, $\pm\frac{1}{F-C}$ and trial $\mathfrak{u}(1)_R$ charges $\frac{F-C}{F}$, $2\frac{F-C}{F}$, $\frac{C}{F}$.

- For $0 \le C \le F/3$ the $\mathrm{SU}(C)$ theory is IR-free.
- For $F/3 < C < 2F/3$ flow to SCFT.
- For $2F/3 \le C \le F$ the $\mathrm{SU}(C')$ theory is IR-free.
- For $0 < F < C$ supersymmetry is broken.

Gauge-invariants match: mesons $M^j{}_i = \tilde{Q}^j Q_i$; (anti)baryons $B_{\mathcal{A}} = \bigwedge_{i\in\mathcal{A}} Q_i \leftrightarrow \bigwedge_{i\notin\mathcal{A}} \tilde{q}^i$ and $\tilde{B}_{\mathcal{A}} = \bigwedge_{j\in\mathcal{A}} \tilde{Q}^j \leftrightarrow \bigwedge_{j\notin\mathcal{A}} q_j$ for $\mathcal{A} \subset [\![1,F]\!]$ with $|\mathcal{A}| = C$. Relation $\tilde{B}_{\mathcal{A}'} B_{\mathcal{A}} = \det(M^j{}_i)^{j\in\mathcal{A}'}_{i\in\mathcal{A}}$ of $\mathrm{SU}(C)$ theory only holds quantumly in $\mathrm{SU}(C')$ theory.

A mass term $W = m\tilde{Q}_F Q_F$ decouples a flavour ($F \to F-1$) and Higgses $\mathrm{SU}(C')$ to $\mathrm{SU}(C'-1)$. Dually, Higgsing $\mathrm{SU}(C)$ to $\mathrm{SU}(C-1)$ using $\langle \tilde{Q}_F Q_F \rangle \to \infty$ (so $F \to F-1$) gives mass to q_F, $\tilde{q}_F$ and leaves $\mathrm{SU}(C')$ fixed.

SO: $\boxed{\mathrm{SU}(F)} \xrightarrow{Q} (\mathrm{SO}(C))$ dual to $M \circlearrowleft \boxed{\mathrm{SU}(F)} \xleftarrow{q} (\mathrm{SO}(C'))$ with $C' = F - C + 4$ and $W = Mqq$ (we assume $C, C' \ge 4$).

Non-anomalous global symm. $\mathfrak{u}(1)_R \times (\mathrm{SU}(F) \times \mathbb{Z}_{2F})/\mathbb{Z}_{F,\mathrm{diag}}$; Q, M, q have trial $\mathfrak{u}(1)_R$ charges $\frac{F-C+2}{F}$, $2\frac{F-C+2}{F}$, $\frac{C-2}{F}$ and are $\overline{\square}$, $\mathrm{sym}^2\,\overline{\square}$, $\square$ under $\mathrm{SU}(F)$, while $\mathbb{Z}_{2F}$ acts by $Q \to \mathrm{e}^{\pi\mathrm{i}/F} Q$, $q \to \mathcal{C}\mathrm{e}^{-\pi\mathrm{i}/F} q$ with $\mathcal{C}$ charge conjugation.

- For $F/3 < C-2 < 2F/3$ flow to SCFT.
- For $2F/3 \le C-2 \le F-2$ the $\mathrm{SU}(C')$ theory is IR-free.
- For $F+5 \le C$ supersymmetry is broken.

Gauge-invariants match: mesons $M_{(ij)} = Q_i Q_j$; field strength $W_{\mathcal{A},\alpha} = W_\alpha \wedge \bigwedge_{i\in\mathcal{A}} Q_i \leftrightarrow W_\alpha \wedge \bigwedge_{i\notin\mathcal{A}} q^i$ for $|\mathcal{A}| = C-2$ flavours; baryons $B_{\mathcal{A}} = \bigwedge_{i\in\mathcal{A}} Q_i \leftrightarrow W_\alpha \wedge W_\alpha \wedge \bigwedge_{i\notin\mathcal{A}} q^i$ for $|\mathcal{A}| = C$ and $b_{\mathcal{A}} = W_\alpha \wedge W_\alpha \wedge \bigwedge_{i\in\mathcal{A}} Q_i \leftrightarrow \bigwedge_{i\notin\mathcal{A}} q^i$ for $|\mathcal{A}| = C-4$.

USp: $\boxed{?(2F)} \xrightarrow{Q} (\mathrm{USp}(2C))$ dual to $M \circlearrowleft \boxed{?(2F)} \xleftarrow{\tilde{q}} (\mathrm{USp}(2C'))$ with $W =$ and $C' = F - C - 2$.

Self-duality in the SU, SO, USp dualities for $F = 2C, 2C-4, 2C+2$ respectively, namely $C(R_{\mathrm{chirals}}) = 2C(\mathrm{adj})$; adding an adjoint gives $\mathcal{N} = 2$ SCFTs.

Kutasov–Schwimmer–Seiberg duality [hep-th/9510222] $\boxed{\mathrm{SU}(F)} \xrightarrow{Q} (\mathrm{SU}(C))^{\circlearrowleft X} \xrightarrow{\tilde{Q}} \boxed{\mathrm{SU}(F)}$ dual to $\boxed{\mathrm{SU}(F)} \xleftarrow{\tilde{q}} (\mathrm{SU}(C'))^{\circlearrowleft x} \xleftarrow{q} \boxed{\mathrm{SU}(F)}$, with $M_1, \dots, M_k$ from the right $\boxed{\mathrm{SU}(F)}$ to the left $\boxed{\mathrm{SU}(F)}$,

for $W_{\mathrm{el}} = \operatorname{Tr} P(X) = \sum_{j=1}^{k+1} s_{k+1-j} \operatorname{Tr} X^j/j$ (with $s_1 = 0$) and $W_{\mathrm{mag}} = \alpha(s) - \operatorname{Tr} P(x) + \frac{1}{\mu^2} \sum_{1\le i\le j\le k} s_{k-j} M_i \tilde{q} x^{j-i} q$, for some function α and mass μ. Here, $C' = kF - C \ge 0$.

