Tables for supersymmetry.

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§1 Special functions

Multiple gamma function. For $a_i \in \mathbb{C}$ with $\operatorname{Re} a_i > 0$, $\Gamma_N(x|\vec{a}) = \prod_{\vec{n}}^{\text{reg.}} (x + \vec{n}\cdot\vec{a})^{-1} = \exp(\partial_s \sum_{\vec{n}} (x+\vec{n}\cdot\vec{a})^{-s}|_{s=0})$, where $\vec{n} \in \mathbb{Z}_{\geq 0}^N$. Here, we zeta-regularized the product; the sum is analytically continued from $\operatorname{Re} s > N$. The meromorphic $x \mapsto \Gamma_N(x|\vec{a})$ has no zero and poles at $x = -\vec{n}\cdot\vec{a}$ (simple poles for generic $\vec{a}$). $\Gamma_0(x|) = 1/x$, $\Gamma_1(x|a) = a^{x/a-1/2}\Gamma(x/a)/\sqrt{2\pi}$, $\Gamma_N(x|\vec{a}) = \Gamma_{N-1}(x|a_1,\dots,a_{N-1})\Gamma_N(x+a_N|\vec{a})$ and it is invariant under permutations of $\vec{a}$.

Plethystic exponential. Let $\mathbf{m} \subset R[[x_1,\dots,x_n]]$ be series with no constant term over a ring R. Then $\operatorname{plexp} : \mathbf{m} \to 1+\mathbf{m}$ obeys $\operatorname{plexp}[x_i^p] = 1/(1-x_i^p)$, $\operatorname{plexp}[f+g] = \operatorname{plexp}[f]\operatorname{plexp}[g]$ and $\operatorname{plexp}[\lambda f] = \operatorname{plexp}[f]^\lambda$ for $\lambda \in R$. It maps an index of single-particle states $f(x)$ to that of multiparticle states $\operatorname{plexp} f(x) = \exp \sum_{k\geq 1} \frac{1}{k} f(x_1^k,\dots,x_n^k)$.

q-Pochhammer $(a;q)_\infty = \operatorname{plexp}\frac{-a}{1-q} = \prod_{k=0}^\infty (1-aq^k)$ and finite version $(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty$. Products are often denoted $(a_1,\dots,a_N;q)_n = (a_1;q)_n \cdots (a_N;q)_n$. Properties: $(a;q)_{-n}(q/a;q)_n = (-q/a)^n q^{n(n-1)/2}$ and q-binomial theorem $(ax;q)_\infty/(x;q)_\infty = \sum_{n=0}^\infty x^n (a;q)_n/(q;q)_n$.

q-gamma (or basic gamma) function for $|q|<1$, $\Gamma_q(x) = (1-q)^{1-x}(q;q)_\infty/(q^x;q)_\infty$ obeys $\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$ and $\Gamma_q(x) \xrightarrow{q\to 1} \Gamma(x)$. It has simple poles at $x \in \mathbb{Z}_{\leq 0}$ and no zero.

Modular form of weight k: holomorphic on $\mathbf{H} = \{\operatorname{Im}\tau > 0\}$ and as $\tau \to \mathrm{i}\infty$ and obeys $f(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^k f(\tau)$.

Dedekind eta function: $\eta(\tau) = q^{1/24}(q;q)_\infty$ for $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. $\Delta = \eta^{24}$ is a modular form of weight 12.

Theta functions: q-theta $\theta(z;q) = (z;q)_\infty (q/z;q)_\infty$ obeys $\theta(z;q) = \theta(q/z;q) = -z\theta(1/z;q)$. Variant $\theta_1(z;q) = \theta_1(\tau|u) = \mathrm{i}z^{-1/2}q^{1/12}\eta(\tau)\theta(z;q) = -\mathrm{i}z^{1/2}q^{1/8}(q;q)_\infty (qz;q)_\infty (\frac{1}{z};q)_\infty$ with $z = \mathrm{e}^{2\pi\mathrm{i}u}$.

Eisenstein series $(k\geq 1)$ $E_{2k} = 1 + \frac{2}{\zeta(1-2k)}\sum_{n=1}^\infty n^{2k-1}\frac{q^n}{1-q^n}$ obeys $E_{2k}(\frac{a\tau+b}{c\tau+d}) = (c\tau+d)^{2k}E_{2k}(\tau) + \frac{6}{\pi\mathrm{i}}c(c\tau+d)\delta_{k=1}$. For $k\geq 2$ it is a modular form and $E_{2k} = \frac{1}{2\zeta(2k)}\sum_{0\neq\lambda\in\mathbb{Z}+\tau\mathbb{Z}}\lambda^{-2k}$.

Elliptic gamma function $\Gamma(z;p,q) = \operatorname{plexp}\frac{z-pq/z}{(1-p)(1-q)} = \prod_{m=0}^\infty\prod_{n=0}^\infty (1-p^{m+1}q^{m+1}z^{-1})/(1-p^m q^n z)$. Obeys $\Gamma(z;p,q) = \Gamma(z;q,p) = 1/\Gamma(pq/z;p,q)$ and $\Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q)$ and $\Gamma(z;0,q) = 1/(z;q)_\infty$.

