Tables for supersymmetry.

Based on [1-2].

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1. Lie (super)algebras

Simple complex Lie algebras. Infinite series $A_{n\geq 1}$, $B_{n\geq 1}$, $C_{n\geq 1}$, $D_{n\geq 2}$ with $A_1=B_1=C_1$, $B_2=C_2$, $D_2=A_1\oplus A_1$, $D_3=A_3$. Five exceptions with $\dim(E_6)=78$, $\dim(E_7)=133$, $\dim(E_8)=248$, $\dim(F_4)=52$, $\dim(G_2)=14$.

Type	Dimension	Lie algebra
A_n	$n(n+2)$	$sl(n+1,\mathbb{C})=\{\text{traceless}\}$
B_n	$n(2n+1)$	$so(2n+1,\mathbb{C})=\{\text{antisymmetric}\}$
C_n	$n(2n+1)$	$sp(2n,\mathbb{C})=\{\left(\begin{smallmatrix}0 & \mathbb{1}_n\\ -\mathbb{1}_n & 0\end{smallmatrix}\right)\times\text{symmetric}\}$
D_n	$n(2n-1)$	$so(2n,\mathbb{C})=\{\text{antisymmetric}\}$

Real forms. Denote $sl(n)=sl(n,\mathbb{R})$, $sp(2n)=sp(2n,\mathbb{R})$, $su^*(2n)=sl(n,\mathbb{H})$, $so^*(2n)=o(n,\mathbb{H})$, $usp(2m,2n)=u(m,n,\mathbb{H})$. A Lie algebra is "compact" if it exponentiates to a compact Lie group. In $E_{r(s)}$, s is the number of (non-compact) − (compact) generators.

	Real form	Max "compact" subalgebra	Range
$sl(n)$	$su(n)$	"compact"	
	$sl(n)$	$so(n)$	
	$su(n-p,p)$	$su(n-p)\oplus su(p)\oplus u(1)$	$0<p<n$
	$su^*(n)$	$usp(n)$	n even
$so(n)$	$so(n)$	"compact"	
	$so(p,n-p)$	$so(p)\oplus so(n-p)$	$0<p<n$
	$so^*(2n)$	$u(n)$	n even
$sp(2n)$	$usp(2n)$	"compact"	
	$sp(2n)$	$u(n)$	
	$usp(2n-2p,2p)$	$usp(2n-2p)\oplus usp(2p)$	$0<p<n$

Real form	Max "compact" subalgebra	Real form	Max "compact" subalgebra
$E_{6(-78)}$	"compact"	$E_{8(-248)}$	"compact"
$E_{6(-26)}$	F_4	$E_{8(-24)}$	$E_{7,-133}\oplus su(2)$
$E_{6(-14)}$	$so(10)\oplus so(2)$	$E_{8(8)}$	$so(16)$
$E_{6(2)}$	$su(6)\oplus su(2)$	$G_{2(-14)}$	"compact"
$E_{6(6)}$	$usp(8)$	$G_{2(2)}$	$su(2)\oplus su(2)$
$E_{7(-133)}$	"compact"	$F_{4(-52)}$	"compact"
$E_{7(-25)}$	$E_{6,-78}\oplus so(2)$	$F_{4(-20)}$	$so(9)$
$E_{7(-5)}$	$so(12)\oplus su(2)$	$F_{4(4)}$	$usp(6)\oplus su(2)$
$E_{7(7)}$	$su(8)$		

Accidental isomorphisms.

$so(2)=u(1)$, $so(1,1)=\mathbb{R}$ — $so(4,1)=usp(2,2)$

$so(3)=su(2)=su^*(2)$ — $so(3,2)=sp(4)$

$so(2,1)=su(1,1)=sl(2)=sp(2)$ — $so(6)=su(4)$

$so(4)=su(2)\oplus su(2)$ — $so(5,1)=su^*(4)$

$so(3,1)=sl(2,\mathbb{C})=sp(2,\mathbb{C})$ — $so(4,2)=su(2,2)$

$so(2,2)=sl(2)\oplus sl(2)$ — $so(3,3)=sl(4)$

$so^*(4)=su(1,1)\oplus su(2)$ — $so^*(6)=su(3,1)$

$so(5)=usp(4)$ — $so^*(8)=so(6,2)$

Classical Lie superalgebras: the bosonic algebra acts on the fermionic generators in a completely reducible representation. This excludes Cartan-type superalgebras $W(n)$, $S(n)$, $\tilde{S}(n)$ and $H(n)$. In this table, $m,n\geq 1$ and we do not list purely bosonic Lie algebras. The factor $\mathbb{C}$ of $sl(m|n)$ must be removed if $m=n$.