Non-anomalous global symmetries $\mathfrak{su}(F)^2 \times \mathfrak{u}(1)_B \times \mathfrak{u}(1)_R$ where $(Q, \tilde{Q})$, X, $(q, \tilde{q})$, x have $\mathfrak{u}(1)_B$ charges $\pm\frac{1}{C}$, 0, $\pm\frac{1}{C'}$, 0 and $\mathfrak{u}(1)_R$ charges $1 - \frac{2C}{(k+1)F}$, $\frac{2}{k+1}$, $1 - \frac{2C'}{(k+1)F}$, $\frac{2}{k+1}$.

Gauge-invariants: mesons $M_j = \tilde{Q} X^{j-1} Q$; baryons $B_{\mathcal{A}} = \bigwedge_{(i,j)\in\mathcal{A}} (X^{j-1} Q_i) \leftrightarrow \bigwedge_{(i,j)\notin\mathcal{A}} (x^{j-1} q_i)$ and antibaryons $\tilde{B}_{\mathcal{A}} = \bigwedge_{(i,j)\in\mathcal{A}} (\tilde{Q}_i X^{j-1}) \leftrightarrow \bigwedge_{(i,j)\notin\mathcal{A}} (\tilde{q}_i x^{j-1})$ for $\mathcal{A} \subset [\![1,F]\!] \times [\![1,k]\!]$ with $|\mathcal{A}| = C$; and $\operatorname{Tr} X^j/j = \partial W/\partial s_{k+1-j} = \partial W_{\mathrm{mag}}/\partial s_{k+1-j}$ expressed in terms of $\operatorname{Tr} x^i$ (for this, $\alpha(s)$ matters); ...

Aharony–Sonnenschein–Yankielowicz [hep-th/9504113] $\mathrm{SU}(C)$ with F flavours $(Q, \tilde{Q})$, F' flavours $(Z, \tilde{Z})$ and an adjoint X, with $W_{\mathrm{el}} = \tilde{Z} X Z + \operatorname{Tr} X^3/3$ is dual to $\mathrm{SU}(2F+F'-C)$ with $F+F'$ flavours $q, \tilde{q}, z, \tilde{z}$, an adjoint x, gauge singlets M_j, N, $\tilde{N}$ and $W_{\mathrm{mag}} = \tilde{z} x z + \operatorname{Tr} x^3/3 + M_1 \tilde{q} x q + M_2 \tilde{q} q + N \tilde{z} q + \tilde{N} \tilde{q} z$. Duality likely generalizes to $W = \operatorname{Tr} P(X) + \sum_i \tilde{Z}_i X^{k_i} Z_i$ with an $\mathrm{SU}(F \deg P + \sum_i k_i - C)$ dual.

Other 4d $\mathcal{N} = 1$ dualities Brodie, Intriligator–Pouliot, Argyres–Intriligator–Leigh–Strassler, Klebanov cascade, Intriligator–Leigh–Strassler, Kutasov–Lin.

§6.2 3d dualities

We recall $\mathrm{Sp}(n) = \mathrm{USp}(2n)$.

Level-rank duality of Chern–Simons TQFTs (proven). For $k, N \ge 0$, have $\mathrm{U}(k)_{\pm N} \leftrightarrow \mathrm{SU}(N)_{\mp k}$, $\mathrm{SO}(k)_{\pm N} \leftrightarrow \mathrm{SO}(N)_{\mp k}$, $\mathrm{Sp}(k)_{\pm N} \leftrightarrow \mathrm{Sp}(N)_{\mp k}$ but subtleties [1607.07457].

Chern–Simons matter dualities [1706.08755] where scalars and fermions are in $\mathbb{C}^N$ for $\mathrm{U}(N)$, $\mathbb{R}^N$ for $\mathrm{SO}(N)$, $?^{2N}$ for $\mathrm{Sp}(N)$, and scalars have quartic coupling. Assuming N, k, F, $\pm$ obey $N \in \mathbb{Z}_{\ge 0}$, $F/2 \pm k \in \mathbb{Z}_{\ge 0}$ and an unknown upper bound $F < N_*$ greater than $2|k|$, conjectured dualities (modified for $\mathrm{SO}(1)$ and $\mathrm{SO}(2)$, see reference):

- $\mathrm{SU}(N)_k$ & F fermions $\leftrightarrow$ $\mathrm{U}(F/2 \pm k)_{\mp N}$ & F scalars;
- $\mathrm{SO}(N)_k$ & F fermions $\leftrightarrow$ $\mathrm{SO}(F/2 \pm k)_{\mp N}$ & F scalars;
- $\mathrm{Sp}(N)_k$ & F fermions $\leftrightarrow$ $\mathrm{Sp}(F/2 \pm k)_{\mp N}$ & F scalars.

Denote $X = \mathrm{SU}, \mathrm{SO}, \mathrm{Sp}$ and $X' = \mathrm{U}, \mathrm{SO}, \mathrm{Sp}$ to uniformize cases. Turn on equal mass m for all fermions. For $\mp m \gg g^2$ the theory flows to $X(N)_{k \mp F/2} \leftrightarrow X'(|k \mp F/2|)_{-\operatorname{sign}(k \mp F/2) N}$. If $F \le 2|k|$ the (only) bosonic dual has the same two phases. If $2|k| < F < N_*$ a confining phase sits between these phases; each bosonic dual describes two of the three phases. In the confining phase the global symmetry breaks to $Y(F/2+k) \times Y(F/2-k) \subset Y(F)$, giving an NLSM on the quotient and N times the Wess–Zumino term. Here Y is Sp for SU(2) and otherwise $\tilde{X}$.