Polylogarithm and Riemann zeta $\zeta(s) = \operatorname{Li}_s(1)$ where $\operatorname{Li}_s(z) = \sum_{k\geq 1} z^k/k^s = \big(1/\Gamma(s)\big)\int_0^\infty t^{s-1}\mathrm{d}t/(e^t/z-1)$ obeys $\operatorname{Li}_{s+1}(z) = \int_0^z \operatorname{Li}_s(t)\mathrm{d}t/t$ and $\operatorname{Li}_1(z) = -\log(1-z)$. The dilogarithm obeys $\operatorname{Li}_2(x)+\operatorname{Li}_2(1-x) = \pi^2/6 - \log(x)\log(1-x)$ (reflection formula) and $\operatorname{Li}_2(x)+\operatorname{Li}_2(y)-\operatorname{Li}_2(xy) = \operatorname{Li}_2(t)+\operatorname{Li}_2(u)+\log(1-t)\log(1-u)$ where $x = t/(1-u)$ and $y = u/(1-t)$ (pentagon formula).

§2 Lie algebras and groups (dimension $<\infty$)

§2.1 Lie algebras

Complex simple Lie algebras. Infinite series $\mathfrak{a}_{n\geq 1}$, $\mathfrak{b}_{n\geq 1}$, $\mathfrak{c}_{n\geq 1}$, $\mathfrak{d}_{n\geq 2}$ with $\mathfrak{a}_1 = \mathfrak{b}_1 = \mathfrak{c}_1$, $\mathfrak{b}_2 = \mathfrak{c}_2$, $\mathfrak{d}_2 = \mathfrak{a}_1\oplus\mathfrak{a}_1$, $\mathfrak{d}_3 = \mathfrak{a}_3$.

Five exceptions with dimensions

$\mathfrak{e}_6$	$\mathfrak{e}_7$	$\mathfrak{e}_8$	$\mathfrak{f}_4$	$\mathfrak{g}_2$
78	133	248	52	14

Type	Dimension	Lie algebra
$\mathfrak{a}_n$	$n(n+2)$	$\mathfrak{sl}(n+1,\mathbb{C}) = \{\text{traceless}\}$
$\mathfrak{b}_n$	$n(2n+1)$	$\mathfrak{so}(2n+1,\mathbb{C}) = \{\text{antisymmetric}\}$
$\mathfrak{c}_n$	$n(2n+1)$	$\mathfrak{sp}(2n,\mathbb{C}) = \left\{\left(\begin{smallmatrix}0 & \mathbb{1}_n\\ -\mathbb{1}_n & 0\end{smallmatrix}\right)\times\text{symmetric}\right\}$
$\mathfrak{d}_n$	$n(2n-1)$	$\mathfrak{so}(2n,\mathbb{C}) = \{\text{antisymmetric}\}$

Roots and Weyl group. The Weyl group has $\prod_i d_i$ elements where d_i are degrees of fundamental invariants. (Below, $\mathbb{1}_i$ denotes the i-th unit vector in $\mathbb{Z}^n$ and $1\leq i\neq j\leq n$.)

$\mathfrak{a}_{n-1}$: (note shifted rank) roots $\mathbb{1}_i - \mathbb{1}_j$, simple roots $\mathbb{1}_i - \mathbb{1}_{i+1}$. The Weyl group S_n permutes the $\mathbb{1}_i$. Fundamental invariants: $x_1^k + \cdots + x_n^k$ for $2\leq k\leq n$.

$\mathfrak{b}_n$: roots $\pm\mathbb{1}_i$ and $\pm\mathbb{1}_i\pm\mathbb{1}_j$, simple roots $\mathbb{1}_i - \mathbb{1}_{i+1}$ and $\mathbb{1}_n$. The Weyl group $\{\pm 1\}^n \rtimes S_n$ permutes and changes signs of the $\mathbb{1}_i$. Fundamental invariants: $x_1^{2k}+\cdots+x_n^{2k}$ for $2\leq 2k\leq 2n$.

$\mathfrak{c}_n$: roots $\pm 2\mathbb{1}_i$ and $\pm\mathbb{1}_i\pm\mathbb{1}_j$, simple roots $\mathbb{1}_i - \mathbb{1}_{i+1}$ and $2\mathbb{1}_n$. Same Weyl group and invariants as $\mathfrak{b}_n$.

$\mathfrak{d}_n$: roots $\pm\mathbb{1}_i \pm \mathbb{1}_j$, simple roots $\mathbb{1}_i - \mathbb{1}_{i+1}$ and $\mathbb{1}_{n-1} + \mathbb{1}_n$. The Weyl group $\{\pm 1\}^{n-1} \rtimes S_n$ permutes the $\mathbb{1}_i$ and changes an even number of signs. Fundamental invariants $x_1 \cdots x_n$ and $x_1^{2k} + \cdots + x_n^{2k}$ for $2 \leq 2k \leq 2n-2$.

$\mathfrak{e}_8$: $\{\pm\mathbb{1}_i \pm \mathbb{1}_j\} \cup \{\frac{1}{2}\sum_{k=1}^{8} \epsilon_k \mathbb{1}_k \mid \epsilon_k = \pm 1, \prod_{k=1}^{8} \epsilon_k = -1\}$, simple roots $\mathbb{1}_i - \mathbb{1}_{i+1}$ and $\frac{1}{2}(-\mathbb{1}_1 - \cdots - \mathbb{1}_5 + \mathbb{1}_6 + \mathbb{1}_7 + \mathbb{1}_8)$. The $2^{14}\, 3^5\, 5^2\, 7 = 696729600$-element Weyl group is $O_8^+(\mathbb{F}_2)$. Degrees of invariants are $\{d_i\} = \{2, 8, 12, 14, 18, 20, 24, 30\}$, with mnemonic $1 +$ (primes from 7 to 29).