	Bosonic algebra	Fermionic repr.
$sl(m\|n)$	$sl(m,\mathbb{C})\oplus sl(n,\mathbb{C})\oplus\mathbb{C}$	$(m,\overline{n})\oplus(\overline{m},n)$
$osp(m\|2n)$	$so(m,\mathbb{C})\oplus sp(2n)$	$(m,2n)$
$D(2,1,\alpha)$	$sl(2,\mathbb{C})^3$	$(2,2,2)$
$F(4)$	$so(7,\mathbb{C})\oplus sl(2,\mathbb{C})$	$(8,2)$
$G(3)$	$G_2\oplus sl(2,\mathbb{C})$	$(7,2)$
$P(m)$	$sl(m+1,\mathbb{C})$	sym$\oplus$(antisym)*
$Q(m)$	$sl(m+1,\mathbb{C})$	adjoint

Real forms of Lie superalgebras, starting from their "compact" form ($p=q=0$). $P(m)$ has no "compact" form. Here, $m,n\geq 1$, $0\leq p\leq m/2$, $0\leq q\leq n/2$. The forms su^*, osp^*, Q^* only exist for even rank; sl' only if $m=n$.

Real form	Bosonic algebra
$su(m-p,p\|n-q,q)$	$su(m-p,p)\oplus su(n-q,q)\oplus u(1)^{\S}$
$sl(m\|n)$	$sl(m)\oplus sl(n)\oplus so(1,1)^{\S}$
$sl'(n\|n)$ $(m=n)$	$sl(n,\mathbb{C})$
$su^*(m\|n)$ $(m,n$ even$)$	$su^*(m)\oplus su^*(n)\oplus so(1,1)^{\S}$
$osp(m-p,p\|2n)$	$so(m-p,p)\oplus sp(2n)$
$osp^*(m\|2n-2q,2q)$ $(m$ even$)$	$so^*(m)\oplus usp(2n-2q,2q)$
$D^p(2,1,\alpha)^{\P}$	$so(4-p,p)\oplus sl(2)$ $(p=0,1,2)$
$F^p(4)$ for $p=0,3$	$so(7-p,p)\oplus sl(2)$
$F^p(4)$ for $p=1,2$	$so(7-p,p)\oplus su(2)$
$G_s(3)$ for $s=-14,2$	$G_{2(s)}\oplus sl(2)$
$P(m)$	$sl(m+1)$
$UQ(m-p,p)$	$su(m+1-p,p)$
$Q(m)$	$sl(m+1)$
$Q^*(m)$ $(m$ odd$)$	$su^*(m+1)$

§ For $m=n$, $u(1)$ and $so(1,1)$ factors are absent. Additionally, one can project down to a single bosonic factor.

¶ The three $sl(2)$ bosonic factors of $D(2,1,\alpha)$ appear with weights 1, α and $-1-\alpha$ in fermion anticommutators. For D^0 and D^2, α is real. For D^1, $\alpha=1+ia$ with a real.

Some isomorphisms: $su(1,1|1)=sl(2|1)=osp(2|2)$ and $su(2|1)=osp(2^*|2,0)$. For $\alpha=1$, $D^p(2,1,1)=osp(4-p,p|2)$.

2. Spinors

Clifford algebra. Let h_{ab} be diagonal with s '+1' and t '−1', and $d=s+t$. The Clifford algebra $\{\Gamma_a,\Gamma_b\}=2h_{ab}$ has real dimension 2^d and is isomorphic to a matrix algebra $M_{2^\#}(\bullet)$ with

$s-t$ mod 8	0	1	2	3	4	5	6	7
$\bullet$ is	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$

Charge conjugation. $(-\eta)\Gamma_a^T=\mathcal{C}\Gamma_a\mathcal{C}^{-1}$ are conjugate for $\eta=\pm 1$ because they obey the same algebra. Get $\mathcal{C}^T=-\varepsilon\mathcal{C}$ with $\varepsilon=\pm 1$ by transposing twice. Let $\Gamma^{(n)}=\Gamma_{a_1\dots a_n}$. Using $\left(\mathcal{C}\Gamma^{(n)}\right)^T=-\epsilon(-)^{n(n-1)/2}(-\eta)^n\mathcal{C}\Gamma^{(n)}$ find which n mod 4 give symmetric $\mathcal{C}\Gamma^{(n)}$. The sum of $\binom{d}{n}$ must be $2^{\lfloor d/2\rfloor}(2^{\lfloor d/2\rfloor}+1)/2$. This fixes ϵ,η. Odd d require $\eta=(-1)^{d(d+1)/2}$ to preserve $\Gamma^{(d)}$. Even d allow two choices of signs: consult the rows $d\pm 1$.