Integrating out one fermion of large mass m shifts $F \to F-1$. For $\mp m > 0$ get $k \to k + 1/2$; the dual scalar becomes massive and $k + F/2$ is unchanged. For $\pm m > 0$ get $k \to k - 1/2$; the dual scalar has Mexican potential so $k + F/2 \to k + F/2 - 1$.

With scalars and fermions [1712.00020, 1712.04933] (less firm footing). Denote "ϕ" and "ψ" for scalars and fermions, let (X, X') be one of (SU, U), (SO, SO), (Sp, Sp), and assume $0 \le S \le N$, $0 \le F \le k$ and additionally $(S, F) \ne (N, k)$ for $(X, X') = (\mathrm{SU}, \mathrm{U})$ and $S + F + 3 \le N + k$ for SO. The duality is then $X(N)_{k-F/2}, S\phi, F\psi \leftrightarrow X'(k)_{-N+S/2}, F\phi, S\psi$.

§6.3 Field theory dualities

2d $\mathcal{N} = (0,2)$ Gadde–Gukov–Putrov triality (IR).

2d $\mathcal{N} = (2,2)$ mirror symmetry of Calabi–Yau sigma models (exact).

2d $\mathcal{N} = (2,2)$ Hori–Tong (SU), Hori (Sp, SO groups), plus adjoint (ADE-type and $(2,2)^*$-like) dualities (IR).

2d $\mathcal{N} = (2,2)$ Hori–Vafa/Hori–Kapustin duality of gauged linear sigma models and Landau–Ginzburg models (IR).

3d Chern–Simons level-rank duality.

3d $\mathcal{N} = 2$ Aharony, Giveon–Kutasov, Aharony–Fleischer dualities (IR).

3d $\mathcal{N} = 2$ and $\mathcal{N} = 4$ mirror symmetry exchanging Coulomb and Higgs branches (IR).

S-duality of 4d $\mathcal{N} = 2$ gauge theories (exact).

S-duality of 4d $\mathcal{N} = 4$ SYM (exact).

§6.4 String theory dualities

In this table "type IIA" etc. refer to string theories not supergravities

F-theory on K3	$\Leftrightarrow$ $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic on T^2
M-theory on K3	$\Leftrightarrow$ heterotic or type I on T^3
Type IIA on K3	$\Leftrightarrow$ heterotic or type I on T^4
M-theory on G_2-manifolds[1]	$\Leftrightarrow$ heterotic or type I on CY_3
M-theory on $\mathrm{K3}^2$	$\Leftrightarrow$ type IIA on $T^3/\mathbb{Z}_2$

§7 Manifolds

§7.1 Pseudo-Riemannian geometry

$T_{\dots\lambda[\mu_1\dots\mu_m]\nu\dots} = m!^{-1}\sum_{\sigma\in S_m}\epsilon(\sigma)T_{\dots\lambda\mu_{\sigma(1)}\dots\mu_{\sigma(m)}\nu\dots}$ antisymmetrization, where $\epsilon(\sigma)=\pm1$ is the signature of the permutation; symmetrization $T_{\dots\lambda(\mu_1\dots\mu_m)\nu\dots}$ is without ϵ. Derivatives: $\partial_\mu = \partial/\partial x^\mu$ and $T^{\nu_1\dots\nu_n}_{\rho_1\dots\rho_r\,,\mu_1\dots\mu_m} = \partial_{\mu_1}\cdots\partial_{\mu_m}T^{\nu_1\dots\nu_n}_{\rho_1\dots\rho_r}$ and $T^{\nu_1\dots\nu_n}_{\rho_1\dots\rho_r\,;\mu_1\dots\mu_m} = \nabla_{\mu_1}\cdots\nabla_{\mu_m}T^{\nu_1\dots\nu_n}_{\rho_1\dots\rho_r}$ (namely ";" is ∇).

Connection $\nabla = \partial+\Gamma$ in terms of Christoffel symbols Γ. $T^{\nu_1\dots}_{\rho_1\dots\,;\mu} = T^{\nu_1\dots}_{\rho_1\dots\,,\mu} + (T^{\lambda\nu_2\dots}_{\rho_1\dots}\Gamma^{\nu_1}{}_{\lambda\mu}+\cdots) - (T^{\nu_1\dots}_{\lambda\rho_2\dots}\Gamma^{\lambda}{}_{\rho_1\mu}+\cdots)$ for a tensor. In particular, $\nabla_\mu v^\nu = \partial_\mu v^\nu + v^\lambda\Gamma^\nu{}_{\lambda\mu}$ for a vector and $\nabla_\mu\omega_\nu = \partial_\mu\omega_\nu - \omega_\lambda\Gamma^\lambda{}_{\nu\mu}$ for a one-form. Extra $-w(\log\Omega)_{,\mu}T^{\nu_1\dots}_{\rho_1\dots}$ for a weight w tensor density, where Ω is the volume factor ($|\det g|^{1/2}$ for a metric).

The Levi-Civita connection of a metric g is $\nabla=\partial+\Gamma$ with $\Gamma^\lambda{}_{\nu\rho} = \frac{1}{2}g^{\lambda\mu}(\partial_\rho g_{\mu\nu}+\partial_\nu g_{\mu\rho}-\partial_\mu g_{\nu\rho})$. It is the only torsion-free connection ($\Gamma^\lambda_{[\mu\nu]}=0$) that kills the metric. Note that $\Gamma^\lambda{}_{\lambda\rho} = \frac{1}{2}g^{\lambda\mu}\partial_\rho g_{\lambda\mu} = \frac{1}{2}\partial_\rho\log|\det g|$. Denote $\sqrt{g}=|\det g|^{1/2}$. Then $\sqrt{g}\,\nabla_\nu v^\nu = \partial_\nu(v^\nu\sqrt{g})$ and $\sqrt{g}\,\nabla_\nu F^{[\nu\rho]} = \partial_\nu(F^{[\nu\rho]}\sqrt{g})$ are total derivatives.