$\mathfrak{e}_7$: roots $\sum_{i=1}^{8} a_i \mathbb{1}_i$ of $\mathfrak{e}_8$ with $a_1 = \sum_{i=2}^{8} a_i$, simple roots are those of $\mathfrak{e}_8$ except $\mathbb{1}_1 - \mathbb{1}_2$. The $2^{10} \times 3^4 \times 5 \times 7 = 2903040$-element Weyl group is $\mathbb{Z}_2 \times \mathrm{PSp}_6(\mathbb{F}_2)$. Degrees of invariants are $\{d_i\} = \{2, 6, 8, 10, 12, 14, 18\}$.

$\mathfrak{e}_6$: roots $\sum_{i=1}^{8} a_i \mathbb{1}_i$ of $\mathfrak{e}_8$ with $a_1 = a_2$ and $\sum_{i=3}^{8} a_i = 0$, simple roots are those of $\mathfrak{e}_8$ except $\mathbb{1}_1 - \mathbb{1}_2$ and $\mathbb{1}_2 - \mathbb{1}_3$. The $2^7\, 3^4\, 5 = 51840$-element Weyl group is $\mathrm{Aut}(\mathrm{PSp}_4(\mathbb{F}_3))$. Degrees of invariants are $\{d_i\} = \{2, 5, 6, 8, 9, 12\}$.

$\mathfrak{f}_4$: roots $\pm\mathbb{1}_i$, $\pm\mathbb{1}_i \pm \mathbb{1}_j$, $\frac{1}{2}(\pm\mathbb{1}_1 \pm \mathbb{1}_2 \pm \mathbb{1}_3 \pm \mathbb{1}_4)$, simple roots $\mathbb{1}_1 - \mathbb{1}_2$, $\mathbb{1}_2 - \mathbb{1}_3$, $\mathbb{1}_3$, $-\frac{1}{2}(\mathbb{1}_1 + \mathbb{1}_2 + \mathbb{1}_3 + \mathbb{1}_4)$. It has an 1152-element Weyl group and $\{d_i\} = \{2, 6, 8, 12\}$.

$\mathfrak{g}_2$: 12 roots $\mathrm{e}^{2\pi \mathrm{i} k/6}, \mathrm{e}^{2\pi \mathrm{i}(2k+1)/12}\sqrt{3} \in \mathbb{C}$ for $0 \leq k < 6$, simple roots 1 and $\mathrm{e}^{5\pi \mathrm{i}/6}\sqrt{3}$. The 12-element Weyl group is the dihedral group D_6, and $\{d_i\} = \{2, 6\}$.

The Coxeter number $h(\mathfrak{g}) = (\dim \mathfrak{g} / \operatorname{rank} \mathfrak{g}) - 1$ is the largest d_i. A Coxeter element is the product of all simple reflections, in any order. Its eigenvalues $\mathrm{e}^{2\pi \mathrm{i}(d_i - 1)/h}$ come in conjugate pairs.

A real simple Lie algebra is a complex algebra (see above) or a real form of it. Let $\mathfrak{sp}(m,n) = \mathfrak{usp}(2m,2n) = \mathfrak{u}(m,n,\mathbb{H})$, $\mathfrak{su}^*(2n) = \mathfrak{sl}(n,\mathbb{H}) = \{\operatorname{Re}\operatorname{Tr} M = 0 \text{ in } \mathfrak{gl}(n,\mathbb{H})\} \simeq \mathfrak{gl}(n,\mathbb{H})/\mathbb{R}$, $\mathfrak{so}^*(2n) = \mathfrak{o}(n,\mathbb{H})$. A Lie algebra is called compact if it exponentiates to a compact Lie group. In $\mathfrak{e}_{r(s)}$, s is the number of (non-compact) $-$ (compact) generators.

	Real form	Max compact subalgebra	Range
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{su}(n)$	compact	
	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(n)$	
	$\mathfrak{su}(n-p,p)$	$\mathfrak{su}(n-p) \oplus \mathfrak{su}(p) \oplus \mathfrak{u}(1)$	$0 < p < n$
	$\mathfrak{su}^*(n)$	$\mathfrak{usp}(n)$	n even
$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{so}(n)$	compact	
	$\mathfrak{so}(p,n-p)$	$\mathfrak{so}(p) \oplus \mathfrak{so}(n-p)$	$0 < p < n$
	$\mathfrak{so}^*(n)$	$\mathfrak{u}(n/2)$	n even
$\mathfrak{sp}(2n,\mathbb{C})$	$\mathfrak{usp}(2n)$	compact	
	$\mathfrak{sp}(2n,\mathbb{R})$	$\mathfrak{u}(n)$	
	$\mathfrak{usp}(2n-2p,2p)$	$\mathfrak{usp}(2n-2p) \oplus \mathfrak{usp}(2p)$	$0 < p < n$

