Tables for supersymmetry.

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1 Special functions

Multiple gamma function. For $a_i \in \mathbb{C}$ with $\operatorname{Re} a_i > 0$, $\Gamma_N(x|\vec{a}) = \prod_{\vec{n}}^{\text{reg.}}(x+\vec{n}\cdot\vec{a})^{-1} = \exp(\partial_s \sum_{\vec{n}}(x+\vec{n}\cdot\vec{a})^{-s}|_{s=0})$, where $\vec{n} \in \mathbb{Z}_{\geq 0}^N$. Here, we zeta-regularized the product; the sum is analytically continued from $\operatorname{Re} s > N$. The meromorphic $x \mapsto \Gamma_N(x|\vec{a})$ has no zero and poles at $x = -\vec{n}\cdot\vec{a}$ (simple poles for generic $\vec{a}$). $\Gamma_0(x|) = 1/x$, $\Gamma_1(x|a) = a^{x/a-1/2}\Gamma(x/a)/\sqrt{2\pi}$, $\Gamma_N(x|\vec{a}) = \Gamma_{N-1}(x|a_1,\dots,a_{N-1})\Gamma_N(x+a_N|\vec{a})$ and it is invariant under permutations of $\vec{a}$.

Plethystic exponential. Let $\mathbf{m} \subset R[[x_1,\dots,x_n]]$ be series with no constant term over a ring R. Then $\operatorname{plexp}: \mathbf{m} \to 1+\mathbf{m}$ obeys $\operatorname{plexp}[x_i^p] = 1/(1-x_i^p)$, $\operatorname{plexp}[f+g] = \operatorname{plexp}[f]\operatorname{plexp}[g]$ and $\operatorname{plexp}[\lambda f] = \operatorname{plexp}[f]^\lambda$ for $\lambda \in R$. It maps an index of single-particle states $f(x)$ to that of multiparticle states $\operatorname{plexp} f(x) = \exp\sum_{k\geq 1}\frac{1}{k}f(x_1^k,\dots,x_n^k)$.

q-Pochhammer $(a;q)_\infty = \operatorname{plexp}\frac{-a}{1-q} = \prod_{k=0}^\infty(1-aq^k)$ and finite version $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$. Products are often denoted $(a_1,\dots,a_N;q)_n = (a_1;q)_n\cdots(a_N;q)_n$. Properties: $(a;q)_{-n}(q/a;q)_n = (-q/a)^n q^{n(n-1)/2}$ and q-binomial theorem $(ax;q)_\infty/(x;q)_\infty = \sum_{n=0}^\infty x^n(a;q)_n/(q;q)_n$.

q-gamma (or basic gamma) function for $|q|<1$, $\Gamma_q(x) = (1-q)^{1-x}(q;q)_\infty/(q^x;q)_\infty$ obeys $\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$ and $\Gamma_q(x) \xrightarrow{q\to 1} \Gamma(x)$. It has simple poles at $x \in \mathbb{Z}_{\leq 0}$ and no zero.

Modular form of weight k: holomorphic on $\mathbf{H} = \{\operatorname{Im}\tau > 0\}$ and as $\tau \to \mathrm{i}\infty$ and obeys $f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$.

Dedekind eta function: $\eta(\tau) = q^{1/24}(q;q)_\infty$ for $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. $\Delta = \eta^{24}$ is a modular form of weight 12.

Theta functions: q-theta $\theta(z;q) = (z;q)_\infty(q/z;q)_\infty$ obeys $\theta(z;q) = \theta(q/z;q) = -z\theta(1/z;q)$. Variant $\theta_1(z;q) = \theta_1(\tau|u) = \mathrm{i}z^{-1/2}q^{1/12}\eta(\tau)\theta(z;q) = -\mathrm{i}z^{1/2}q^{1/8}(q;q)_\infty(qz;q)_\infty(\frac{1}{z};q)_\infty$ with $z = \mathrm{e}^{2\pi\mathrm{i}u}$.

Eisenstein series $(k\geq 1)$ $E_{2k} = 1+\frac{2}{\zeta(1-2k)}\sum_{n=1}^\infty n^{2k-1}\frac{q^n}{1-q^n}$ obeys $E_{2k}(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{2k}E_{2k}(\tau)+\frac{6}{\pi\mathrm{i}}c(c\tau+d)\delta_{k=1}$. For $k\geq 2$ it is a modular form and $E_{2k} = \frac{1}{2\zeta(2k)}\sum_{0\neq\lambda\in\mathbb{Z}+\tau\mathbb{Z}}\lambda^{-2k}$.

Elliptic gamma function $\Gamma(z;p,q) = \operatorname{plexp}\frac{z-pq/z}{(1-p)(1-q)} = \prod_{m=0}^\infty\prod_{n=0}^\infty(1-p^{m+1}q^{m+1}z^{-1})/(1-p^m q^n z)$. Obeys $\Gamma(z;p,q) = \Gamma(z;q,p) = 1/\Gamma(pq/z;p,q)$ and $\Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q)$ and $\Gamma(z;0,q) = 1/(z;q)_\infty$.

2 Lie algebras and groups (dimension $<\infty$)

2.1 Lie algebras

Complex simple Lie algebras. Infinite series $\mathfrak{a}_{n\geq 1}$, $\mathfrak{b}_{n\geq 1}$, $\mathfrak{c}_{n\geq 1}$, $\mathfrak{d}_{n\geq 2}$ with $\mathfrak{a}_1 = \mathfrak{b}_1 = \mathfrak{c}_1$, $\mathfrak{b}_2 = \mathfrak{c}_2$, $\mathfrak{d}_2 = \mathfrak{a}_1\oplus\mathfrak{a}_1$, $\mathfrak{d}_3 = \mathfrak{a}_3$.

Five exceptions with dimensions

$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{f}_4$	$\mathfrak{g}_2$
78	133	248	52	14

.

Type	Dimension	Lie algebra
$\mathfrak{a}_n$	$n(n+2)$	$\mathfrak{sl}(n+1,\mathbb{C}) = \{\text{traceless}\}$
$\mathfrak{b}_n$	$n(2n+1)$	$\mathfrak{so}(2n+1,\mathbb{C}) = \{\text{antisymmetric}\}$
$\mathfrak{c}_n$	$n(2n+1)$	$\mathfrak{sp}(2n,\mathbb{C}) = \{\left(\begin{smallmatrix}0 & \mathbb{1}_n\\ -\mathbb{1}_n & 0\end{smallmatrix}\right)\times\text{symmetric}\}$
$\mathfrak{d}_n$	$n(2n-1)$	$\mathfrak{so}(2n,\mathbb{C}) = \{\text{antisymmetric}\}$

Roots and Weyl group. The Weyl group has $\prod_i d_i$ elements where d_i are degrees of fundamental invariants. (Below, $\mathbb{1}_i$ denotes the i-th unit vector in $\mathbb{Z}^n$ and $1\leq i\neq j\leq n$.)

- $\mathfrak{a}_{n-1}$**:** (note shifted rank) roots $\mathbb{1}_i-\mathbb{1}_j$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$. The Weyl group S_n permutes the $\mathbb{1}_i$. Fundamental invariants: $x_1^k+\cdots+x_n^k$ for $2\leq k\leq n$.
- $\mathfrak{b}_n$**:** roots $\pm\mathbb{1}_i$ and $\pm\mathbb{1}_i\pm\mathbb{1}_j$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$ and $\mathbb{1}_n$. The Weyl group $\{\pm 1\}^n\rtimes S_n$ permutes and changes signs of the $\mathbb{1}_i$. Fundamental invariants: $x_1^{2k}+\cdots+x_n^{2k}$ for $2\leq 2k\leq 2n$.
- $\mathfrak{c}_n$**:** roots $\pm 2\mathbb{1}_i$ and $\pm\mathbb{1}_i\pm\mathbb{1}_j$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$ and $2\mathbb{1}_n$. Same Weyl group and invariants as $\mathfrak{b}_n$.
- $\mathfrak{d}_n$**:** roots $\pm\mathbb{1}_i\pm\mathbb{1}_j$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$ and $\mathbb{1}_{n-1}+\mathbb{1}_n$. The Weyl group $\{\pm 1\}^{n-1}\rtimes S_n$ permutes the $\mathbb{1}_i$ and changes an even number of signs. Fundamental invariants $x_1\cdots x_n$ and $x_1^{2k}+\cdots+x_n^{2k}$ for $2\leq 2k\leq 2n-2$.
- $\mathfrak{e}_8$**:** $\{\pm\mathbb{1}_i\pm\mathbb{1}_j\}\cup\{\frac{1}{2}\sum_{k=1}^8\epsilon_k\mathbb{1}_k \mid \epsilon_k = \pm 1, \prod_{k=1}^8\epsilon_k = -1\}$, simple roots $\mathbb{1}_i-\mathbb{1}_{i+1}$ and $\frac{1}{2}(-\mathbb{1}_1-\cdots-\mathbb{1}_5+\mathbb{1}_6+\mathbb{1}_7+\mathbb{1}_8)$. The $2^{14}\,3^5\,5^2\,7 = 696729600$-element Weyl group is $O_8^+(\mathbb{F}_2)$. Degrees of invariants are $\{d_i\} = \{2,8,12,14,18,20,24,30\}$, with mnemonic $1+$(primes from 7 to 29).
- $\mathfrak{e}_7$**:** roots $\sum_{i=1}^8 a_i\mathbb{1}_i$ of $\mathfrak{e}_8$ with $a_1 = \sum_{i=2}^8 a_i$, simple roots are those of $\mathfrak{e}_8$ except $\mathbb{1}_1-\mathbb{1}_2$. The $2^{10}\times 3^4\times 5\times 7 = 2903040$-element Weyl group is $\mathbb{Z}_2\times\mathrm{PSp}_6(\mathbb{F}_2)$. Degrees of invariants are $\{d_i\} = \{2,6,8,10,12,14,18\}$.

$\mathfrak{e}_6$**:** roots $\sum_{i=1}^{8} a_i \mathbb{1}_i$ of $\mathfrak{e}_8$ with $a_1 = a_2$ and $\sum_{i=3}^{8} a_i = 0$, simple roots are those of $\mathfrak{e}_8$ except $\mathbb{1}_1 - \mathbb{1}_2$ and $\mathbb{1}_2 - \mathbb{1}_3$. The $2^7\, 3^4\, 5 = 51840$-element Weyl group is $\mathrm{Aut}(\mathrm{PSp}_4(\mathbb{F}_3))$. Degrees of invariants are $\{d_i\} = \{2, 5, 6, 8, 9, 12\}$.