Killing vector k_μ such that $\nabla_{(\mu}k_{\nu)}=0$. For a symmetric conserved stress-tensor T we have $\nabla_\mu(k_\nu T^{\mu\nu})=0$, giving conserved quantities.

§7.2 G-structures, holonomy

Structure group. A G-structure on a manifold X (with $n=\dim_{\mathbb{R}}X$) is a G-subbundle of the $\mathrm{GL}(n,\mathbb{R})$-principal bundle $\mathrm{GL}(TX)$ of tangent frames, namely a global section of $\mathrm{GL}(TX)/G$.

A $\mathrm{GL}_+(n,\mathbb{R})=\{\det>0\}$ structure on X is equivalent to an orientation. A $\mathrm{SL}(n,\mathbb{R})$ structure is equivalent to a volume form. Similar definitions for Riemannian manifolds etc.:

G-structure	Manifold type	Other characterization‡
$\mathrm{O}(n)$	Riemannian	metric $g>0$
$\mathrm{SO}(n)$	oriented, Riemannian	
$\mathrm{O}(p,q)$	pseudo-Riemannian	metric of signature (p,q)
$\mathrm{SO}_+(p,q)$	pseudo-Riemannian, oriented, time-oriented	
$\mathrm{Pin}_\pm$ or Spin	(pseudo)-Riemannian $\mathrm{pin}_\pm$ or spin manifold	
$\mathrm{GL}(n/2,\mathbb{C})$	Almost complex	$\mathbb{C}\circlearrowright TX$ (i.e., $J^2=-1$)
$\mathrm{Sp}(2n/2,\mathbb{R})$	Almost symplectic	Non-degenerate $\omega\in\Omega^2X$
$\mathrm{U}(n/2)$	Almost Hermitian	Two compatible (g,J,ω)§
$\mathrm{U}^*(n/2)$	Almost hypercomplex¶	$J_1,J_2,J_3\circlearrowright TX$
$\mathrm{USp}(n/2)$	Almost hyperHermitian	$(g,J_{1,2,3},\omega_{1,2,3})$
$\mathrm{U}^*(n/2)\mathrm{USp}(2)$	Almost quaternionic¶	$\mathbb{H}\circlearrowright TX$
$\mathrm{USp}(n/2)\mathrm{USp}(2)$	Almost quaternion-Hermitian	$(g,\mathbb{H},\omega_{1,2,3})$

‡ All sections are global. For instance, an almost complex structure is a global section J of $\operatorname{End}TX$ with $J^2=-1$. A metric is a global section g of $S^2(T^*X)$.

§ Any two of (g,J,ω) fix the third by $\omega_{ik}=J_i{}^jg_{jk}$ if they are compatible: $J_i{}^jJ_l{}^k\omega_{jk}=\omega_{il}$ or $J_i{}^jJ_l{}^kg_{jk}=g_{il}$ namely ω or g is J-invariant, or $\omega_{ij}g^{jk}\omega_{kl}=-g_{il}$. In a basis $e^\beta,\bar{e}^{\bar\gamma}$ ($=\mathrm{d}z^\beta,\mathrm{d}\bar{z}^{\bar\gamma}$ for Hermitian manifolds) of $(1,0)$ and $(0,1)$ forms, $\omega=\frac{\mathrm{i}}{2}h_{\beta\bar\gamma}\,e^\beta\wedge\bar{e}^{\bar\gamma}$ and $g=\frac{1}{2}h_{\beta\bar\gamma}(e^\beta\otimes\bar{e}^{\bar\gamma}+\bar{e}^{\bar\gamma}\otimes e^\beta)$.

On an almost complex manifold, (p,q)-forms are wedge products $\Omega^{(p,q)}X=\bigwedge^p\big(\Omega^{(1,0)}X\big)\wedge\bigwedge^q\big(\Omega^{(0,1)}X\big)$ where J acts by $\pm\mathrm{i}$ on $\Omega^1X=\Omega^{(1,0)}X\oplus\Omega^{(0,1)}X$. The exterior derivative is $\mathrm{d}=\mathrm{d}^{2,-1}+\mathrm{d}^{1,0}+\mathrm{d}^{0,1}+\mathrm{d}^{-1,2}$ with $\mathrm{d}^{i,j}:\Omega^{(p,q)}\to\Omega^{(p+i,q+j)}$. Dolbeault differential operators are $\partial=\mathrm{d}^{1,0}$ and $\overline{\partial}=\mathrm{d}^{0,1}$.

An almost symplectic $2m$-manifold admits the volume form $\omega^m/m!$. On an almost Hermitian manifold X it is equal to the Riemannian volume form and belongs to $\Omega^{(m,m)}X$.

¶ While almost quaternionic manifolds have a 3d subbundle of $\operatorname{End}TX$ locally spanned by J_1,J_2,J_3 with $J_i^2=J_1J_2J_3=-1$, almost hypercomplex manifolds require J_1,J_2,J_3 to be global.

Integrability. A G-structure is k-integrable (resp. integrable) near $x\in X$ if it can be trivialized to order k (resp. all orders) in a neighborhood of x. We automatically have 0-integrability.

Any Riemannian structure is 1-integrable thanks to Riemann normal coordinates. Integrability is equivalent to the Riemann curvature vanishing.

An almost complex structure is complex if (equivalently) it is integrable; it is 1-integrable; it has a vanishing Nijenhuis tensor $N_J:\bigwedge^2X\to TX$ defined on vector fields u, v by the Lie brackets $N_J(u,v)=-J^2[u,v]+J[Ju,v]+J[u,Jv]-[Ju,Jv]$; the Lie bracket of $(1,0)$ vector fields is a $(1,0)$ vector field; $\mathrm{d}=\partial+\overline{\partial}$ namely $\mathrm{d}^{2,-1}=0=\mathrm{d}^{-1,2}$; or $\overline{\partial}^2=0$.