$\mathfrak{e}_{6(-78)}$	compact	$\mathfrak{e}_{8(-248)}$	compact
$\mathfrak{e}_{6(-26)}$	$\mathfrak{f}_4$	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_7 \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) \oplus \mathfrak{so}(2)$	$\mathfrak{e}_{8(8)}$	$\mathfrak{so}(16)$
$\mathfrak{e}_{6(2)}$	$\mathfrak{su}(6) \oplus \mathfrak{su}(2)$	$\mathfrak{g}_{2(-14)}$	compact
$\mathfrak{e}_{6(6)}$	$\mathfrak{usp}(8)$	$\mathfrak{g}_{2(2)}$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{7(-133)}$	compact	$\mathfrak{f}_{4(-52)}$	compact
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6 \oplus \mathfrak{so}(2)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{so}(9)$
$\mathfrak{e}_{7(-5)}$	$\mathfrak{so}(12) \oplus \mathfrak{su}(2)$	$\mathfrak{f}_{4(4)}$	$\mathfrak{usp}(6) \oplus \mathfrak{su}(2)$
$\mathfrak{e}_{7(7)}$	$\mathfrak{su}(8)$		

Accidental isomorphisms.

$\mathfrak{so}(2) = \mathfrak{u}(1)$, $\mathfrak{so}(1,1) = \mathbb{R}$	$\mathfrak{so}(4,1) = \mathfrak{usp}(2,2)$
$\mathfrak{so}(3) = \mathfrak{su}(2) = \mathfrak{su}^*(2) = \mathfrak{usp}(2)$	$\mathfrak{so}(3,2) = \mathfrak{sp}(4,\mathbb{R})$
$\mathfrak{so}(2,1) = \mathfrak{su}(1,1) = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{sp}(2,\mathbb{R})$	$\mathfrak{so}(6) = \mathfrak{su}(4)$
$\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$	$\mathfrak{so}(5,1) = \mathfrak{su}^*(4)$
$\mathfrak{so}(3,1) = \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{sp}(2,\mathbb{C})$	$\mathfrak{so}(4,2) = \mathfrak{su}(2,2)$
$\mathfrak{so}(2,2) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{so}(3,3) = \mathfrak{sl}(4,\mathbb{R})$
$\mathfrak{so}^*(4) = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{su}(2)$	$\mathfrak{so}^*(6) = \mathfrak{su}(3,1)$
$\mathfrak{so}(5) = \mathfrak{usp}(4)$	$\mathfrak{so}^*(8) = \mathfrak{so}(6,2)$

ADE classification of symmetric matrices with eigenvalues in $(-2,2)$ and $\mathbb{Z}_{\geq 0}$ entries (adjacency matrices of ADE diagrams), of simply laced simple Lie algebras, of binary polyhedral groups Γ (discrete subgroups of $\mathrm{SU}(2)$) and du Val singularities $\mathbb{C}^2/\Gamma \simeq$(zeros of Kleinian polynomial), of integers $1 \leq p \leq q \leq r$ with $1/p + 1/q + 1/r > 1$, of singularities with no moduli (Arnold) hence of $\mathcal{N} = 2$ minimal models ($c < 3$), of $\mathcal{N} = 0$ unitary minimal models ($c < 1$), of quivers of finite type,...

$\mathfrak{g}$	(p,q,r)	Kleinian polynomial
$\mathfrak{a}_k$	$(1, q, 1+k-q)$	$w^2 + x^2 + y^{k+1}$
$\mathfrak{d}_k$	$(2, 2, k-2)$	$w^2 + x^2 y + y^{k-1}$
$\mathfrak{e}_6$	$(2, 3, 3)$	$w^2 + x^3 + y^4$
$\mathfrak{e}_7$	$(2, 3, 4)$	$w^2 + x^3 + xy^3$
$\mathfrak{e}_8$	$(2, 3, 5)$	$w^2 + x^3 + y^5$

§2.2 Lie groups

Basics. The identity component G_0 is a normal subgroup: G/G_0 is the group of components. The maximal compact subgroup K is unique up to conjugation.

Every compact connected Lie group K is a quotient of $\mathrm{U}(1)^n \times \prod_{i=1}^{m} K_i$ by a finite subgroup Γ of its center, where K_i are simple, compact, simply-connected, connected. Then $\pi_1(K)/\mathbb{Z}^n \simeq \Gamma$ for some embedding $\mathbb{Z}^n \hookrightarrow \pi_1(K)$, and the center of K is $Z(K) = \big(\mathrm{U}(1)^n \times \prod_{i=1}^{m} Z(K_i)\big)/\Gamma$.

Center of all such K_i: $Z(\mathrm{SU}(n)) = \mathbb{Z}_n$, $Z(\mathrm{USp}(2n)) = \mathbb{Z}_2$, $Z(\mathrm{Spin}(n \geq 3)) = (\mathbb{Z}_2 \text{ for } n \text{ odd}, \mathbb{Z}_4 \text{ for } n/2 \text{ odd}, \mathbb{Z}_2^2 \text{ otherwise})$, $Z(\widetilde{\mathrm{E}}_{6(-78)}) = \mathbb{Z}_3$, $Z(\widetilde{\mathrm{E}}_{7(-133)}) = \mathbb{Z}_2$, while $\mathrm{E}_{8(-248)}$, $\mathrm{F}_{4(-52)}$, $\mathrm{G}_{2(-14)}$ have no center.

Named quotients: $\mathrm{SO}(n) = \mathrm{Spin}(n)/\mathbb{Z}_2$ and $\mathrm{P}G = G/Z(G)$ for $G = \mathrm{SU}, \mathrm{USp}, \mathrm{SO}$ (also $\mathrm{U}, \mathrm{GL}, \mathrm{SL}$). The other two quotients $\mathrm{Spin}(4n)/\mathbb{Z}_2$ have no name.