d mod 8	n	ϵ	η
0			
1	0, 1	−1	−1
2			
3	1, 2	+1	+1
4			
5	2, 3	+1	−1
6			
7	0, 3	−1	+1

Reduced spinors. $M_{ab} \in so(s,t)$ acts as $\gamma_a\gamma_b$ on representations of the Clifford algebra. But the $2^{\lceil d/2 \rceil}$-dimensional representation is not irreducible as a representation of $so(s,t)$.

In even d, Weyl (or chiral) spinors $\Gamma^{(d)}\lambda = \pm\lambda$ have $2^{d/2-1}$ real components. Let B be defined by $\Gamma_a^* = -\eta(-1)^t B\Gamma_a B^{-1}$. Majorana spinors $\lambda^* = B\lambda$ exist for $s-t \equiv 0,\pm1,\pm2$ mod 8; the case $s-t\equiv\pm2$ requires $\eta = \mp(-1)^{d/2}$. When $s-t\equiv 3,4,5$, a set of $2n$ spinors can be symplectic Majorana: $(\lambda^I)^* = B\Omega_{IJ}\lambda^J$ for $\Omega = ((0,\mathbb{1}_n);(-\mathbb{1}_n,0))$. (Symplectic) Majorana–Weyl spinors exist for $s-t\equiv 0,4$ mod 8. The table also includes the real dimension of the minimal spinor.

d	$t\equiv 0$		1		2		3 mod 4	
1	M	1	M	1				
2	M$^-$	2	MW	1	M$^+$	2		
3	s	4	M	2	M	2	s	4
4	sW	4	M$^+$	4	MW	2	M$^-$	4
5	s	8	s	8	M	4	M	4
6	M$^+$	8	sW	8	M$^-$	8	MW	4
7	M	8	s	16	s	16	M	8
8	MW	8	M$^-$	16	sW	16	M$^+$	16
9	M	16	M	16	s	32	s	32
10	M$^-$	32	MW	16	M$^+$	32	sW	32
11	s	64	M	32	M	32	s	64
12	sW	64	M$^+$	64	MW	32	M$^-$	64

Flavour symmetries of N minimal spinors. This is also the R-symmetry of the N-extended superalgebra. For (symplectic) Majorana Weyl spinors, specify $N=(N_L,N_R)$ left/right-handed.

$$\text{M} \begin{cases} u(N) & \text{if } d \text{ even} \\ so(N) & \text{if } d \text{ odd} \end{cases}$$
$$\text{MW}: so(N_L)\times so(N_R)$$
$$\text{s}\ : usp(N)$$
$$\text{sW}\ : usp(N_L)\times usp(N_R)$$

Products of spinor representations. For odd $d=2m+1$, let $\mathcal{S}$ be a spinor representation of complex dimension 2^m. The symmetric product $S^2\mathcal{S}$ consists of k-forms with $k\equiv m$ mod 4. Since k-forms and $(d-k)$-forms are the same representation, other descriptions can be given. For the antisymmetric product $\Lambda^2\mathcal{S}$, take $k\equiv m-1$ mod 4. See the list of forms in the table.

d	1	3	5	7	9	11
$\dim_{\mathbb{C}}\mathcal{S}$	1	2	4	8	16	32
$S^2\mathcal{S}$	0	1	2	0,3	0,1,4	1,2,5
$\Lambda^2\mathcal{S}$	.	0	0,1	1,2	2,3	0,3,4

For even $d=2m$, let $\mathcal{S}_\pm$ be the Weyl spinor representations of complex dimension 2^{m-1}. The tensor product $\mathcal{S}_+\otimes\mathcal{S}_-$ consists of $(m-1-2j)$-forms for $0\le j\le (m-1)/2$. The symmetric products $S^2\mathcal{S}_\pm$ decompose into the (anti)-self-dual m-forms and $(m-4j)$-forms for $0<j\le m/4$. The antisymmetric products $\Lambda^2\mathcal{S}_\pm$ decompose into $(m-2-4j)$-forms for $0\le j\le (m-2)/4$.