$\mathfrak{f}_4$**:** roots $\pm\mathbb{1}_i$, $\pm\mathbb{1}_i \pm \mathbb{1}_j$, $\frac{1}{2}(\pm\mathbb{1}_1 \pm \mathbb{1}_2 \pm \mathbb{1}_3 \pm \mathbb{1}_4)$, simple roots $\mathbb{1}_1 - \mathbb{1}_2$, $\mathbb{1}_2 - \mathbb{1}_3$, $\mathbb{1}_3$, $-\frac{1}{2}(\mathbb{1}_1 + \mathbb{1}_2 + \mathbb{1}_3 + \mathbb{1}_4)$. It has an 1152-element Weyl group and $\{d_i\} = \{2, 6, 8, 12\}$.

$\mathfrak{g}_2$**:** 12 roots $\mathrm{e}^{2\pi \mathrm{i} k/6}, \mathrm{e}^{2\pi \mathrm{i}(2k+1)/12}\sqrt{3} \in \mathbb{C}$ for $0 \le k < 6$, simple roots 1 and $\mathrm{e}^{5\pi \mathrm{i}/6}\sqrt{3}$. The 12-element Weyl group is the dihedral group D_6, and $\{d_i\} = \{2, 6\}$.

The Coxeter number $h(\mathfrak{g}) = (\dim \mathfrak{g} / \operatorname{rank} \mathfrak{g}) - 1$ is the largest d_i. A Coxeter element is the product of all simple reflections, in any order. Its eigenvalues $\mathrm{e}^{2\pi \mathrm{i}(d_i - 1)/h}$ come in conjugate pairs.

A real simple Lie algebra is a complex algebra (see above) or a real form of it. Let $\mathfrak{sp}(m, n) = \mathfrak{usp}(2m, 2n) = \mathfrak{u}(m, n, \mathbb{H})$, $\mathfrak{su}^*(2n) = \mathfrak{sl}(n, \mathbb{H}) = \{\operatorname{Re}\operatorname{Tr} M = 0 \text{ in } \mathfrak{gl}(n, \mathbb{H})\} \simeq \mathfrak{gl}(n, \mathbb{H})/\mathbb{R}$, $\mathfrak{so}^*(2n) = \mathfrak{o}(n, \mathbb{H})$. A Lie algebra is called compact if it exponentiates to a compact Lie group. In $\mathfrak{e}_{r(s)}$, s is the number of (non-compact) − (compact) generators.

	Real form	Max compact subalgebra	Range
$\mathfrak{sl}(n, \mathbb{C})$	$\mathfrak{su}(n)$	compact	
	$\mathfrak{sl}(n, \mathbb{R})$	$\mathfrak{so}(n)$	
	$\mathfrak{su}(n-p, p)$	$\mathfrak{su}(n-p) \oplus \mathfrak{su}(p) \oplus \mathfrak{u}(1)$	$0 < p < n$
	$\mathfrak{su}^*(n)$	$\mathfrak{usp}(n)$	n even
$\mathfrak{so}(n, \mathbb{C})$	$\mathfrak{so}(n)$	compact	
	$\mathfrak{so}(p, n-p)$	$\mathfrak{so}(p) \oplus \mathfrak{so}(n-p)$	$0 < p < n$
	$\mathfrak{so}^*(n)$	$\mathfrak{u}(n/2)$	n even
$\mathfrak{sp}(2n, \mathbb{C})$	$\mathfrak{usp}(2n)$	compact	
	$\mathfrak{sp}(2n, \mathbb{R})$	$\mathfrak{u}(n)$	
	$\mathfrak{usp}(2n-2p, 2p)$	$\mathfrak{usp}(2n-2p) \oplus \mathfrak{usp}(2p)$	$0 < p < n$

$\mathfrak{e}_{6(-78)}$	compact	$\mathfrak{e}_{8(-248)}$	compact
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_4$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) \oplus \mathfrak{so}(2)$	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{g}_{2(-14)}$	compact
$\mathfrak{e}_{6(6)}$	$\mathfrak{usp}(8)$	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{7(-133)}$	compact	$\mathfrak{f}_{4(-52)}$	compact
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6 \oplus \mathfrak{so}(2)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{f}_{4(4)}$	$\mathfrak{usp}(6) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(8)$		

Accidental isomorphisms.

$$\mathfrak{so}(2) = \mathfrak{u}(1), \quad \mathfrak{so}(1,1) = \mathbb{R} \qquad \mathfrak{so}(4,1) = \mathfrak{usp}(2,2)$$
$$\mathfrak{so}(3) = \mathfrak{su}(2) = \mathfrak{su}^*(2) = \mathfrak{usp}(2) \qquad \mathfrak{so}(3,2) = \mathfrak{sp}(4,\mathbb{R})$$
$$\mathfrak{so}(2,1) = \mathfrak{su}(1,1) = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{sp}(2,\mathbb{R}) \qquad \mathfrak{so}(6) = \mathfrak{su}(4)$$
$$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \qquad \mathfrak{so}(5,1) = \mathfrak{su}^*(4)$$
$$\mathfrak{so}(3,1) = \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sp}(2,\mathbb{C}) \qquad \mathfrak{so}(4,2) = \mathfrak{su}(2,2)$$
$$\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}) \qquad \mathfrak{so}(3,3) = \mathfrak{sl}(4,\mathbb{R})$$
$$\mathfrak{so}^*(4) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2) \qquad \mathfrak{so}^*(6) = \mathfrak{su}(3,1)$$
$$\mathfrak{so}(5) = \mathfrak{usp}(4) \qquad \mathfrak{so}^*(8) = \mathfrak{so}(6,2)$$

ADE classification of symmetric matrices with eigenvalues in $(-2, 2)$ and $\mathbb{Z}_{\ge 0}$ entries (adjacency matrices of ADE diagrams), of simply laced simple Lie algebras, of binary polyhedral groups Γ (discrete subgroups of SU(2)) and du Val singularities $\mathbb{C}^2/\Gamma \simeq$(zeros of Kleinian polynomial), of integers $1 \le p \le q \le r$ with $1/p + 1/q + 1/r > 1$, of singularities with no moduli (Arnold) hence of $\mathcal{N} = 2$ minimal models ($c < 3$), of $\mathcal{N} = 0$ unitary minimal models ($c < 1$), of quivers of finite type,. . .

$\mathfrak{g}$	(p, q, r)	Kleinian polynomial
$\mathfrak{a}_k$	$(1, q, 1+k-q)$	$w^2 + x^2 + y^{k+1}$
$\mathfrak{d}_k$	$(2, 2, k-2)$	$w^2 + x^2 y + y^{k-1}$
$\mathfrak{e}_6$	$(2, 3, 3)$	$w^2 + x^3 + y^4$
$\mathfrak{e}_7$	$(2, 3, 4)$	$w^2 + x^3 + xy^3$
$\mathfrak{e}_8$	$(2, 3, 5)$	$w^2 + x^3 + y^5$

2.2 Lie groups

Basics. The identity component G_0 is a normal subgroup: G/G_0 is the group of components. The maximal compact subgroup K is unique up to conjugation.

Every compact connected Lie group K is a quotient of $\mathrm{U}(1)^n \times \prod_{i=1}^{m} K_i$ by a finite subgroup Γ of its center, where K_i are simple, compact, simply-connected, connected. Then $\pi_1(K)/\mathbb{Z}^n \simeq \Gamma$ for some embedding $\mathbb{Z}^n \hookrightarrow \pi_1(K)$, and the center of K is $Z(K) = \big(U(1)^n \times \prod_{i=1}^{m} Z(K_i)\big)/\Gamma$.

Center of all such K_i: $Z(\mathrm{SU}(n)) = \mathbb{Z}_n$, $Z(\mathrm{USp}(2n)) = \mathbb{Z}_2$, $Z(\mathrm{Spin}(n \ge 3)) = (\mathbb{Z}_2 \text{ for } n \text{ odd}, \mathbb{Z}_4 \text{ for } n/2 \text{ odd}, \mathbb{Z}_2^2 \text{ otherwise})$, $Z(\widetilde{\mathrm{E}}_{6(-78)}) = \mathbb{Z}_3$, $Z(\widetilde{\mathrm{E}}_{7(-133)}) = \mathbb{Z}_2$, while $\mathrm{E}_{8(-248)}$, $\mathrm{F}_{4(-52)}$, $\mathrm{G}_{2(-14)}$ have no center.

Named quotients: $\mathrm{SO}(n) = \mathrm{Spin}(n)/\mathbb{Z}_2$ and $\mathrm{P}G = G/Z(G)$ for $G = \mathrm{SU}, \mathrm{USp}, \mathrm{SO}$ (also $\mathrm{U}, \mathrm{GL}, \mathrm{SL}$). The other two quotients $\mathrm{Spin}(4n)/\mathbb{Z}_2$ have no name.

Real simple Lie groups are the simply-connected G (classified by simple Lie algebras) and their quotients by a subgroup $\Gamma \subset Z(G)$. One has $Z(G/\Gamma) = Z(G)/\Gamma$ and $\pi_1(G/\Gamma) = \Gamma$. All G/Γ are covers of the center-free $G_{\mathrm{cf}} = G/Z(G)$, and are classified by quotients of $\pi_1(G_{\mathrm{cf}}) = \pi_1(K)$ where $K \subset G_{\mathrm{cf}}$ is the maximal compact subgroup.

For each real simple Lie algebra $\mathfrak{g}$, we write: G_{cf} as a quotient of its algebraic universal cover $\widetilde{G}_{\mathrm{alg}}$ (largest embeddable in $\mathrm{GL}(N < \infty)$) by the algebraic π_1; the (topological) π_1; the real rank r_{Re}; and K. Below, $\iota(l) = (1 \text{ for } l \text{ odd}, 2 \text{ otherwise})$, $p + q = n$ with $p, q \ge 1$, and $2k = n$ when n is even. For $\mathfrak{sl}(2)$ use $\mathrm{SU}(2) = \mathrm{Sp}(2)$, $\mathrm{SL}(2, \mathbb{R}) = \mathrm{Sp}(2, \mathbb{R})$, $\mathrm{SL}(2, \mathbb{C}) = \mathrm{Sp}(2, \mathbb{C})$.