Real connected simple Lie groups are the simply-connected $\widetilde{G}$ (classified by simple Lie algebras) and their quotients by a subgroup $\Gamma \subset Z(\widetilde{G})$ of the center; equivalently, covers of the center-free $G_{\mathrm{cf}} = \widetilde{G}/Z(\widetilde{G})$. One has $\pi_1(\widetilde{G}/\Gamma) = \Gamma$ and $Z(\widetilde{G}/\Gamma) = Z(\widetilde{G})/\Gamma$. The algebraic universal cover $\widetilde{G}_{\mathrm{alg}}$ (largest with a faithful finite-dimensional representation) may be a quotient of $\widetilde{G}$. We define $\pi_1^{\mathrm{alg}}(\widetilde{G}_{\mathrm{alg}}/\Gamma) = \Gamma$. For each real simple Lie algebra $\mathfrak{g}$, we tabulate: G_{cf} as a quotient of $\widetilde{G}_{\mathrm{alg}}$; the (topological) π_1; the real rank r_{Re}; and the maximal compact subgroup $K \subset G_{\mathrm{cf}}$. Below, $\iota(l) = (1 \text{ for } l \text{ odd}, 2 \text{ otherwise})$, $p + q = n$ with $p, q \geq 1$, and $2k = n$ when n is even. For $\mathfrak{sl}(2)$ use $\mathrm{SU}(2) = \mathrm{Sp}(2)$, $\mathrm{SL}(2,\mathbb{R}) = \mathrm{Sp}(2,\mathbb{R})$, $\mathrm{SL}(2,\mathbb{C}) = \mathrm{Sp}(2,\mathbb{C})$.

	$\widetilde{G}_{\mathrm{alg}}/\pi_1^{\mathrm{alg}}(G_{\mathrm{cf}})$	K	π_1	r_{Re}
$\mathfrak{sl}(n\geq 3)$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathbb{Z}_n$	0
	$\mathrm{SL}(n,\mathbb{R})/\mathbb{Z}_{\iota(n)}$	$\mathrm{PSpin}(n)^{\ddagger\S}$	$Z(\mathrm{Spin}(n))^{\ddagger\S}$	$n-1$
	$\mathrm{SU}(p,q)/\mathbb{Z}_{p+q}$	$\frac{\mathrm{SU}(p)\times\mathrm{SU}(q)\times\mathrm{U}(1)}{\mathbb{Z}_{pq/\gcd(p,q)}}$ ¶	$\mathbb{Z}$	$\min(p,q)$
	$\mathrm{SU}^*(2k)/\mathbb{Z}_2$	$\mathrm{USp}(2k)/\mathbb{Z}_2$	$\mathbb{Z}_2$	$k-1$
	$\mathrm{SL}(n,\mathbb{C})/\mathbb{Z}_n$	$\mathrm{SU}(n)/\mathbb{Z}_n$	$\mathbb{Z}_n$	$n-1$
$\mathfrak{so}(n\geq 3)$	$\mathrm{PSpin}(n)^{\ddagger}$	$\mathrm{PSpin}(n)$	$Z(\mathrm{Spin}(n))^{\ddagger}$	0
	$\mathrm{PSpin}(p,q)^{\ddagger}$	$\frac{\mathrm{SO}(p)\times\mathrm{SO}(q)}{\mathbb{Z}_2 \text{ if } p,\, q \text{ even}}$	$\Gamma^{\parallel}$	$\min(p,q)$
	$\mathrm{SO}^*(2k)/\mathbb{Z}_2$	$\mathrm{U}(k)/\mathbb{Z}_2$	$\mathbb{Z}_{\iota(k)}\times\mathbb{Z}$	$\lfloor k/2\rfloor$
	$\mathrm{PSpin}(n,\mathbb{C})$	$\mathrm{PSpin}(n)$	$Z(\mathrm{Spin}(n))^{\ddagger}$	$\lfloor n/2\rfloor$
$\mathfrak{sp}(2n\geq 2)$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathbb{Z}_2$	0
	$\mathrm{Sp}(2n,\mathbb{R})/\mathbb{Z}_2$	$\mathrm{U}(n)/\mathbb{Z}_2$	$\mathbb{Z}_{\iota(n)}\times\mathbb{Z}$	n
	$\mathrm{USp}(2p,2q)/\mathbb{Z}_2$	$\frac{\mathrm{USp}(2p)\times\mathrm{USp}(2q)}{\mathbb{Z}_2}$	$\mathbb{Z}_2$	$\min(p,q)$
	$\mathrm{Sp}(2n,\mathbb{C})/\mathbb{Z}_2$	$\mathrm{USp}(2n)/\mathbb{Z}_2$	$\mathbb{Z}_2$	n

‡ For $r+s\geq 3$, $\mathrm{PSpin}(r,s)=\mathrm{Spin}(r,s)/Z(\mathrm{Spin}(r,s))$ and $Z(\mathrm{Spin}(r,s)) = (\mathbb{Z}_2$ if r or s odd, $\mathbb{Z}_4$ if $\frac{r+s}{2}$ odd, else $\mathbb{Z}_2^2)$.

§ Exception: for $n=2$, $K=\mathrm{SO}(2)/\mathbb{Z}_2$ and $\pi_1=\mathbb{Z}$.