A symplectic structure is an integrable almost symplectic structure. Equivalently, it is 1-integrable: $\mathrm{d}\omega=0$. Altogether,

$$\begin{tikzcd}
\text{general} \arrow[rr, "\omega"] \arrow[dr, "g‡"] \arrow[dd, "J"'] & & \text{almost symplectic} \arrow[rr, "\mathrm{d}\omega=0"] \arrow[dd, "J\text{ or }g"] & & \text{symplectic} \arrow[dd, "J"] \\
& \text{Riemannian} \arrow[dr, "\omega\text{ or }J"] & & & \\
\text{almost complex} \arrow[rr, "\omega\text{ or }g‡"] \arrow[d, "N_J=0"'] & & \text{almost Hermitian} \arrow[rr, "\mathrm{d}\omega=0"] \arrow[d, "N_J=0"] \arrow[drr, "§"] & & \text{almost Kähler} \arrow[d, "N_J=0"] \\
\text{complex} \arrow[rr, "\omega\text{ or }g‡"] & & \text{Hermitian} \arrow[rr, "\mathrm{d}\omega=0"] & & \text{Kähler}
\end{tikzcd}$$

(Almost) quaternionic/quaternionHermitian/quaternionKähler and (almost) hypercomplex/hyperHermitian/hyperKähler manifolds are defined by replacing J by a 3d subbundle of $\operatorname{End}TX$ or by global sections J_1,J_2,J_3 as in the table of G-structures.

‡ Since $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$ is contractible, any manifold admits (non-canonically) an $\mathrm{O}(n)$-structure, namely a smooth choice of which frames are orthonormal, i.e., a Riemannian metric g. Similarly $\mathrm{GL}(n/2,\mathbb{C})/\mathrm{U}(n/2)$ is contractible so almost complex manifolds admit almost Hermitian structures.

§ An almost Hermitian manifold is Kähler if (equivalently) its $\mathrm{U}(n/2)$-structure is 1-integrable; $\mathrm{d}\omega=0$ and $N_J=0$; $\nabla\omega=0$; $\nabla J=0$; or the holonomy group is in $\mathrm{U}(n/2)$. Locally, $\omega=\mathrm{i}\partial\overline{\partial}\rho$ for some real-valued Kähler potentials ρ, and ω is invariant under Kähler transformations $\rho\to\rho+f(z)+\bar{f}(\bar{z})$.

The holonomy group at $x\in X$ of a connection ∇ on a bundle $E\to X$ is the group of symmetries of E_x arising from parallel transport along closed curves based at x.

For Riemannian manifolds X the holonomy group is defined as that of the Levi-Civita connection on the tangent bundle. It is a subgroup of $\mathrm{O}(n)$ $\big($or $\mathrm{SO}(n)$ for X orientable$\big)$ since parallel transport preserves orthogonality ($\nabla g=0$).

If the holonomy group acts reducibly on the tangent space then X is locally (globally if X is geodesically complete) a product. Simply connected X that are locally neither products nor symmetric spaces (we give a list later) can have the following special holonomy subgroups of $\mathrm{SO}(n)$ (Berger's theorem)

Holonomy	Manifold type	$\dim_{\mathbb{R}}$
$\mathrm{U}(m)$	Kähler	$2m$
$\mathrm{SU}(m)$	Calabi–Yau CY_m	$2m$
$\big(\mathrm{USp}(2k)\times\mathrm{USp}(2)\big)/\mathbb{Z}_2$	quaternionic Kähler	$4k$
$\mathrm{USp}(2k)$	hyperKähler	$4k$
$\mathrm{Spin}(7)$	$\mathrm{Spin}(7)$ manifold	8
G_2	G_2 manifold	7

Note that hyperKähler $\implies$ Calabi–Yau $\implies$ Kähler since $\mathrm{USp}(m)\subset\mathrm{SU}(m)\subset\mathrm{U}(m)$. In contrast, quaternionic-Kähler manifolds are not Kähler.

A Calabi–Yau manifold is a Kähler manifold such that (equivalently) some Kähler metric has global holonomy group in $\mathrm{SU}(m)$; the structure group can be reduced to $\mathrm{SU}(m)$; or the holomorphic canonical bundle is trivial i.e., there exists a nowhere vanishing holomorphic top-form.

For simply connected manifolds, the conditions above are equivalent to the following (always equivalent) conditions on X: some Kähler metric has local holonomy group in $\mathrm{SU}(m)$; some Kähler metric has vanishing Ricci curvature; the first real Chern class vanishes; a positive power of the holomorphic canonical bundle is trivial; X has a finite cover with trivial holomorphic canonical bundle; X has a finite cover equal to the product of a torus and a simply connected manifold with trivial holomorphic canonical bundle.

Spin structures

Symmetric spaces

K3 surfaces are the only CY_2: they have holonomy $\mathrm{SU}(2)$.

Yau's theorem. Fix a complex structure on a compact complex manifold X of $\dim_{\mathbb{C}} X>1$ and vanishing real first Chern class. Any real class $H^{1,1}(X,\mathbb{C})$ of positive norm contains a unique Kähler form whose metric is Ricci flat.

(from Wikipedia on Calabi conjecture: "The Calabi conjecture states that a compact Khler manifold has a unique Khler metric in the same class whose Ricci form is any given 2-form representing the first Chern class.")