	$\widetilde{G}_{\mathrm{alg}}/\pi_1^{\mathrm{alg}}(G_{\mathrm{cf}})$	K	π_1	r_{Re}
$\mathfrak{sl}(n \ge 3)$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathbb{Z}_n$	0
	$\mathrm{SL}(n, \mathbb{R})/\mathbb{Z}_{\iota(n)}$	$\mathrm{PSpin}(n)^{\ddagger\S}$	$Z(\mathrm{Spin}(n))^{\ddagger\S}$	$n-1$
	$\mathrm{SU}(p, q)/\mathbb{Z}_{p+q}$	$\frac{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}{\mathbb{Z}_{pq/\gcd(p,q)}}\P$	$\mathbb{Z}$	$\min(p, q)$
	$\mathrm{SU}^*(2k)/\mathbb{Z}_2$	$\mathrm{USp}(2k)/\mathbb{Z}_2$	$\mathbb{Z}_2$	$k-1$
	$\mathrm{SL}(n, \mathbb{C})/\mathbb{Z}_n$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathbb{Z}_n$	$n-1$
$\mathfrak{so}(n \ge 3)$	$\mathrm{PSpin}(n)^{\ddagger}$	$\mathrm{PSpin}(n)$	$Z(\mathrm{Spin}(n))^{\ddagger}$	0
	$\mathrm{PSpin}(p, q)^{\ddagger}$	$\frac{\mathrm{SO}(p)\times\mathrm{SO}(q)}{\mathbb{Z}_2 \text{ if } p,\, q \text{ even}}$	$\Gamma^{\Vert}$	$\min(p, q)$
	$\mathrm{SO}^*(2k)/\mathbb{Z}_2$	$\mathrm{U}(k)/\mathbb{Z}_2$	$\mathbb{Z}_{\iota(k)} \times \mathbb{Z}$	$\lfloor k/2 \rfloor$
	$\mathrm{PSpin}(n, \mathbb{C})$	$\mathrm{PSpin}(n)$	$Z(\mathrm{Spin}(n))^{\ddagger}$	$\lfloor n/2 \rfloor$
$\mathfrak{sp}(2n \ge 2)$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	$\mathrm{Sp}(2n, \mathbb{R})/\mathbb{Z}_2$	$\mathrm{U}(n)/\mathbb{Z}_2$	$\mathbb{Z}_{\iota(n)} \times \mathbb{Z}$	n
	$\mathrm{USp}(2p, 2q)/\mathbb{Z}_2$	$\frac{\mathrm{USp}(2p)\times\mathrm{USp}(2q)}{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\min(p, q)$
	$\mathrm{Sp}(2n, \mathbb{C})/\mathbb{Z}_2$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathbb{Z}_2$	n

‡ For $r + s \ge 3$, $\mathrm{PSpin}(r, s) = \mathrm{Spin}(r, s)/Z(\mathrm{Spin}(r, s))$ and $Z(\mathrm{Spin}(r, s)) = (\mathbb{Z}_2 \text{ if } r \text{ or } s \text{ odd}, \mathbb{Z}_4 \text{ if } \frac{r+s}{2} \text{ odd, else } \mathbb{Z}_2^2)$.

§ Exception: for $n = 2$, $K = \mathrm{SO}(2)/\mathbb{Z}_2$ and $\pi_1 = \mathbb{Z}$.

¶ $K \ni \overline{(A,B,\lambda)} \mapsto \begin{pmatrix} \lambda^{q/(p+q)}A & 0 \\ 0 & \lambda^{-p/(p+q)}B \end{pmatrix} \in \mathrm{PSU}(p,q)$.

‖ $\Gamma = \pi_1(\mathrm{SO}(p)) \times \pi_1(\mathrm{SO}(q))$ for p or q odd (each factor is $\mathbb{Z}_2$ except $\pi_1(\mathrm{SO}(1)) = 0$ and $\pi_1(\mathrm{SO}(2)) = \mathbb{Z}$); otherwise $\Gamma \subset \pi_1(\mathrm{SO}(p)/\mathbb{Z}_2) \times \pi_1(\mathrm{SO}(q)/\mathbb{Z}_2)$ consists of (γ_p, γ_q) such that both or neither γ is in the corresponding $\pi_1(\mathrm{SO}) \subset \pi_1(\mathrm{SO}/\mathbb{Z}_2)$.

Discrete groups in this table should not be trusted.

$\widetilde{G}_{\mathrm{alg}}/\pi_1^{\mathrm{alg}}(G_{\mathrm{cf}})$	K	π_1	r_{Re}
$\widetilde{\mathrm{E}}_{6(-78)}/\mathbb{Z}_3$	$= \mathrm{E}_{6(-78)}$	$\mathbb{Z}_3$	0
$\mathrm{E}_{6(-26)}$	$\mathrm{F}_{4(-52)}$	1	2
$\widetilde{\mathrm{E}}_{6(-14)}/\mathbb{Z}$	$\mathrm{Spin}(10) \times \mathrm{U}(1)/?$	$\mathbb{Z}$	2
$\widetilde{\mathrm{E}}_{6(2)}/\mathbb{Z}_6$	$(\mathrm{SU}(6)/\mathbb{Z}_6) \times \mathrm{SU}(2)$	$\mathbb{Z}_6$	4
$\widetilde{\mathrm{E}}_{6(6)}/\mathbb{Z}_2$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathbb{Z}_2$	6
$\widetilde{\mathrm{E}}_6^{\mathbb{C}}/\mathbb{Z}_3$	$\mathrm{E}_{6(-78)}$	$\mathbb{Z}_3$	6
$\widetilde{\mathrm{E}}_{7(-133)}/\mathbb{Z}_2$	$= \mathrm{E}_{7(-133)}$	$\mathbb{Z}_2$	0
$\widetilde{\mathrm{E}}_{7(-25)}/\mathbb{Z}$	$\mathrm{E}_{6(-78)} \times \mathrm{U}(1)/?$	$\mathbb{Z}$	3
$\widetilde{\mathrm{E}}_{7(-5)}/\mathbb{Z}_2^2$	$\mathrm{Spin}(12) \times \mathrm{SU}(2)/\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	4
$\widetilde{\mathrm{E}}_{7(7)}/\mathbb{Z}_4$	$\mathrm{SU}(8)/\mathbb{Z}_4$	$\mathbb{Z}_4$	7
$\widetilde{\mathrm{E}}_7^{\mathbb{C}}/\mathbb{Z}_2$	$\mathrm{E}_{7(-133)}$	$\mathbb{Z}_2$	7
$\mathrm{E}_{8(-248)}$	$\mathrm{E}_{8(-248)}$	1	0
$\widetilde{\mathrm{E}}_{8(-24)}/\mathbb{Z}_2$	$\widetilde{\mathrm{E}}_{7(-133)} \times \mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\widetilde{\mathrm{E}}_{8(8)}/\mathbb{Z}_2$	$\mathrm{SO}(16)/\mathbb{Z}_2$	$\mathbb{Z}_2$	8
$\mathrm{E}_8^{\mathbb{C}}$	$\mathrm{E}_{8(-248)}$	1	8
$\mathrm{F}_{4(-52)}$	$\mathrm{F}_{4(-52)}$	1	0
$\widetilde{\mathrm{F}}_{4(-20)}/\mathbb{Z}_2$	$\mathrm{Spin}(9)/\mathbb{Z}_2$	$\mathbb{Z}_2$	1
$\widetilde{\mathrm{F}}_{4(4)}$	$\mathrm{USp}(6) \times \mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\mathrm{F}_4^{\mathbb{C}}$	$\mathrm{F}_{4(-52)}$	1	4
$\mathrm{G}_{2(-14)}$	$\mathrm{G}_{2(-14)}$	1	0
$\widetilde{\mathrm{G}}_{2(2)}/\mathbb{Z}_2$	$\mathrm{SU}(2) \times \mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\mathrm{G}_2^{\mathbb{C}}$	$\mathrm{G}_{2(-14)}$	1	4

Classical Lie groups $\pi_0(\mathrm{O}(p,q)) = \pi_0(\mathrm{O}(p)) \times \pi_0(\mathrm{O}(q))$ is $\mathbb{Z}_2^2$ for $p, q \geq 1$; the identity component $\mathrm{SO}_+(p,q)$ has a double cover $\mathrm{Spin}(p,q)$.

Accidental isomorphisms (low-rank real reductive Lie groups) $\mathbb{R}/\mathbb{Z} = \mathrm{U}(1)$; $\mathrm{SU}(2) = \mathrm{Spin}(3) \twoheadrightarrow \mathrm{SO}(3)$; ...

Homotopy. Any connected Lie group is homeomorphic to its maximal compact subgroup K times a Euclidean space $\mathbb{R}^p$. All $\pi_{j\geq 1}(K)$ are abelian and finitely generated, $\pi_2(K) = 0$, $\pi_3(K) = \mathbb{Z}^m$ where m counts simple factors in a finite cover $\mathrm{U}(1)^n \times \prod_{i=1}^m K_i \twoheadrightarrow K$, and $\pi_j(K) = \prod_{i=1}^m \pi_j(K_i)$ for $j \geq 2$.

For any G there exists $\prod_{i=1}^{\mathrm{rank}\, G} S^{2d_i - 1} \to G$ which induces isomorphisms of rational (i.e., torsion-free part of) homotopy/cohomology groups where d_i are the degrees of fundamental invariants. For compact simple K,

Group	$(2d_i - 1)$
A_n	$3, 5, \ldots, 2n+1$
$\mathrm{B}_n, \mathrm{C}_n$	$3, 7, \ldots, 4n-1$
D_n	$3, 7, \ldots, 4n-5, 2n-1$
E_6	$3, 9, 11, 15, 17, 23$
E_7	$3, 11, 15, 19, 23, 27, 35$
E_8	$3, 15, 23, 27, 35, 39, 47, 59$
F_4	$3, 11, 15, 23$
G_2	$3, 11$

$\pi_{j\geq 2}(G)$ has a factor $\mathbb{Z}$ for each S^j above, and some torsion. Explicitly, $\pi_j(\mathrm{SU}(n))$ is $\mathbb{Z}$ for odd $j < 2n$, 0 for even $j < 2n$, and is pure torsion for $j \geq 2n$. Similarly, $\pi_{j<4n+2}(\mathrm{USp}(2n))$ is $\mathbb{Z}$ for $j \equiv 3, 7 \bmod 8$, $\mathbb{Z}_2$ for $j \equiv 4, 5 \bmod 8$, and 0 otherwise.

2.3 Simple Lie superalgebras

Classical Lie superalgebras: the bosonic algebra acts on the fermionic generators in a completely reducible representation. This excludes Cartan-type superalgebras $\mathfrak{w}(n)$, $\mathfrak{s}(n)$, $\tilde{\mathfrak{s}}(n)$ and $\mathfrak{h}(n)$. In this table, $m, n \geq 1$ and we do not list purely bosonic Lie algebras. The factor $\mathbb{C}$ of $\mathfrak{sl}(m|n)$ must be removed if $m = n$.