¶ $K \ni \overline{(A,B,\lambda)} \mapsto \begin{pmatrix} \lambda^{q/(p+q)}A & 0 \\ 0 & \lambda^{-p/(p+q)}B \end{pmatrix} \in \mathrm{PSU}(p,q)$.

∥ $\Gamma=\pi_1(\mathrm{SO}(p))\times\pi_1(\mathrm{SO}(q))$ for p or q odd (each factor is $\mathbb{Z}_2$ except $\pi_1(\mathrm{SO}(1))=0$ and $\pi_1(\mathrm{SO}(2))=\mathbb{Z}$); otherwise $\Gamma\subset\pi_1(\mathrm{SO}(p)/\mathbb{Z}_2)\times\pi_1(\mathrm{SO}(q)/\mathbb{Z}_2)$ consists of (γ_p,γ_q) such that both or neither γ is in the corresponding $\pi_1(\mathrm{SO})\subset\pi_1(\mathrm{SO}/\mathbb{Z}_2)$.

Discrete groups in this table should not be trusted.

$\widetilde{G}_{\mathrm{alg}}/\pi_1^{\mathrm{alg}}(G_{\mathrm{cf}})$	K	π_1	r_{Re}
$\widetilde{\mathrm{E}}_{6(-78)}/\mathbb{Z}_3$	$=\mathrm{E}_{6(-78)}$	$\mathbb{Z}_3$	0
$\mathrm{E}_{6(-26)}$	$\mathrm{F}_{4(-52)}$	1	2
$\widetilde{\mathrm{E}}_{6(-14)}/\mathbb{Z}$	$\mathrm{Spin}(10)\times\mathrm{U}(1)/?$	$\mathbb{Z}$	2
$\widetilde{\mathrm{E}}_{6(2)}/\mathbb{Z}_6$	$(\mathrm{SU}(6)/\mathbb{Z}_6)\times\mathrm{SU}(2)$	$\mathbb{Z}_6$	4
$\widetilde{\mathrm{E}}_{6(6)}/\mathbb{Z}_2$	$\mathrm{USp}(8)/\mathbb{Z}_2$	$\mathbb{Z}_2$	6
$\widetilde{\mathrm{E}}_6^{\mathbb{C}}/\mathbb{Z}_3$	$\mathrm{E}_{6(-78)}$	$\mathbb{Z}_3$	6
$\widetilde{\mathrm{E}}_{7(-133)}/\mathbb{Z}_2$	$=\mathrm{E}_{7(-133)}$	$\mathbb{Z}_2$	0
$\widetilde{\mathrm{E}}_{7(-25)}/\mathbb{Z}$	$\mathrm{E}_{6(-78)}\times\mathrm{U}(1)/?$	$\mathbb{Z}$	3
$\widetilde{\mathrm{E}}_{7(-5)}/\mathbb{Z}_2^2$	$\mathrm{Spin}(12)\times\mathrm{SU}(2)/\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$	4
$\widetilde{\mathrm{E}}_{7(7)}/\mathbb{Z}_4$	$\mathrm{SU}(8)/\mathbb{Z}_4$	$\mathbb{Z}_4$	7
$\widetilde{\mathrm{E}}_7^{\mathbb{C}}/\mathbb{Z}_2$	$\mathrm{E}_{7(-133)}$	$\mathbb{Z}_2$	7
$\mathrm{E}_{8(-248)}$	$\mathrm{E}_{8(-248)}$	1	0
$\widetilde{\mathrm{E}}_{8(-24)}/\mathbb{Z}_2$	$\widetilde{\mathrm{E}}_{7(-133)}\times\mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\widetilde{\mathrm{E}}_{8(8)}/\mathbb{Z}_2$	$\mathrm{SO}(16)/\mathbb{Z}_2$	$\mathbb{Z}_2$	8
$\mathrm{E}_8^{\mathbb{C}}$	$\mathrm{E}_{8(-248)}$	1	8
$\mathrm{F}_{4(-52)}$	$\mathrm{F}_{4(-52)}$	1	0
$\widetilde{\mathrm{F}}_{4(-20)}/\mathbb{Z}_2$	$\mathrm{Spin}(9)/\mathbb{Z}_2$	$\mathbb{Z}_2$	1
$\widetilde{\mathrm{F}}_{4(4)}$	$\mathrm{USp}(6)\times\mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\mathrm{F}_4^{\mathbb{C}}$	$\mathrm{F}_{4(-52)}$	1	4
$\mathrm{G}_{2(-14)}$	$\mathrm{G}_{2(-14)}$	1	0
$\widetilde{\mathrm{G}}_{2(2)}/\mathbb{Z}_2$	$\mathrm{SU}(2)\times\mathrm{SU}(2)/\mathbb{Z}_2$	$\mathbb{Z}_2$	4
$\mathrm{G}_2^{\mathbb{C}}$	$\mathrm{G}_{2(-14)}$	1	4

Spin and Pin groups. $\mathrm{SO}(n)$ has a double cover $\mathrm{Spin}(n)$. Since $\pi_0(\mathrm{O}(n))=\mathbb{Z}_2$ there are two double covers: $\mathrm{Pin}_+(n)$ in which a reflection R obeys $R^2=1$, and $\mathrm{Pin}_-(n)$ in which $R^2=(-1)^F$. For $p,q\geq 1$, $\pi_0(\mathrm{O}(p,q))=\pi_0(\mathrm{O}(p))\times\pi_0(\mathrm{O}(q))=\mathbb{Z}_2^2$; the identity component $\mathrm{SO}_+(p,q)$ has a double cover $\mathrm{Spin}(p,q)$. The eight double covers of $\mathrm{O}(p,q)$ differ in whether R^2, T^2 and $(RT)^2$ are $+1$ or $(-1)^F$.