§8 Misc

§8.1 Special functions

Multiple gamma function. For $a_i\in\mathbb{C}$ with $\operatorname{Re} a_i>0$, $\Gamma_N(x|\vec{a})=\prod_{\vec{n}}^{\text{reg.}}(x+\vec{n}\cdot\vec{a})^{-1}=\exp(\partial_s\sum_{\vec{n}}(x+\vec{n}\cdot\vec{a})^{-s}|_{s=0})$, where $\vec{n}\in\mathbb{Z}_{\geq 0}^N$. Here, we zeta-regularized the product; the sum is analytically continued from $\operatorname{Re} s>N$. The meromorphic $x\mapsto\Gamma_N(x|\vec{a})$ has no zero and poles at $x=-\vec{n}\cdot\vec{a}$ (simple poles for generic $\vec{a}$). $\Gamma_0(x|)=1/x$, $\Gamma_1(x|a)=a^{x/a-1/2}\Gamma(x/a)/\sqrt{2\pi}$, $\Gamma_N(x|\vec{a})=\Gamma_{N-1}(x|a_1,\dots,a_{N-1})\Gamma_N(x+a_N|\vec{a})$ and it is invariant under permutations of $\vec{a}$.

Plethystic exponential. Let $\mathbf{m}\subset R[[x_1,\dots,x_n]]$ be series with no constant term over a ring R. Then $\operatorname{plexp}:\mathbf{m}\to 1+\mathbf{m}$ obeys $\operatorname{plexp}[x_i^p]=1/(1-x_i^p)$, $\operatorname{plexp}[f+g]=\operatorname{plexp}[f]\operatorname{plexp}[g]$ and $\operatorname{plexp}[\lambda f]=\operatorname{plexp}[f]^\lambda$ for $\lambda\in R$. It maps an index of single-particle states $f(x)$ to that of multiparticle states $\operatorname{plexp} f(x)=\exp\sum_{k\geq 1}\frac{1}{k}f(x_1^k,\dots,x_n^k)$.

q-Pochhammer $(a;q)_\infty=\operatorname{plexp}\frac{-a}{1-q}=\prod_{k=0}^\infty(1-aq^k)$ and finite version $(a;q)_n=(a;q)_\infty/(aq^n;q)_\infty$. Products are often denoted $(a_1,\dots,a_N;q)_n=(a_1;q)_n\cdots(a_N;q)_n$. Properties: $(a;q)_{-n}(q/a;q)_n=(-q/a)^n q^{n(n-1)/2}$ and q-binomial theorem $(ax;q)_\infty/(x;q)_\infty=\sum_{n=0}^\infty x^n(a;q)_n/(q;q)_n$.

q-gamma (or basic gamma) function for $|q|<1$, $\Gamma_q(x)=(1-q)^{1-x}(q;q)_\infty/(q^x;q)_\infty$ obeys $\Gamma_q(x+1)=\frac{1-q^x}{1-q}\Gamma_q(x)$ and $\Gamma_q(x)\xrightarrow{q\to 1}\Gamma(x)$. It has simple poles at $x\in\mathbb{Z}_{\leq 0}$ and no zero.

Modular form of weight k: holomorphic on $\mathbf{H}=\{\operatorname{Im}\tau>0\}$ and as $\tau\to\mathrm{i}\infty$ and obeys $f(\frac{a\tau+b}{c\tau+d})=(c\tau+d)^k f(\tau)$.

Dedekind eta function: $\eta(\tau)=q^{1/24}(q;q)_\infty$ for $q=\mathrm{e}^{2\pi\mathrm{i}\tau}$. $\Delta=\eta^{24}$ is a modular form of weight 12.

Theta functions: q-theta $\theta(z;q)=(z;q)_\infty(q/z;q)_\infty$ obeys $\theta(z;q)=\theta(q/z;q)=-z\theta(1/z;q)$. Variant $\theta_1(z;q)=\theta_1(\tau|u)=\mathrm{i}z^{-1/2}q^{1/12}\eta(\tau)\,\theta(z;q)=-\mathrm{i}z^{1/2}q^{1/8}(q;q)_\infty(qz;q)_\infty(1/z;q)_\infty$ with $z=\mathrm{e}^{2\pi\mathrm{i}u}$.

Eisenstein series ($k\geq 1$) $E_{2k}=1+\frac{2}{\zeta(1-2k)}\sum_{n=1}^\infty n^{2k-1}\frac{q^n}{1-q^n}$ obeys $E_{2k}(\frac{a\tau+b}{c\tau+d})=(c\tau+d)^{2k}E_{2k}(\tau)+\frac{6}{\pi\mathrm{i}}c(c\tau+d)\delta_{k=1}$. For $k\geq 2$ it is a modular form and $E_{2k}=\frac{1}{2\zeta(2k)}\sum_{0\neq\lambda\in\mathbb{Z}+\tau\mathbb{Z}}\lambda^{-2k}$.

Elliptic gamma function $\Gamma(z;p,q)=\operatorname{plexp}\frac{z-pq/z}{(1-p)(1-q)}=\prod_{m=0}^\infty\prod_{n=0}^\infty(1-p^{m+1}q^{m+1}z^{-1})/(1-p^mq^nz)$ obeys $\Gamma(z;p,q)=\Gamma(z;q,p)=1/\Gamma(pq/z;p,q)$ and $\Gamma(pz;p,q)=\theta(z;q)\Gamma(z;p,q)$ and $\Gamma(z;0,q)=1/(z;q)_\infty$.

Polylogarithm and Riemann zeta $\zeta(s)=\mathrm{Li}_s(1)$ where $\mathrm{Li}_s(z)=\sum_{k\geq 1}z^k/k^s=\big(1/\Gamma(s)\big)\int_0^\infty t^{s-1}\mathrm{d}t/(\mathrm{e}^t/z-1)$ obeys $\mathrm{Li}_{s+1}(z)=\int_0^z\mathrm{Li}_s(t)\mathrm{d}t/t$ and $\mathrm{Li}_1(z)=-\log(1-z)$. The dilogarithm obeys $\mathrm{Li}_2(x)+\mathrm{Li}_2(1-x)=\pi^2/6-\log(x)\log(1-x)$ (reflection formula) and $\mathrm{Li}_2(x)+\mathrm{Li}_2(y)-\mathrm{Li}_2(xy)=\mathrm{Li}_2(t)+\mathrm{Li}_2(u)+\log(1-t)\log(1-u)$ where $x=t/(1-u)$ and $y=u/(1-t)$ (pentagon formula).