	Bosonic algebra	Fermionic repr.
$\mathfrak{sl}(m\|n)$	$\mathfrak{sl}(m,\mathbb{C}) \oplus \mathfrak{sl}(n,\mathbb{C}) \oplus \mathbb{C}$	$(m, \overline{n}) \oplus (\overline{m}, n)$
$\mathfrak{osp}(m\|2n)$	$\mathfrak{so}(m,\mathbb{C}) \oplus \mathfrak{sp}(2n,\mathbb{R})$	$(m, 2n)$
$\mathfrak{d}(2,1,\alpha)$	$\mathfrak{sl}(2,\mathbb{C})^3$	$(2,2,2)$
$\mathfrak{f}(4)$	$\mathfrak{so}(7,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C})$	$(8,2)$
$\mathfrak{g}(3)$	$\mathfrak{g}_2 \oplus \mathfrak{sl}(2,\mathbb{C})$	$(7,2)$
$\mathfrak{p}(m)$	$\mathfrak{sl}(m+1,\mathbb{C})$	sym $\oplus$ (antisym)*
$\mathfrak{q}(m)$	$\mathfrak{sl}(m+1,\mathbb{C})$	adjoint

Real forms of Lie superalgebras, starting from their compact form ($p = q = 0$). $\mathfrak{p}(m)$ has no compact form. Here, $m, n \geq 1$, $0 \leq p \leq m/2$, $0 \leq q \leq n/2$. The forms $\mathfrak{su}^*$, $\mathfrak{osp}^*$, $\mathfrak{q}^*$ only exist for even rank; $\mathfrak{sl}'$ only if $m = n$.

Real form	Bosonic algebra
$\mathfrak{su}(m-p,p\|n-q,q)$	$\mathfrak{su}(m-p,p) \oplus \mathfrak{su}(n-q,q) \oplus \mathfrak{u}(1)^\ddagger$
$\mathfrak{sl}(m\|n)$	$\mathfrak{sl}(m,\mathbb{R}) \oplus \mathfrak{sl}(n,\mathbb{R}) \oplus \mathfrak{so}(1,1)^\ddagger$
$\mathfrak{sl}'(n\|n)$ $(m = n)$	$\mathfrak{sl}(n,\mathbb{C})$
$\mathfrak{su}^*(m\|n)$ $(m, n$ even$)$	$\mathfrak{su}^*(m) \oplus \mathfrak{su}^*(n) \oplus \mathfrak{so}(1,1)^\ddagger$
$\mathfrak{osp}(m-p,p\|2n)$	$\mathfrak{so}(m-p,p) \oplus \mathfrak{sp}(2n,\mathbb{R})$
$\mathfrak{osp}^*(m\|2n-2q,2q)$ $(m$ even$)$	$\mathfrak{so}^*(m) \oplus \mathfrak{usp}(2n-2q,2q)$
$\mathfrak{d}^p(2,1,\alpha)$ §	$\mathfrak{so}(4-p,p) \oplus \mathfrak{sl}(2,\mathbb{R})$ $(p = 0,1,2)$
$\mathfrak{f}^p(4)$ for $p = 0, 3$	$\mathfrak{so}(7-p,p) \oplus \mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{f}^p(4)$ for $p = 1, 2$	$\mathfrak{so}(7-p,p) \oplus \mathfrak{su}(2)$
$\mathfrak{g}_s(3)$ for $s = -14, 2$	$\mathfrak{g}_{2(s)} \oplus \mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{p}(m)$	$\mathfrak{sl}(m+1,\mathbb{R})$
$\mathfrak{uq}(m-p,p)$	$\mathfrak{su}(m+1-p,p)$
$\mathfrak{q}(m)$	$\mathfrak{sl}(m+1,\mathbb{R})$
$\mathfrak{q}^*(m)$ $(m$ odd$)$	$\mathfrak{su}^*(m+1)$

‡ For $m = n$, $\mathfrak{u}(1)$ and $\mathfrak{so}(1,1)$ factors are absent. Additionally, one can project down to a single bosonic factor.

§ The three $\mathfrak{sl}(2)$ bosonic factors of $\mathfrak{d}(2,1,\alpha)$ appear with weights 1, α and $-1-\alpha$ in fermion anticommutators. For $\mathfrak{d}^0$ and $\mathfrak{d}^2$, α is real. For $\mathfrak{d}^1$, $\alpha = 1 + ia$ with a real.

Some isomorphisms: $\mathfrak{su}(1,1|1) = \mathfrak{sl}(2|1) = \mathfrak{osp}(2|2)$ and $\mathfrak{su}(2|1) = \mathfrak{osp}^*(2|2,0)$ and $\mathfrak{d}^p(2,1,\alpha = 1) = \mathfrak{osp}(4-p,p|2)$ and $\mathfrak{osp}(6,2|4) = \mathfrak{osp}^*(8|4)$.

2.4 Lie supergroups

2.5 Representations of Lie (super)algebras/groups

3 Spinors

Clifford algebra. Let h_{ab} be diagonal with s '+1' and t '−1', and $d = s + t$. The Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2h_{ab}$ has real dimension 2^d and is isomorphic to a matrix algebra $M_{2^\#}(\bullet)$ with

$s - t \bmod 8$	0	1	2	3	4	5	6	7
$\bullet$ is	$\mathbb{R}$	$\mathbb{R} \oplus \mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$

Charge conjugation. $(-\eta)\Gamma_a^T = \mathcal{C}\Gamma_a\mathcal{C}^{-1}$ are conjugate for $\eta = \pm 1$ because they obey the same algebra. Get $\mathcal{C}^T = -\varepsilon\mathcal{C}$ with $\varepsilon = \pm 1$ by transposing twice. Let $\Gamma^{(n)} = \Gamma_{a_1\dots a_n}$. Using $\left(\mathcal{C}\Gamma^{(n)}\right)^T = -\epsilon(-)^{n(n-1)/2}(-\eta)^n\mathcal{C}\Gamma^{(n)}$ find which n mod 4 give symmetric $\mathcal{C}\Gamma^{(n)}$. The sum of $\binom{d}{n}$ must be $2^{\lfloor d/2\rfloor}(2^{\lfloor d/2\rfloor}+1)/2$. This fixes ϵ, η. Odd d require $\eta = (-1)^{d(d+1)/2}$ to preserve $\Gamma^{(d)}$. Even d allow two choices of signs: consult the rows $d \pm 1$.

d mod 8	n	ϵ	η
$0\langle 1$	0, 1	-1	-1
$2\langle 3$	1, 2	$+1$	$+1$
$4\langle 5$	2, 3	$+1$	-1
$6\langle 7$	0, 3	-1	$+1$

Reduced spinors. $M_{ab} \in \mathfrak{so}(s,t)$ acts as $\gamma_a\gamma_b$ on representations of the Clifford algebra. But the $2^{\lceil d/2\rceil}$-dimensional representation is not irreducible as a representation of $\mathfrak{so}(s,t)$.

In even d, Weyl (or chiral) spinors $\Gamma^{(d)}\lambda = \pm\lambda$ have $2^{d/2-1}$ real components. Let B be defined by $\Gamma_a^* = -\eta(-1)^t B\Gamma_a B^{-1}$. Majorana spinors $\lambda^* = B\lambda$ exist for $s-t \equiv 0, \pm 1, \pm 2$ mod 8; the case $s-t \equiv \pm 2$ requires $\eta = \mp(-1)^{d/2}$. When $s-t \equiv 3,4,5$, a set of $2n$ spinors can be symplectic Majorana: $(\lambda^I)^* = B\Omega_{IJ}\lambda^J$ for $\Omega = ((0,\mathbb{1}_n);(-\mathbb{1}_n,0))$. (Symplectic) Majorana–Weyl spinors exist for $s-t \equiv 0,4$ mod 8. The table also includes the real dimension of the minimal spinor.

d		$t \equiv 0$		1		2		3 mod 4	
1	(D 2)	M	1	M	1				
2	(W 2)	M^-	2	MW	1	M^+	2		
3	(D 4)	s	4	M	2	M	2	s	4
4	(W 4)	sW	4	M^+	4	MW	2	M^-	4
5	(D 8)	s	8	s	8	M	4	M	4
6	(W 8)	M^+	8	sW	8	M^-	8	MW	4
7	(D 16)	M	8	s	16	s	16	M	8
8	(W16)	MW	8	M^-	16	sW	16	M^+	16
9	(D 32)	M	16	M	16	s	32	s	32
10	(W32)	M^-	32	MW	16	M^+	32	sW	32
11	(D 64)	s	64	M	32	M	32	s	64
12	(W64)	sW	64	M^+	64	MW	32	M^-	64

Flavour symmetries of N minimal spinors. This is also the R-symmetry of the N-extended superalgebra. For (symplectic) Majorana Weyl spinors, specify $N = (N_L, N_R)$ left/right-handed.

$$\mathrm{M}\ \begin{cases} \mathfrak{u}(N) & \text{if } d \text{ even} \\ \mathfrak{so}(N) & \text{if } d \text{ odd} \end{cases}$$

MW: $\mathfrak{so}(N_L) \times \mathfrak{so}(N_R)$
s : $\mathfrak{usp}(2N)$
sW : $\mathfrak{usp}(2N_L) \times \mathfrak{usp}(2N_R)$

E.g., Lorentzian 6d (2,0) has $\mathfrak{usp}(4) \times \mathfrak{usp}(0)$ R-symmetry.

Products of spinor representations. For odd $d = 2m+1$, let $\mathcal{S}$ be a spinor representation of complex dimension 2^m. The symmetric product $S^2\mathcal{S}$ consists of k-forms with $k \equiv m$ mod 4. Since k-forms and $(d-k)$-forms are the same representation, other descriptions can be given. For the antisymmetric product $\bigwedge^2\mathcal{S}$, take $k \equiv m-1$ mod 4. See the list of forms in the table.

d	1	3	5	7	9	11
$\dim_{\mathbb{C}}\mathcal{S}$	1	2	4	8	16	32
$S^2\mathcal{S}$	0	1	2	0, 3	0, 1, 4	1, 2, 5
$\bigwedge^2\mathcal{S}$	.	0	0, 1	1, 2	2, 3	0, 3, 4

For even $d = 2m$, let $\mathcal{S}_\pm$ be the Weyl spinor representations of complex dimension 2^{m-1}. The tensor product $\mathcal{S}_+ \otimes \mathcal{S}_-$ consists of $(m-1-2j)$-forms for $0 \le j \le (m-1)/2$. The symmetric products $S^2\mathcal{S}_\pm$ decompose into the (anti)-self-dual m-forms and $(m-4j)$-forms for $0 < j \le m/4$. The antisymmetric products $\bigwedge^2\mathcal{S}_\pm$ decompose into $(m-2-4j)$-forms for $0 \le j \le (m-2)/4$.

d	2	4	6	8	10	12
$\dim_{\mathbb{C}}\mathcal{S}_\pm$	1	2	4	8	16	32
$S^2\mathcal{S}_\pm$	$1^\dagger$	$2^\dagger$	$3^\dagger$	$0, 4^\dagger$	$1, 5^\dagger$	$2, 6^\dagger$
$\bigwedge^2\mathcal{S}_\pm$	.	0	1	2	3	0, 4
$\mathcal{S}_+ \otimes \mathcal{S}_-$	0	1	0, 2	1, 3	0, 2, 4	1, 3, 5

Note that $S^2(\mathcal{S}_+ \oplus \mathcal{S}_-) = S^2\mathcal{S}_+ \oplus (\mathcal{S}_+ \otimes \mathcal{S}_-) \oplus S^2\mathcal{S}_-$

$$\bigwedge^2(\mathcal{S}_+ \oplus \mathcal{S}_-) = \bigwedge^2\mathcal{S}_+ \oplus (\mathcal{S}_+ \otimes \mathcal{S}_-) \oplus \bigwedge^2\mathcal{S}_-$$

4 Supersymmetry algebras

The Poincaré algebra is $\mathbb{R}^{s,t} \rtimes \mathfrak{so}(s,t)$, the semi-direct product of translations by rotations. Namely, $[P_a, P_b] = 0$, $[M_{ab}, P_c] = 2ih_{c[a}P_{b]}$, and $[M_{ab}, M^{cd}] = 4ih^{[c}_{[a}M^{d]}_{b]}$.