Accidental isomorphisms (low-rank real reductive Lie groups) $\mathbb{R}/\mathbb{Z}=\mathrm{U}(1)$; $\mathrm{SU}(2)=\mathrm{Spin}(3)\twoheadrightarrow\mathrm{SO}(3)$; …

Homotopy. Any connected Lie group is homeomorphic to its maximal compact subgroup K times a Euclidean space $\mathbb{R}^p$. All $\pi_{j\geq 1}(K)$ are abelian and finitely generated, $\pi_2(K)=0$, $\pi_3(K)=\mathbb{Z}^m$ where m counts simple factors in a finite cover $\mathrm{U}(1)^n\times\prod_{i=1}^m K_i\twoheadrightarrow K$, and $\pi_j(K)=\prod_{i=1}^m\pi_j(K_i)$ for $j\geq 2$.

For any G there exists $\prod_{i=1}^{\operatorname{rank} G} S^{2d_i-1}\to G$ which induces isomorphisms of rational (i.e., torsion-free part of) homotopy/cohomology groups where d_i are the degrees of fundamental invariants. For compact simple K,

Group	$(2d_i-1)$
A_n	$3,5,\dots,2n+1$
$\mathrm{B}_n,\mathrm{C}_n$	$3,7,\dots,4n-1$
D_n	$3,7,\dots,4n-5,2n-1$
E_6	$3,9,11,15,17,23$
E_7	$3,11,15,19,23,27,35$
E_8	$3,15,23,27,35,39,47,59$
F_4	$3,11,15,23$
G_2	$3,11$

$\pi_{j\geq 2}(G)$ has a factor $\mathbb{Z}$ for each S^j above, and some torsion. Explicitly, $\pi_j(\mathrm{SU}(n))$ is $\mathbb{Z}$ for odd $j<2n$, 0 for even $j<2n$, and is pure torsion for $j\geq 2n$. Similarly, $\pi_{j<4n+2}(\mathrm{USp}(2n))$ is $\mathbb{Z}$ for $j\equiv 3,7 \bmod 8$, $\mathbb{Z}_2$ for $j\equiv 4,5 \bmod 8$, and 0 otherwise.

§2.3 Simple Lie superalgebras

Classical Lie superalgebras: the bosonic algebra acts on the fermionic generators in a completely reducible representation. This excludes Cartan-type superalgebras $\mathfrak{w}(n)$, $\mathfrak{s}(n)$, $\tilde{\mathfrak{s}}(n)$ and $\mathfrak{h}(n)$. In this table, $m,n\geq 1$ and we do not list purely bosonic Lie algebras. The factor $\mathbb{C}$ of $\mathfrak{sl}(m|n)$ must be removed if $m=n$.

	Bosonic algebra	Fermionic repr.
$\mathfrak{sl}(m\vert n)$	$\mathfrak{sl}(m,\mathbb{C})\oplus\mathfrak{sl}(n,\mathbb{C})\oplus\mathbb{C}$	$(m,\overline{n})\oplus(\overline{m},n)$
$\mathfrak{osp}(m\vert 2n)$	$\mathfrak{so}(m,\mathbb{C})\oplus\mathfrak{sp}(2n,\mathbb{R})$	$(m,2n)$
$\mathfrak{d}(2,1,\alpha)$	$\mathfrak{sl}(2,\mathbb{C})^3$	$(2,2,2)$
$\mathfrak{f}(4)$	$\mathfrak{so}(7,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$	$(8,2)$
$\mathfrak{g}(3)$	$\mathfrak{g}_2\oplus\mathfrak{sl}(2,\mathbb{C})$	$(7,2)$
$\mathfrak{p}(m)$	$\mathfrak{sl}(m+1,\mathbb{C})$	sym $\oplus$ (antisym)*
$\mathfrak{q}(m)$	$\mathfrak{sl}(m+1,\mathbb{C})$	adjoint

Real forms of Lie superalgebras, starting from their compact form ($p=q=0$). $\mathfrak{p}(m)$ has no compact form. Here, $m,n\geq 1$, $0\leq p\leq m/2$, $0\leq q\leq n/2$. The forms $\mathfrak{su}^*$, $\mathfrak{osp}^*$, $\mathfrak{q}^*$ only exist for even rank; $\mathfrak{sl}'$ only if $m=n$.