Gauss hypergeometric function is given by ${}_2F_1(a,b;c;z)=\sum_{n\geq 0}z^n(a)_n(b)_n/\big(n!(c)_n\big)$ converging for $|z|<1$, continued to $\mathbb{C}\setminus[1,\infty)$ with branch cut. Here $(a)_n=a(a+1)\dots(a+n-1)$.

Generalized hypergeometric functions: let $a_i,b_i\notin\mathbb{Z}_{\leq 0}$. Then ${}_jF_k(a;b;z)=\sum_{n\geq 0}z^n(a_1)_n\dots(a_j)_n/\big(n!(b_1)_n\dots(b_k)_n\big)$ converges if $j=k+1$ and $|z|<1$, or if $j\leq k$. Differential equation $z\prod_{i=1}^j(z\partial_z+a_i)F(z)=z\partial_z\prod_{i=1}^k(z\partial_z+b_i-1)F(z)$. Physically: vortex partition function of the 2d $\mathcal{N}=(2,2)$ U(1) theory with j charge $+1$ and $k+1$ charge -1 chiral multiplets. Fox–Wright, Appell, Kampé de Fériet and Lauricella functions are vortex partition functions of specific $\mathrm{U}(1)^n$ theories.

Basic hypergeometric series in terms of q-Pochhammer ${}_j\phi_k(a;b;q,z)=\sum_{n\geq 0}\big(-q^{(n-1)/2}\big)^{n(1+k-j)}z^n(a_1,\dots,a_j;q)_n/(b_1,\dots,b_k;q)_n$.

§8.2 Physics of field theories

Phases characterized by potential $V(R)$ (up to a constant) between quarks at distance R: Coulomb $1/R$, free electric $1/\big(R\log(R\Lambda)\big)$, free magnetic $\log(R\Lambda)/R$, Higgs (constant), confining σR.

A solitary wave is a solution (of some wave equation) whose energy density is localized in space and travels at some speed u. A soliton additionally keeps its shape asymptotically after collisions.

Instantons are anti-self-dual ($F=-\star F$) connections on $\mathbb{R}^4$ that decay at infinity and extend to S^4. The bundles are classified by $\pi_3(G)$ so an instanton number $k\in\mathbb{Z}$ for simple gauge groups; for fixed $k\geq 0$ ($k\leq 0$) (anti)instantons minimize the action. The k-instanton moduli space for $G=\mathrm{SU}(N)$ is a $4Nk$-dimensional [inspire:128223] hyperKähler manifold, in bijection with rank N framed locally free sheaves on $\mathbb{CP}^2$ [inspire:125044].

A 't Hooft–Polyakov monopole is a smooth codimension 3 topological soliton where the gauge group G is spontaneously broken by a Higgs field to H, far from the soliton. Monopole charge is in $\pi_2(G/H) = \mathrm{Ker}\big(\pi_1(H) \to \pi_1(G)\big)$ "Dirac monopole of H trivialized in G".

Example: SU(2) gauge theory with adjoint scalar ϕ of potential $(a^2 - \mathrm{Tr}\,\phi^2)^2$. Residual U(1) gauge field $F_{\mathrm{U}(1)} = \mathrm{Tr}(\hat\phi F) - 2\,\mathrm{Tr}(\hat\phi D\hat\phi D\hat\phi)$ with $\hat\phi = \phi/\sqrt{\mathrm{Tr}\,\phi^2}$ obeys $\mathrm{d}F_{\mathrm{U}(1)} = \mathrm{Tr}(\mathrm{d}\hat\phi)^3$, whose integral is the monopole charge.

§8.3 Homotopy, homology and cohomology

Basic properties. $\pi_0(X,x)$ is the set of connected components. $\pi_1(X,x)$ is the fundamental group. For $k \geq 1$, $\pi_k(X,x)$ only depends on the connected component of x. $\pi_k(X\times Y,(x,y)) = \pi_k(X,x)\times\pi_k(Y,y)$.

Quotient. If G acts on connected simply-connected X then $\pi_1(X/G) = \pi_0(G)$ ($= G$ for G discrete).

Long exact sequence for a fiber bundle $F \hookrightarrow E \twoheadrightarrow B$: for base-points $b_0 \in B$ and $e_0 = f_0 \in F = p^{-1}(b_0) \subset E$, $\cdots \to \pi_{i+1}(B) \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \cdots \to \pi_0(E)$ is exact, namely each image equals the next kernel (inverse image of the constant map).

Homotopy groups of spheres are finite except $\pi_n(S^n) = \mathbb{Z}$ and $\pi_{4n-1}(S^{2n}) = \mathbb{Z}\times$ finite. For $k < n$, $\pi_k(S^n) = 0$, and $\pi_{n+k}(S^n)$ is independent of n for $n \geq k+2$. All $\pi_k(S^0) = 0$, $\pi_k(S^1) = 0$ for $k \neq 1$, and $\pi_k(S^3) = \pi_k(S^2)$ for $k \neq 2$.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
S^0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$

$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$ for $n \geq 2$ and $\pi_k(\mathbb{RP}^n) = \pi_k(S^n)$ for $k \geq 2$. $\pi_1(\mathbb{CP}^n) = 0$, $\pi_2(\mathbb{CP}^n) = \mathbb{Z}$, $\pi_k(\mathbb{CP}^n) = \pi_k(S^{2n+1})$ for $k \geq 3$.