Super-Poincaré algebra. Add supercharges in some spinor representation Q of the Poincaré algebra (so $[P_a, Q] = 0$). Their anticommutator transforms in the representation S^2Q and should include the one-form P. Depending on s,t they can include other k-forms Z, called central charges because $[P,Z] = [Z,Z] = 0$. The super-Poincaré algebra is $((\mathbb{R}^{s,t} \times Z).Q) \rtimes (\mathfrak{so}(s,t) \times R)$, where the R-symmetry acts on Q. This Lie superalgebra is graded: $\mathrm{gr}(\mathbb{R}^{s,t} \times Z) = -2$, $\mathrm{gr}(Q) = -1$, and $\mathrm{gr}(\mathfrak{so}(s,t) \times R) = 0$. The supertranslations consist of $(\mathbb{R}^{s,t} \times Z).Q$.

Example: M-theory algebra. $d = 10+1$ super-Poincaré algebra with $Q =$ Majorana. Since S^2Q has 1, 2, and 5-forms, there are 2-form and 5-form central charges $Z_{(2)}$ and $Z_{(5)}$ (under which M2 and M5 branes are charged):

$$\{Q_\alpha, Q_\beta\} = (\gamma^M C)_{\alpha\beta}P_M + \frac{1}{2}(\gamma_{MN}C)_{\alpha\beta}Z^{MN}_{(2)} + \frac{1}{5!}(\gamma_{MNPQR}C)_{\alpha\beta}Z^{MNPQR}_{(5)}$$

Altogether the M-theory algebra is $\mathfrak{osp}(1|32)$.

Superconformal algebras are the same as super AdS_{d+1}. The bosonic part is $\mathfrak{so}(d,2)$ and R-symmetries. As a supermatrix: $\begin{pmatrix} \mathfrak{so}(d,2) & Q+S \\ Q-S & R \end{pmatrix}$ or $\mathfrak{so}(d,2) \leftrightarrow R$. Note that $\{Q,S\}$ contains R. For $d=2$, the finite conformal algebra is $\mathfrak{so}(2,2) = \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$, sum of two $d=1$ algebras, so the superalgebra is sum of two $d=1$ superalgebras.

d	Superalgebra	R-symmetries	#Q+#S
1	$\mathfrak{osp}(N\|2)$	$\mathfrak{o}(N)$	$2N$
	$\mathfrak{su}(N\|1,1)$	$\mathfrak{su}(N)\oplus\mathfrak{u}(1)$ for $N\neq 2$	$4N$
	$\mathfrak{su}(2\|1,1)$	$\mathfrak{su}(2)$	8
	$\mathfrak{osp}(4^*\|2N)$	$\mathfrak{su}(2)\oplus\mathfrak{usp}(2N)$	$8N$
	$\mathfrak{g}(3)$	$\mathfrak{g}_2$	14
	$\mathfrak{f}^0(4)$	$\mathfrak{so}(7)$	16
	$\mathfrak{d}^0(2,1,\alpha)$	$\mathfrak{su}(2)\oplus\mathfrak{su}(2)$	8
3	$\mathfrak{osp}(N\|4)$	$\mathfrak{so}(N)$	$4N$
4	$\mathfrak{su}(2,2\|N)$	$\mathfrak{su}(N)\oplus\mathfrak{u}(1)$ for $N\neq 4$	$8N$
	$\mathfrak{su}(2,2\|4)$	$\mathfrak{su}(4)$	32
5	$\mathfrak{f}^2(4)$	$\mathfrak{su}(2)$	16
6	$\mathfrak{osp}(8^*\|N)$	$\mathfrak{usp}(N)$ (N even)	$8N$

Dimensional reduction of Euclidean/Lorentzian supersymmetry algebras. 10d $\mathcal{N}=1\to$ 6d $\mathcal{N}=(1,1)$ or $(2,0)$? $\to$ 5d $\mathcal{N}=2\to$ 4d $\mathcal{N}=4\to$ 3d $\mathcal{N}=8$. Also 6d $\mathcal{N}=(1,0)\to$ 5d $\mathcal{N}=1\to$ 4d $\mathcal{N}=2\to$ 3d $\mathcal{N}=4\to$ 2d $\mathcal{N}=(4,4)$. Also 4d $\mathcal{N}=1\to$ 3d $\mathcal{N}=2\to$ 2d $\mathcal{N}=(2,2)$.

Explicit supersymmetry algebras 4d $\mathcal{N}=2$ $\{Q^A_\alpha,\overline{Q}^B_{\dot\alpha}\}=\epsilon^{AB}P_{\alpha\dot\alpha}$

Supersymmetry on symmetric curved spaces 4d $\mathcal{N}=2$ supersymmetry on S^4 is $\mathfrak{osp}(2|4)$. 2d $\mathcal{N}=(2,2)$ supersymmetry on S^2 is $\mathfrak{osp}(2|2)$.

5 Supermultiplets

5.1 Spin ≤ 1 supermultiplets

For 16 **supercharges**, there is only the vector.

For 8 **supercharges**, vector and hyper.

For 4 **supercharges**, vector, chiral, linear multiplets.

For 2 **supercharges**, vector, chiral, linear, Fermi, …

5.2 Other supermultiplets

6d $\mathcal{N}=(1,0)$ tensor multiplet (contains one scalar), reduces to 4d $\mathcal{N}=2$ vector.

6d $\mathcal{N}=(1,0)$ supergravity multiplet, reduces to 4d $\mathcal{N}=2$ supergravity multiplet and two vectors.

6 Supersymmetric gauge theories

A gauge group is a compact reductive Lie group G such as $(SU(3)\times SU(2)\times U(1))/\mathbb{Z}_6$. Gauge couplings are one real parameter per simple factor in $\mathfrak{g}$.

6.1 Maximal super Yang–Mills

Data: gauge group.

Lorentzian 10d $\mathcal{N}=1$ SYM is anomalous unless the gauge group is abelian. Its dimensional reductions are anomaly-free and have one gauge field, $10-d$ scalars and $\mathcal{N}$ (symplectic or Majorana, and Weyl or not) spinors. The Lagrangian's R-symmetry Spin$(10-d)$ is contained in the automorphism group of the superalgebra (they coincide for $d\geq 5$).

dim.	$\mathcal{N}$ spinors	autom. $\supset$ R-sym.
10d	$(1,0)$ MW	
9d	1 M	
8d	1 M	$\mathrm{U}(1)=\mathrm{Spin}(2)$
7d	1 s	$\mathrm{USp}(2)=\mathrm{Spin}(3)$
6d	$(1,1)$ sW	$\mathrm{USp}(2)^2=\mathrm{Spin}(4)$
5d	2 s	$\mathrm{USp}(4)=\mathrm{Spin}(5)$
4d	4 M	$\mathrm{U}(4)\supset\mathrm{Spin}(6)$
3d	8 M	$\mathrm{Spin}(8)\supset\mathrm{Spin}(7)$
2d	$(8,8)$ MW	$\mathrm{Spin}(8)^2\supset\mathrm{Spin}(8)$
1d	16 M	$\mathrm{Spin}(16)\supset\mathrm{Spin}(9)$

6.2 5d $\mathcal{N}=1$ SCFTs

built from 5-brane diagrams or UV fixed point of gauge theory.

SU(2) SYM with $N_f\leq 7$ fundamental hypermultiplets has $\mathrm{SO}(2N_f)\times\mathrm{U}(1)_T\subset\mathrm{E}_{N_f+1}$ flavor symmetry enhancement. For $N_f=0$, non-trivial "θ" in $\pi_4(\mathrm{SU}(2))=\mathbb{Z}_2$ gives the $\widetilde{E}_1$ theory with $\mathrm{U}(1)_T$ symmetry only.

6.3 4d $\mathcal{N}=4$

Data: gauge group, and for each simple fact a gauge coupling and theta angle: $\tau=\theta/(2\pi)+4\pi i/g^2$.

6.4 4d $\mathcal{N}=2$

Data: gauge group, representation for half-hypermultiplets.

There can be no continuous flavor symmetry enhancement.

The theory on $\mathbb{R}^4_{\epsilon_1,0}$ (Nekrasov–Shatashvili limit) $\leftrightarrow$ quantum integrable system with Planck constant ϵ_1.

Coulomb moduli $\leftrightarrow$ action variables.

Supersymmetric vacua $\leftrightarrow$ eigenstates.

Lift to $\mathbb{R}^4\times S^1$ gives K-theoretic Nekrasov partition function. The 5d theory $\leftrightarrow$ relativistic version of the integrable system.

(G,G') **Argyres–Douglas theories** (with G and G' among A_k, D_k, $\mathrm{E}_{6,7,8}$) are engineered as IIB strings on three-fold singularity $f_G(x_1,x_2)+f_{G'}(x_3,x_4)=0$ where $f_{A_k}(x,y)=x^2+y^{k+1}$ etc. (see page 2).

6.5 4d $\mathcal{N}=1$

Superpotential term $\int d^2\theta\, W$ gives a potential for scalars and Yukawa-type interactions. W is holomorphic in chiral fields and in couplings seen as background fields. Example: the kinetic term $\mathrm{Im}\int d^2\theta[\tau W_\alpha^2]$ of an abelian gauge field: W_α^2 is a chiral field.

Wess-Zumino model: chiral multiplet ϕ with $W=m\phi^2+g\phi^3$.