d	2	4	6	8	10	12
$\dim_{\mathbb{C}}\mathcal{S}_\pm$	1	2	4	8	16	32
$S^2\mathcal{S}_\pm$	$1^\dagger$	$2^\dagger$	$3^\dagger$	$0,4^\dagger$	$1,5^\dagger$	$2,6^\dagger$
$\Lambda^2\mathcal{S}_\pm$	.	0	1	2	3	0,4
$\mathcal{S}_+\otimes\mathcal{S}_-$	0	1	0,2	1,3	0,2,4	1,3,5

Note that $S^2(\mathcal{S}_+\oplus\mathcal{S}_-) = S^2\mathcal{S}_+\oplus(\mathcal{S}_+\otimes\mathcal{S}_-)\oplus S^2\mathcal{S}_-$
$$\Lambda^2(\mathcal{S}_+\oplus\mathcal{S}_-) = \Lambda^2\mathcal{S}_+\oplus(\mathcal{S}_+\otimes\mathcal{S}_-)\oplus\Lambda^2\mathcal{S}_-$$

3. Supersymmetry algebras

The Poincaré algebra is $\mathbb{R}^{s,t}\rtimes so(s,t)$, the semi-direct product of translations by rotations. Namely, $[P_a,P_b]=0$, $[M_{ab},P_c] = 2ih_{c[a}P_{b]}$, and $[M_{ab},M^{cd}] = 4ih^{[c}_{[a}M^{d]}_{b]}$.

Super-Poincaré algebra. Add supercharges in some spinor representation Q of the Poincaré algebra (so $[P_a,Q]=0$). Their anticommutator transforms in the representation S^2Q and should include the one-form P. Depending on s,t they can include other k-forms Z, called central charges because $[P,Z]=[Z,Z]=0$. The super-Poincaré algebra is $((\mathbb{R}^{s,t}\times Z).Q)\rtimes(so(s,t)\times R)$, where the R-symmetry acts on Q. This Lie superalgebra is graded: $\mathrm{gr}(\mathbb{R}^{s,t}\times Z) = -2$, $\mathrm{gr}(Q)=-1$, and $\mathrm{gr}(so(s,t)\times R) = 0$. The supertranslations consist of $(\mathbb{R}^{s,t}\times Z).Q$.

Example: M-theory algebra. $d=10+1$ super-Poincaré algebra with $Q=$ Majorana. Since S^2Q has 1, 2, and 5-forms, there are 2-form and 5-form central charges $Z_{(2)}$ and $Z_{(5)}$ (under which M2 and M5 branes are charged):

$$\{Q_\alpha,Q_\beta\} = (\gamma^M C)_{\alpha\beta}P_M + \frac{1}{2}(\gamma_{MN}C)_{\alpha\beta}Z^{MN}_{(2)} + \frac{1}{5!}(\gamma_{MNPQR}C)_{\alpha\beta}Z^{MNPQR}_{(5)}$$

Altogether the M-theory algebra is $osp(1|32)$.

Superconformal algebras are the same as super AdS_{d+1}. The bosonic part is $so(d,2)$ and R-symmetries. As a supermatrix: $\begin{pmatrix} so(d,2) & Q+S \\ Q-S & R\end{pmatrix}$ or $so(d,2)\leftrightarrow R$. Note that $\{Q,S\}$ contains R. For $d=2$, the finite conformal algebra is $so(2,2) = so(2,1)\oplus so(2,1)$, sum of two $d=1$ algebras, so the superalgebra is sum of two $d=1$ superalgebras.

d	Superalgebra	R-symmetries	#Q+#S
1	$osp(N\|2)$	$o(N)$	$2N$
	$su(N\|1,1)$	$su(N)\oplus u(1)$ for $N\neq 2$	$4N$
	$su(2\|1,1)$	$su(2)$	8
	$osp(4^*\|2N)$	$su(2)\oplus usp(2N)$	$8N$
	$G(3)$	G_2	14
	$F^0(4)$	$so(7)$	16
	$D^0(2,1,\alpha)$	$su(2)\oplus su(2)$	8
3	$osp(N\|4)$	$so(N)$	$4N$
4	$su(2,2\|N)$	$su(N)\oplus u(1)$ for $N\neq 4$	$8N$
	$su(2,2\|4)$	$su(4)$	32
5	$F^2(4)$	$su(2)$	16
6	$osp(8^*\|N)$	$usp(N)$ (N even)	$8N$

4. Supermultiplets with spins ≤ 1

For 16 **supercharges**, there is only the vector.
For 8 **supercharges**, vector and hyper.
For 4 **supercharges**, vector, chiral, linear multiplets.
For 2 **supercharges**, vector, chiral, linear, Fermi, ...

[1] Tools for supersymmetry by Antoine Van Proeyen
[2] Various Wikipedia articles.