Topological groups have abelian $\pi_1(G)$. Proofs. 1. The multiplication in G (point-wise) and concatenation of loops are two compatible group structures, hence (by Eckmann–Hilton theorem) coincide and are commutative. 2. Explicitly, for $\alpha_1,\alpha_2 \in \pi_1(G)$ loops, $(t_1,t_2) \mapsto \alpha_1(t_1)\alpha_2(t_2) \in G$ is a homotopy between $\alpha_1 \star \alpha_2$ (concatenation) along bottom and right edges, $\alpha_1\cdot\alpha_2$ (point-wise multiplication) along the diagonal, and $\alpha_2\star\alpha_1$ along left and top edges.

$H_k(\mathbb{CP}^n, M) = M$ for $0 \leq k \leq 2n$ even, else 0.

§8.4 Kähler 4-manifolds

K3 surfaces are (the only besides T^4) compact complex surfaces of trivial canonical bundle. They have $h^{1,0} = 0$ (in contrast to T^4). Their first Chern class $c_1 \in H^2(X,\mathbb{Z})$ thus vanishes. By Yau's theorem there exists a Ricci flat metric, whose holonomy is then $\mathrm{SU}(2) = \mathrm{USp}(2)$ by Berger's classification. K3 surfaces are thus Calabi–Yau (CY_2) and hyperKähler (hK_4). Their moduli space is connected and they are all diffeomorphic.

Examples of K3 surfaces. Quartic hypersurface in $\mathbb{P}^4$. Kummer surface namely resolution of $T^4/\mathbb{Z}_2$.

Non-simply connected Ricci-flat Kähler manifolds may fail to be CY_n when the restricted holonomy group is $\mathrm{SU}(n)$ but the global holonomy group is disconnected. For example an Enriques surface $\mathrm{K3}/\mathbb{Z}_2$ has a non-trivial canonical bundle.

A gravitational instanton is a metric with (anti-)self-dual curvature. A simply-connected Riemannian 4-manifold is hyperKähler if and only if it is a gravitational instanton. Compact hK_4 are K3 and T^4. Non-compact hK_4 are ALE (asymptotically locally Euclidean), ALF (asymptotically locally flat), ALG, ALH if their volume growth rate is of order 4, 3, 2, 1. ALE spaces are resolutions of $\mathbb{H}/\Gamma$ for a finite subgroup $\Gamma < \mathrm{USp}(2)$. The quotient $\mathbb{H}/\Gamma$ can appear as a local model of an orbifold singularity in a K3 surface.

ALE hyperKähler 4-manifolds X are diffeomorphic to the minimal resolution of $\mathbb{H}/\Gamma$ for some finite $\Gamma \subset \mathrm{SU}(2)$. The metric is fixed (up to isometry) by cohomology classes $\alpha_1,\alpha_2,\alpha_3 \in H^2(X,\mathbb{R})$ such that there is no two-cycle Σ such that $\Sigma\cdot\Sigma = -2$ and all $\alpha_i(\Sigma) = 0$.

§8.5 Some algebraic constructions

The Weil algebra of a Lie algebra $\mathfrak{g}$ is the differential graded algebra $W(\mathfrak{g}) = \bigwedge^\bullet \mathfrak{g}^* \otimes S^\bullet\mathfrak{g}^*$. Generators $T^a \in \mathfrak{g}^*$ map to $A^a \in \bigwedge^1 \mathfrak{g}^*$ of grading 1 and $F^a \in S^1\mathfrak{g}^*$ of grading 2. The differential obeys $\mathrm{d}A^a = F^a - \frac{1}{2}f^a{}_{bc}A^b\wedge A^c$ and $\mathrm{d}F^a = -f^a{}_{bc}A^b\otimes F^c \in \bigwedge^1\mathfrak{g}^*\otimes S^1\mathfrak{g}^*$ where f are structure constants of $\mathfrak{g}$.

Reduction of a Lie (super)algebra $\mathfrak{g}$. If $\mathfrak{g} = V_1\oplus V_2$ with $[V_1,V_2]\subseteq V_2$ then the bracket of $\mathfrak{g}$ restricted and projected to V_1 defines a Lie (super)algebra.

S-expansion of a Lie (super)algebra $\mathfrak{g}$ by an abelian multiplicative semigroup S: Lie (super)algebra $\mathfrak{g}\times S$ with bracket $[(x,\alpha),(y,\beta)] = ([x,y],\alpha\beta)$. If $S = S_1\cup S_2$ with $S_1S_2\subseteq S_2$ (in particular if there is a zero element $0_S = 0_S\alpha = \alpha 0_S$) then by reduction we get a Lie (super)algebra structure on $\mathfrak{g}\times S_1$.

A color (super)algebra is a graded vector space with a bracket such that (for X, Y, Z with definite grading) $\mathrm{gr}[X,Y] = \mathrm{gr}\,X + \mathrm{gr}\,Y$ and $[X,Y] = -(-1)^{(\mathrm{gr}\,X,\mathrm{gr}\,Y)}[Y,X]$ and Jacobi identity $[X,[Y,Z]](-1)^{(\mathrm{gr}\,Z,\mathrm{gr}\,X)} + [Y,[Z,X]](-1)^{(\mathrm{gr}\,X,\mathrm{gr}\,Y)} + [Z,[X,Y]](-1)^{(\mathrm{gr}\,Y,\mathrm{gr}\,Z)} = 0$, where $(\bullet,\bullet)$ is some bilinear mapping into $\mathbb{C}/(2\mathbb{Z})$.

§8.6 Other

A fuzzy space is d Hermitian matrices X^a ("coordinates") acting on some Hilbert space H. The dispersion of $\psi\in H$ is $\delta_\psi = \sum_a\big(\langle\psi|(X^a)^2|\psi\rangle - \langle\psi|X^a|\psi\rangle^2\big)$.