Pure supersymmetric Yang–Mills (SYM) classically has $\mathrm{U}(1)_R$ symmetry, broken by instantons to $\mathbb{Z}_{2h}$ with $h=C_2(\mathrm{adj})$. It confines, is mass-gapped, and has $C_2(A)$ vacua associated to breaking $\mathbb{Z}_{2h}$ to $\mathbb{Z}_2$ by gaugino condensation $\langle\lambda\lambda\rangle$. Witten index $\mathrm{Tr}(-1)^F=h$.

6.6 3d $\mathcal{N}=4$

Gauge group G and finite-dimensional symplectic representation $\mathbb{M}$ of G.

6.7 3d $\mathcal{N}=2$

Data: gauge group G, quantized trace on $\mathfrak{g}$ for the Chern–Simons term, representation V of G for chiral multiplets, G-invariant superpotential (may break R-symmetry).

6.8 1d $\mathcal{N}=4$

Data: gauge group G, representation V of G for chiral multiplets. Gauge couplings, FI parameters, superpotential W. Flavour Wilson line, twisted and real masses $v, m_1+im_2, m_3 \in \mathfrak{g}_F$ that commute.

R-symmetry: $SU(2)$, times $U(1)$ if W has charge 2. Mixing with flavour symmetries not fixed by superconformal algebra.

6.9 1d $\mathcal{N}=2$

Discrete data: gauge group G, chiral multiplets in a representation V of G, Wilson line in a unitary representation $M = M_0 \oplus M_1$ of $\mathfrak{g}$, flavour symmetry group $G_F \subseteq U(V)\times U(M_0)\times U(M_1)$ commuting with G. Gauge anomaly cancellation: $M \otimes \det^{1/2} V$ must be a representation of G.

Continuous data: gauge couplings, FI parameters, flavor Wilson line and real mass $v, \sigma \in \mathfrak{g}_F$ that commute, $\mathfrak{g}$-equivariant holomorphic odd map $Q\colon V \to \operatorname{End} M$ with $Q^2 = 0$ describing how supercharges act on M.

Special case: Fermi multiplets in representation V_f of G with G-equivariant holomorphic maps $E\colon V \to V_\mathrm{f}$ and $J\colon V \to V_\mathrm{f}^\vee$ obeying $J \cdot E = 0$ are equivalent to Wilson line in $M = \wedge V_\mathrm{f} \otimes \det^{-1/2} V_\mathrm{f}$ with $Q = E\wedge + J \lrcorner$.

R-symmetry: $U(1)$ if $Q\colon V \to \operatorname{End} M$ has charge 1. Mixing with flavour symmetries not fixed by superconformal algebra.

NLSM Chiral multiplet: scalar ϕ in a Kähler target X and fermion in holomorphic bundle $\phi^* T_X$. Wilson line depends on a complex of vector bundles $\mathcal{F}$. Fermi multiplet takes values in a holomorphic vector bundle $\mathcal{E}$ with hermitian metric, equivalent to Wilson line with $\mathcal{F} = \det^{-1/2}\mathcal{E}\otimes\wedge\mathcal{E}$. Anomaly cancellation: $\sqrt{K_X}\otimes\wedge T_X\otimes\det^{-1/2}\mathcal{E}\otimes\wedge\mathcal{E}\otimes\mathcal{F}$ is a well-defined vector bundle on X.

7 Other theories

7.1 Two-dimensional conformal field theories

Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$ where $m\in\mathbb{Z}$. Adjoint $L_n^\dagger = L_{-n}$ and $c^\dagger = c$.

$\mathcal{N}=1$ **super-Virasoro algebra** additionally $[L_m, G_r] = (m/2-r)G_{m+r}$ and $\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2-1/4)\delta_{r+s,0}$ where either $r\in\mathbb{Z}$ (Ramond algebra) or $r \in \mathbb{Z}+1/2$ (Neveu–Schwarz algebra). Adjoint $G_r^\dagger = G_{-r}$.

$\mathcal{N}=2$ **super-Virasoro algebra** $[L_m, J_n] = -nJ_{m+n}$,
$[J_m, J_n] = \frac{c}{3}m\delta_{m+n,0}$, $[L_m, G_r^\pm] = (m/2-r)G_{m+r}^\pm$,
$[J_m, G_r^\pm] = \pm G_{m+r}^\pm$, $\{G_r^+, G_s^+\} = \{G_r^-, G_s^-\} = 0$,
$\{G_r^+, G_s^-\} = L_{r+s} + \frac{1}{2}(r-s)J_{r+s} + \frac{c}{6}(r^2-1/4)\delta_{r+s,0}$.
Adjoint $L_m^\dagger = L_{-m}$, $J_m^\dagger = J_{-m}$, $(G_r^\pm)^\dagger = G_{-r}^\mp$, $c^\dagger = c$. The algebras with $r\in\mathbb{Z}$ (Ramond) or $r\in\mathbb{Z}+1/2$ (Neveu–Schwarz) are isomorphic under spectral shift $\alpha_{\pm1/2}$ where $\alpha_\eta(L_n) = L_n + \eta J_n + \frac{c}{6}\eta^2\delta_{n,0}$, $\alpha_\eta(J_n) = J_n + \frac{c}{3}\eta\delta_n$, $\alpha_\eta(G_r^\pm) = G_{r\pm\eta}^\pm$. Another automorphism is $G_r^+ \leftrightarrow G_r^-$, $J_m \mapsto -J_m - \frac{c}{3}\delta_{m,0}$. We get a $\mathbb{Z}\rtimes\mathbb{Z}_2$ automorphism group.

$SW(3/2,2)$ **super-Virasoro algebra** has L, G, W, U

bc **system,** $\beta\gamma$ **system**

Liouville CFT has $c = 1+6(b+1/b)^2$ and primary operators with $h(\alpha) = \alpha(b+1/b-\alpha)$ for "momentum" $\alpha \in \frac{1}{2}(b+1/b)+i\mathbb{R}$.

Minimal model $\mathcal{M}_{p,q}$ for $p>q$ coprime is a quotient of $b = i\sqrt{p/q}$ Liouville CFT. It has $c = 1 - \frac{6(p-q)^2}{pq}$ and primary operators with $h_{r,s} = \frac{(ps-qr)^2-(p-q)^2}{4pq}$ for $0<r<p$ and $0<s<q$; no degeneracy besides $h_{r,s} = h_{p-r,q-s}$. Example: Ising model $\mathcal{M}_{4,3}$, tricritical Ising model $\mathcal{M}_{5,4}$, Yang–Lee singularity $\mathcal{M}_{5,2}$.

Unitary minimal model $\mathcal{M}_{k+2,k+1}$ is coset $\frac{\hat{\mathfrak{su}}(2)_{k-1}\times\hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_k}$

7.2 Chern–Simons

Chern–Simons $(2m-1)$-form $m\operatorname{Tr}\bigl(A\int_0^1 \mathrm{d}t(tdA+t^2A^2)^{m-1}\bigr)$.

7.3 Supergravity and strings

String actions Polyakov action $L_\mathrm{P} = \lambda^{mn}[(\partial_m X)(\partial_n X) - g_{mn}] + \frac{1}{\alpha'}\sqrt{-g}$. Using equations of motion get Nambu–Goto action $L_\mathrm{NG} = \frac{1}{\alpha'}\sqrt{-\det[(\partial_m X)(\partial_n X)]}$ or Brink–di Veccia–Howe–Deser–Zumino action $L_\mathrm{BdVHDZ} = \frac{1}{2\alpha'}\sqrt{-g}[g^{mn}(\partial_m X)(\partial_n X) - (d-2)]$ with $d=2$ the world-sheet dimension.

Pure supergravities in $4\le d\le 11$. Gravity is topological in $d=3$. The maximum number of supercharges $Q=32$ forbids $d>11$. A priori, all $Q=4k$ are possible. Focus on $32, 16, 8, 4$.

d	$Q=32$	16	8	4
11	✓			
10	$\overset{IIB}{(2,0)}$ $\overset{IIA}{(1,1)}$	$\overset{I}{(1,0)}$		
9	✓	✓		
8	✓	✓		
7	✓	✓		
6	$(2,2)$	$(2,0)$ $(1,1)$	$(1,0)$	
5	✓	✓	✓	
4	$N=8$	$N=4$	$N=2$	$N=1$

M-theory has as its low-energy limit 11d supergravity, which has two $\frac{1}{2}$-BPS membrane solutions (with 16 Killing spinors): M2-brane $\mathrm{d}s^2 = \Lambda^4\,\mathrm{d}x^2 + \frac{\mathrm{d}y^2}{\Lambda^2}$ with $\Lambda = (1+\frac{c_2 N_2 l^6}{|y|^6})^{-1/6}$, and M5-brane $\mathrm{d}s^2 = \Lambda\,\mathrm{d}x^2 + \mathrm{d}y^2/\Lambda^2$ with $\Lambda = (1+\frac{c_5 N_5 l^3}{|y|^3})^{-1/3}$, where $x\in\mathbb{R}^{p,1}$ and $y\in\mathbb{R}^{10-p}$. In the near horizon $y\to0$ these become $\mathrm{AdS}_4\times S^7$ and $\mathrm{AdS}_7\times S^4$ with 32 Killing spinors.

Branes IIA strings: D0, F1 (strings), D2, D4, $\mathrm{O4}^\pm$, $\widetilde{\mathrm{O4}}^+$, NS5, D6, D8 (wall), O8 (wall), etc.. IIB strings: D(-1), F1 (strings), D1, D3, (p,q) 5-branes (includes D5 and NS5), $\mathrm{O5}^\pm$, $\widetilde{\mathrm{O5}}^+$, D7, $\mathrm{O7}^\pm$, ON^0, etc.. M-theory: M2, M5, OM5, M9.

7.4 Integrable models

Relativistic quantum Toda chain. $H = \sum_{n=1}^N\bigl(\cos(2\eta\hat{p}_n) + g^2\cos(\eta\hat{p}_n+\eta\hat{p}_{n+1})\mathrm{e}^{x_{n+1}-x_n}\bigr)$. Its non-relativistic limit is $\eta\to0$ imaginary with $g/(i\eta\sqrt{2}) = c$ fixed.

7.5 Localization results

3d $\mathcal{N}=2$**:** $Z = \int_\mathfrak{t}\mathrm{d}u\,\frac{\prod_{\alpha\text{ root}}(2\sinh(\alpha u/2))^2}{\prod_{w\in\mathcal{R}}\cosh(wu/2)}\mathrm{e}^{\mathrm{i}k\operatorname{Tr}u^2/(4\pi)}$.

8 Manifolds

8.1 Riemannian geometry

8.2 Types of manifolds: G-structures, holonomy

Structure group. A G-structure on a manifold X (with $n=\dim_\mathbb{R} X$) is a G-subbundle of the $\mathrm{GL}(n,\mathbb{R})$-principal bundle $\mathrm{GL}(TX)$ of tangent frames, namely a global section of $\mathrm{GL}(TX)/G$.

A manifold is oriented if it has a $\mathrm{GL}_+(n,\mathbb{R}) = \{\det>0\}$ structure. Similar definitions for Riemannian manifolds etc.:

G-structure	Manifold type	Other characterization‡
$\mathrm{O}(n)$	Riemannian	metric $g > 0$
$\mathrm{SO}(n)$	oriented, Riemannian	
$\mathrm{O}(p, q)$	pseudo-Riemannian	metric of signature (p, q)
$\mathrm{SO}_+(p, q)$	pseudo-Riemannian, oriented, time-oriented	
$\mathrm{Pin}_\pm$ or Spin	(pseudo)-Riemannian $\mathrm{pin}_\pm$ or spin manifold	
$\mathrm{GL}(n/2, \mathbb{C})$	Almost complex	$\mathbb{C} \circlearrowright TX$ (i.e., $J^2 = -1$)
$\mathrm{Sp}(2n/2, \mathbb{R})$	Almost symplectic	Non-degenerate $\omega \in \Omega^2 X$
$\mathrm{U}(n/2)$	Almost Hermitian	Two compatible (g, J, ω)§
$\mathrm{U}^*(n/2)$	Almost hypercomplex¶	$J_1, J_2, J_3 \circlearrowright TX$
$\mathrm{USp}(n/2)$	Almost hyperHermitian	$(g, J_{1,2,3}, \omega_{1,2,3})$
$\mathrm{U}^*(n/2)\mathrm{USp}(2)$	Almost quaternionic¶	$\mathbb{H} \circlearrowright TX$
$\mathrm{USp}(n/2)\mathrm{USp}(2)$	Almost quaternion-Hermitian	$(g, \mathbb{H}, \omega_{1,2,3})$

‡ All sections are global. For instance, an almost complex structure is a global section J of $\operatorname{End} TX$ with $J^2 = -1$. A metric is a global section g of $S^2(T^*X)$.

§ Any two of (g, J, ω) fix the third by $\omega_{ik} = J_i{}^j g_{jk}$ if they are compatible: $J_i{}^j J_l{}^k \omega_{jk} = \omega_{il}$ or $J_i{}^j J_l{}^k g_{jk} = g_{il}$ namely ω or g is J-invariant, or $\omega_{ij} g^{jk} \omega_{kl} = -g_{il}$. In a basis $e^\beta, \bar{e}^{\bar\gamma}$ ($= \mathrm{d}z^\beta, \mathrm{d}\bar{z}^{\bar\gamma}$ for Hermitian manifolds) of $(1, 0)$ and $(0, 1)$ forms, $\omega = \frac{\mathrm{i}}{2} h_{\beta\bar\gamma}\, e^\beta \wedge \bar{e}^{\bar\gamma}$ and $g = \frac{1}{2} h_{\beta\bar\gamma}(e^\beta \otimes \bar{e}^{\bar\gamma} + \bar{e}^{\bar\gamma} \otimes e^\beta)$.

On an almost complex manifold, (p, q)-forms are wedge products $\Omega^{(p,q)} X = \bigwedge^p \big(\Omega^{(1,0)} X\big) \wedge \bigwedge^q \big(\Omega^{(0,1)} X\big)$ where J acts by $\pm\mathrm{i}$ on $\Omega^1 X = \Omega^{(1,0)} X \oplus \Omega^{(0,1)} X$. The exterior derivative is $\mathrm{d} = \mathrm{d}^{2,-1} + \mathrm{d}^{1,0} + \mathrm{d}^{0,1} + \mathrm{d}^{-1,2}$ with $\mathrm{d}^{i,j} : \Omega^{(p,q)} \to \Omega^{(p+i,q+j)}$. Dolbeault differential operators are $\partial = \mathrm{d}^{1,0}$ and $\overline{\partial} = \mathrm{d}^{0,1}$.

An almost symplectic $2m$-manifold admits the volume form $\omega^m/m!$. On an almost Hermitian manifold X it is equal to the Riemannian volume form and belongs to $\Omega^{(m,m)} X$.

¶ While almost quaternionic manifolds have a 3d subbundle of $\operatorname{End} TX$ locally spanned by J_1, J_2, J_3 with $J_i^2 = J_1 J_2 J_3 = -1$, almost hypercomplex manifolds require J_1, J_2, J_3 to be global.

Integrability. A G-structure is k-integrable (resp. integrable) near $x \in X$ if it can be trivialized to order k (resp. all orders) in a neighborhood of x. We automatically have 0-integrability.

Any Riemannian structure is 1-integrable thanks to Riemann normal coordinates. Integrability is equivalent to the Riemann curvature vanishing.

An almost complex structure is complex if (equivalently) it is integrable; it is 1-integrable; it has a vanishing Nijenhuis tensor $N_J : \bigwedge^2 X \to TX$ defined on vector fields u, v by the Lie brackets $N_J(u, v) = -J^2[u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv]$; the Lie bracket of $(1, 0)$ vector fields is a $(1, 0)$ vector field; $\mathrm{d} = \partial + \overline{\partial}$ namely $\mathrm{d}^{2,-1} = 0 = \mathrm{d}^{-1,2}$; or $\overline{\partial}^2 = 0$.

A symplectic structure is an integrable almost symplectic structure. Equivalently, it is 1-integrable: $\mathrm{d}\omega = 0$. Altogether,

$$\begin{tikzcd}
\text{general} \arrow[rr, "\omega"] \arrow[dr, "g\ddagger"] \arrow[dd, "J"'] & & \text{almost symplectic} \arrow[r, "\mathrm{d}\omega = 0"] \arrow[dd, "J \text{ or } g"] & \text{symplectic} \arrow[dd, "J"] \\
& \text{Riemannian} \arrow[dr, "\omega \text{ or } J"] & & \\
\text{almost complex} \arrow[rr, "\omega \text{ or } g\ddagger"] \arrow[d, "N_J = 0"] & & \text{almost Hermitian} \arrow[r, "\mathrm{d}\omega = 0"] \arrow[d, "N_J = 0"] \arrow[dr, "\S"] & \text{almost Kähler} \arrow[d, "N_J = 0"] \\
\text{complex} \arrow[rr, "\omega \text{ or } g\ddagger"] & & \text{Hermitian} \arrow[r, "\mathrm{d}\omega = 0"] & \text{Kähler}
\end{tikzcd}$$

(Almost) quaternionic/quaternionHermitian/quaternionKähler and (almost) hypercomplex/hyperHermitian/hyperKähler manifolds are defined by replacing J by a 3d subbundle of $\operatorname{End} TX$ or by global sections J_1, J_2, J_3 as in the table of G-structures.

‡ Since $\mathrm{GL}(n, \mathbb{R})/\mathrm{O}(n)$ is contractible, any manifold admits (non-canonically) an $\mathrm{O}(n)$-structure, namely a smooth choice of which frames are orthonormal, i.e., a Riemannian metric g. Similarly $\mathrm{GL}(n/2, \mathbb{C})/\mathrm{U}(n/2)$ is contractible so almost complex manifolds admit almost Hermitian structures.

§ An almost Hermitian manifold is Kähler if (equivalently) its $\mathrm{U}(n/2)$-structure is 1-integrable; $\mathrm{d}\omega = 0$ and $N_J = 0$; $\nabla\omega = 0$; $\nabla J = 0$; or the holonomy group is in $\mathrm{U}(n/2)$. Locally, $\omega = \mathrm{i}\partial\overline{\partial}\rho$ for some real-valued Kähler potentials ρ, and ω is invariant under Kähler transformations $\rho \to \rho + f(z) + \bar{f}(\bar{z})$.

The holonomy group at $x \in X$ of a connection ∇ on a bundle $E \to X$ is the group of symmetries of E_x arising from parallel transport along closed curves based at x.

For Riemannian manifolds X the holonomy group is defined as that of the Levi-Civita connection on the tangent bundle. It is a subgroup of $\mathrm{O}(n)$ $\big($or $\mathrm{SO}(n)$ for X orientable$\big)$ since parallel transport preserves orthogonality ($\nabla g = 0$).

If the holonomy group acts reducibly on the tangent space then X is locally (globally if X is geodesically complete) a product. Simply connected X that are locally neither products nor symmetric spaces (we give a list later) can have the following special holonomy subgroups of $\mathrm{SO}(n)$ (Berger's theorem)

Holonomy	Manifold type	$\dim_\mathbb{R}$
$\mathrm{U}(m)$	Kähler	$2m$
$\mathrm{SU}(m)$	Calabi–Yau CY_m	$2m$
$\big(\mathrm{USp}(2k) \times \mathrm{USp}(2)\big)/\mathbb{Z}_2$	quaternionic Kähler	$4k$
$\mathrm{USp}(2k)$	hyperKähler	$4k$
$\mathrm{Spin}(7)$	Spin(7) manifold	8
G_2	G_2 manifold	7

Note that $\mathrm{U}(m) \supset \mathrm{SU}(m) \supset \mathrm{USp}(m)$ implies that all hyperKähler manifolds are Calabi–Yau and thus Kähler. In general, quaternionic-Kähler manifolds are not Kähler.

A Calabi–Yau manifold is a Kähler manifold such that (equivalently) some Kähler metric has global holonomy group in $\mathrm{SU}(m)$; the structure group can be reduced to $\mathrm{SU}(m)$; or the holomorphic canonical bundle is trivial i.e., there exists a nowhere vanishing holomorphic top-form. A weaker set of equivalent conditions

todo: here

For simply connected manifolds, the conditions above are equivalent to the following (always equivalent) conditions on X: some Kähler metric has local holonomy group in $\mathrm{SU}(m)$; some Kähler metric has vanishing Ricci curvature; the first real Chern class vanishes; a positive power of the holomorphic canonical bundle is trivial; X has a finite cover with trivial holomorphic canonical bundle; X has a finite cover equal to the product of a torus and a simply connected manifold with trivial holomorphic canonical bundle.

Spin structures *todo: see `http://mathoverflow.net/questions/220502/`*

Symmetric spaces *todo: list missing*

K3 surfaces are the only CY_2: they have holonomy $\mathrm{SU}(2)$.

Yau's theorem. Fix a complex structure on a compact complex manifold X of $\dim_{\mathbb{C}} X > 1$ and vanishing real first Chern class. Any real class $H^{1,1}(X,\mathbb{C})$ of positive norm contains a unique Kähler form whose metric is Ricci flat.

(from Wikipedia on Calabi conjecture: "The Calabi conjecture states that a compact Khler manifold has a unique Khler metric in the same class whose Ricci form is any given 2-form representing the first Chern class.")

9 Dualities

9.1 Field theory dualities

2d $\mathcal{N} = (0,2)$ Gadde–Gukov–Putrov triality (IR).

2d $\mathcal{N} = (2,2)$ mirror symmetry of Calabi–Yau sigma models (exact).

2d $\mathcal{N} = (2,2)$ Hori–Tong (SU), Hori (Sp, SO groups), plus adjoint (ADE-type and $(2,2)^*$-like) dualities (IR).

2d $\mathcal{N} = (2,2)$ Hori–Vafa/Hori–Kapustin duality of gauged linear sigma models and Landau–Ginzburg models (IR).

3d Chern–Simons level-rank duality.

3d $\mathcal{N} = 2$ Aharony, Giveon–Kutasov, Aharony–Fleischer dualities (IR).

3d $\mathcal{N} = 2$ and $\mathcal{N} = 4$ mirror symmetry exchanging Coulomb and Higgs branches (IR).

4d $\mathcal{N} = 1$ Seiberg, Kutasov–Schwimmer, Brodie, Intriligator–Pouliot, Argyres–Intriligator–Leigh–Strassler, Klebanov cascade, Intriligator–Leigh–Strassler, duality (IR).

S-duality of 4d $\mathcal{N} = 2$ gauge theories (exact).

S-duality of 4d $\mathcal{N} = 4$ SYM (exact).

9.2 4d $\mathcal{N} = 1$ dualities

Seiberg: $\mathrm{SU}(N_c)$, $N_f\square$, $N_f\overline{\square}$ $\Leftrightarrow$ $\mathrm{SU}(N_f - N_c)$, $N_f\square$, $N_f\overline{\square}$, N_f^2 free, with $W = M\tilde{Q}Q$.

Seiberg: $\mathrm{SO}(N_c)$, $N_f\square \Leftrightarrow \mathrm{SO}(N_f - N_c + 4)$, $N_f\square$, #? free, $W =?$

Seiberg: $\mathrm{USp}(2N_c)$, $2N_f\square \Leftrightarrow \mathrm{USp}(2N_f - 2N_c - 4)$, $2N_f\square$, #? free, $W =?$

These three cases are self-dual when $C(R_{\text{chirals}}) = 2C(\text{adj})$, namely $N_f = 2N_c$, $N_f = 2(N_c - 2)$ and $N_f = 2(N_c + 1)$ respectively; adding an adjoint gives $\mathcal{N} = 2$ SCFTs.

9.3 String theory dualities

In this table "type IIA" etc. refer to string theories not supergravities

F-theory on K3	$\Leftrightarrow$ $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic on T^2
M-theory on K3	$\Leftrightarrow$ heterotic or type I on T^3
Type IIA on K3	$\Leftrightarrow$ heterotic or type I on T^4
M-theory on G_2-manifolds[1]	$\Leftrightarrow$ heterotic or type I on CY_3
M-theory on K3[2]	$\Leftrightarrow$ type IIA on $T^3/\mathbb{Z}_2$

10 Misc

10.1 Physics of gauge theories

Phases characterized by potential $V(R)$ (up to a constant) between quarks at distance R: Coulomb $1/R$, free electric $1/\big(R\log(R\Lambda)\big)$, free magnetic $\log(R\Lambda)/R$, Higgs (constant), confining σR.

10.2 Homology and cohomology

$H_k(\mathbb{CP}^n, M) = M$ for $0 \le k \le 2n$ even, else 0.

10.3 Homotopy groups π_n

Basic properties. $\pi_0(X,x)$ is the set of connected components. $\pi_1(X,x)$ is the fundamental group. For $k \ge 1$, $\pi_k(X,x)$ only depends on the connected component of x. $\pi_k(X \times Y, (x,y)) = \pi_k(X,x) \times \pi_k(Y,y)$.

Quotient. If G acts on connected simply-connected X then $\pi_1(X/G) = \pi_0(G)$ ($= G$ for G discrete).

Long exact sequence for a fiber bundle $F \hookrightarrow E \twoheadrightarrow B$: for base-points $b_0 \in B$ and $e_0 = f_0 \in F = p^{-1}(b_0) \subset E$, $\dots \to \pi_{i+1}(B) \to \pi_i(F) \to \pi_i(E) \to \pi_i(B) \to \dots \to \pi_0(E)$ is exact, namely each image equals the next kernel (inverse image of the constant map).

Homotopy groups of spheres are finite except $\pi_n(S^n) = \mathbb{Z}$ and $\pi_{4n-1}(S^{2n}) = \mathbb{Z} \times \text{finite}$. For $k < n$, $\pi_k(S^n) = 0$, and $\pi_{n+k}(S^n)$ is independent of n for $n \ge k+2$. All $\pi_k(S^0) = 0$, $\pi_k(S^1) = 0$ for $k \ne 1$, and $\pi_k(S^3) = \pi_k(S^2)$ for $k \ne 2$.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
S^0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z} \times \mathbb{Z}_{12}$	$\mathbb{Z}_2^2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$

$\pi_1(\mathbb{RP}^n) = \mathbb{Z}_2$ for $n \ge 2$ and $\pi_k(\mathbb{RP}^n) = \pi_k(S^n)$ for $k \ge 2$. $\pi_1(\mathbb{CP}^n) = 0$, $\pi_2(\mathbb{CP}^n) = \mathbb{Z}$, $\pi_k(\mathbb{CP}^n) = \pi_k(S^{2n+1})$ for $k \ge 3$.

Topological groups have abelian $\pi_1(G)$**.** Proofs. 1. The multiplication in G (point-wise) and concatenation of loops are two compatible group structures, hence (by Eckmann–Hilton theorem) coincide and are commutative. 2. Explicitly, for $\alpha_1, \alpha_2 \in \pi_1(G)$ loops, $(t_1,t_2) \mapsto \alpha_1(t_1)\alpha_2(t_2) \in G$ is a homotopy between $\alpha_1 \star \alpha_2$ (concatenation) along bottom and right edges, $\alpha_1 \cdot \alpha_2$ (point-wise multiplication) along the diagonal, and $\alpha_2 \star \alpha_1$ along left and top edges.

10.4 Kähler 4-manifolds

K3 surfaces are (the only besides T^4) compact complex surfaces of trivial canonical bundle. They have $h^{1,0} = 0$ (in contrast to T^4 which has *todo: value*). Their first Chern class $c_1 \in H^2(X,\mathbb{Z})$ thus vanishes. By Yau's theorem there exists a Ricci flat metric, whose holonomy is then $\mathrm{SU}(2) = \mathrm{USp}(2)$ by Berger's classification. K3 surfaces are thus Calabi–Yau (CY_2) and hyperKähler (hK_4). Their moduli space is connected and they are all diffeomorphic.

Examples of K3 surfaces. Quartic hypersurface in $\mathbb{P}^4$. Kummer surface namely resolution of $T^4/\mathbb{Z}_2$.

Non-simply connected Ricci-flat Kähler manifolds may fail to be CY_n when the restricted holonomy group is $\mathrm{SU}(n)$ but the global holonomy group is disconnected. For example an Enriques surface $\mathrm{K3}/\mathbb{Z}_2$ has a non-trivial canonical bundle.

A gravitational instanton is a metric with (anti-)self-dual curvature. A simply-connected Riemannian 4-manifold is hyperKähler if and only if it is a gravitational instanton. Compact hK_4 are K3 and T^4. Non-compact hK_4 are asymptotically locally Euclidean (ALE) spaces asymptotic to $\mathbb{H}/\Gamma$ for a finite subgroup $\Gamma < \mathrm{USp}(2)$. Many such ALE spaces are local resolutions of orbifold singularities of K3 surfaces.

ALE hyperKähler 4-manifolds X are diffeomorphic to the minimal resolution of $\mathbb{H}/\Gamma$ for some finite $\Gamma \subset SU(2)$. The metric is fixed (up to isometry) by cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X, \mathbb{R})$ such that there is no two-cycle Σ such that $\Sigma \cdot \Sigma = -2$ and all $\alpha_i(\Sigma) = 0$.

todo: Taub-NUT spaces, multi-Taub-NUT spaces, Eguchi–Hanson spaces, Gibbons–Hawking multicenter spaces. Write metric. todo: Non-explicitly: Atiyah–Hitchin space (moduli space of two SU(2) *'t Hooft-Polyakov monopoles in 4d).*

todo: The only compact CY_2 are T^4 and K3 surfaces.

todo: The only compact hypercomplex 4*-manifolds are T^4, K3 surfaces, and the Hopf surface* $((\mathbb{H}\backslash 0)/(q^{\mathbb{Z}})$ *for a quaternion* $|q| > 1$*; it is diffeomorphic to $S^3 \times S^1$).*

10.5 Some algebraic constructions

Reduction of a Lie (super)algebra $\mathfrak{g}$. If $\mathfrak{g} = V_1 \oplus V_2$ with $[V_1, V_2] \subseteq V_2$ then the bracket of $\mathfrak{g}$ restricted and projected to V_1 defines a Lie (super)algebra.

S-expansion of a Lie (super)algebra $\mathfrak{g}$ by an abelian multiplicative semigroup S: Lie (super)algebra $\mathfrak{g} \times S$ with bracket $[(x, \alpha), (y, \beta)] = ([x, y], \alpha\beta)$. If $S = S_1 \cup S_2$ with $S_1 S_2 \subseteq S_2$ (in particular if there is a zero element $0_S = 0_S \alpha = \alpha 0_S$) then by reduction we get a Lie (super)algebra structure on $\mathfrak{g} \times S_1$.

A color (super)algebra is a graded vector space with a bracket such that (for X, Y, Z with definite grading) $\mathrm{gr}[X, Y] = \mathrm{gr}\, X + \mathrm{gr}\, Y$ and $[X, Y] = -(-1)^{(\mathrm{gr}\, X, \mathrm{gr}\, Y)}[Y, X]$ and $[X, [Y, Z]](-1)^{(\mathrm{gr}\, Z, \mathrm{gr}\, X)} + [Y, [Z, X]](-1)^{(\mathrm{gr}\, X, \mathrm{gr}\, Y)} + [Z, [X, Y]](-1)^{(\mathrm{gr}\, Y, \mathrm{gr}\, Z)} = 0$, where $(\bullet, \bullet)$ is some bilinear mapping into $\mathbb{C}/(2\mathbb{Z})$.

10.6 Other

A fuzzy space is d Hermitian matrices X^a ("coordinates") acting on some Hilbert space H. The dispersion of $\psi \in H$ is $\delta_\psi = \sum_a \bigl(\langle\psi|(X^a)^2|\psi\rangle - \langle\psi|X^a|\psi\rangle^2\bigr)$.

[1] Tools for supersymmetry by Antoine Van Proeyen

[2] Various Wikipedia articles.

[3] Various ncatlab.org articles.