Real form	Bosonic algebra
$\mathfrak{su}(m-p,p\vert n-q,q)$	$\mathfrak{su}(m-p,p)\oplus\mathfrak{su}(n-q,q)\oplus\mathfrak{u}(1)^{\ddagger}$
$\mathfrak{sl}(m\vert n)$	$\mathfrak{sl}(m,\mathbb{R})\oplus\mathfrak{sl}(n,\mathbb{R})\oplus\mathfrak{so}(1,1)^{\ddagger}$
$\mathfrak{sl}'(n\vert n)$ $(m=n)$	$\mathfrak{sl}(n,\mathbb{C})$
$\mathfrak{su}^*(m\vert n)$ $(m,n$ even$)$	$\mathfrak{su}^*(m)\oplus\mathfrak{su}^*(n)\oplus\mathfrak{so}(1,1)^{\ddagger}$
$\mathfrak{osp}(m-p,p\vert 2n)$	$\mathfrak{so}(m-p,p)\oplus\mathfrak{sp}(2n,\mathbb{R})$
$\mathfrak{osp}^*(m\vert 2n-2q,2q)$ $(m$ even$)$	$\mathfrak{so}^*(m)\oplus\mathfrak{usp}(2n-2q,2q)$
$\mathfrak{d}^p(2,1,\alpha)$ §	$\mathfrak{so}(4-p,p)\oplus\mathfrak{sl}(2,\mathbb{R})$ $(p=0,1,2)$
$\mathfrak{f}^p(4)$ for $p=0,3$	$\mathfrak{so}(7-p,p)\oplus\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{f}^p(4)$ for $p=1,2$	$\mathfrak{so}(7-p,p)\oplus\mathfrak{su}(2)$
$\mathfrak{g}_s(3)$ for $s=-14,2$	$\mathfrak{g}_{2(s)}\oplus\mathfrak{sl}(2,\mathbb{R})$
$\mathfrak{p}(m)$	$\mathfrak{sl}(m+1,\mathbb{R})$
$\mathfrak{uq}(m-p,p)$	$\mathfrak{su}(m+1-p,p)$
$\mathfrak{q}(m)$	$\mathfrak{sl}(m+1,\mathbb{R})$
$\mathfrak{q}^*(m)$ $(m$ odd$)$	$\mathfrak{su}^*(m+1)$

‡ For $m=n$, $\mathfrak{u}(1)$ and $\mathfrak{so}(1,1)$ factors are absent. Additionally, one can project down to a single bosonic factor.

§ The three $\mathfrak{sl}(2)$ bosonic factors of $\mathfrak{d}(2,1,\alpha)$ appear with weights 1, α and $-1-\alpha$ in fermion anticommutators. For $\mathfrak{d}^0$ and $\mathfrak{d}^2$, α is real. For $\mathfrak{d}^1$, $\alpha=1+ia$ with a real.

Some isomorphisms: $\mathfrak{su}(1,1|1)=\mathfrak{sl}(2|1)=\mathfrak{osp}(2|2)$ and $\mathfrak{su}(2|1)=\mathfrak{osp}^*(2|2,0)$ and $\mathfrak{d}^p(2,1,\alpha=1)=\mathfrak{osp}(4-p,p|2)$ and $\mathfrak{osp}(6,2|4)=\mathfrak{osp}^*(8|4)$.

§2.4 Lie supergroups

§2.5 Representations

§3 Spinors

Clifford algebra. Let h_{ab} be diagonal with s '+1' and t '−1', and $d=s+t$. The Clifford algebra $\{\Gamma_a,\Gamma_b\}=2h_{ab}$ has real dimension 2^d and is isomorphic to a matrix algebra $M_{2^\#}(\bullet)$ with

$s-t$ mod 8	0	1	2	3	4	5	6	7
$\bullet$ is	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$

Charge conjugation. $(-\eta)\Gamma_a^T=\mathcal{C}\Gamma_a\mathcal{C}^{-1}$ are conjugate for $\eta=\pm1$ because they obey the same algebra. Get $\mathcal{C}^T=-\varepsilon\mathcal{C}$ with $\varepsilon=\pm1$ by transposing twice. Let $\Gamma^{(n)}=\Gamma_{a_1\dots a_n}$. Using $\left(\mathcal{C}\Gamma^{(n)}\right)^T=-\epsilon(-)^{n(n-1)/2}(-\eta)^n\mathcal{C}\Gamma^{(n)}$ find which n mod 4 give symmetric $\mathcal{C}\Gamma^{(n)}$. The sum of $\binom{d}{n}$ must be $2^{\lfloor d/2\rfloor}(2^{\lfloor d/2\rfloor}+1)/2$. This fixes ϵ,η. Odd d require $\eta=(-1)^{d(d+1)/2}$ to preserve $\Gamma^{(d)}$. Even d allow two choices of signs: consult the rows $d\pm1$.

d mod 8	n	ϵ	η
0⟨1	0, 1	−1	−1
2⟨3	1, 2	+1	+1
4⟨5	2, 3	+1	−1
6⟨7	0, 3	−1	+1

Reduced spinors. $M_{ab}\in\mathfrak{so}(s,t)$ acts as $\gamma_a\gamma_b$ on representations of the Clifford algebra. But the $2^{\lceil d/2\rceil}$-dimensional representation is not irreducible as a representation of $\mathfrak{so}(s,t)$.

In even d, Weyl (or chiral) spinors $\Gamma^{(d)}\lambda=\pm\lambda$ have $2^{d/2-1}$ real components. Let B be defined by $\Gamma_a^*=-\eta(-1)^tB\Gamma_aB^{-1}$. Majorana spinors $\lambda^*=B\lambda$ exist for $s-t\equiv0,\pm1,\pm2$ mod 8; the case $s-t\equiv\pm2$ requires $\eta=\mp(-1)^{d/2}$. When $s-t\equiv3,4,5$, a set of $2n$ spinors can be symplectic Majorana: $(\lambda^I)^*=B\Omega_{IJ}\lambda^J$ for $\Omega=((0,\mathbb{1}_n);(-\mathbb{1}_n,0))$. (Symplectic) Majorana–Weyl spinors exist for $s-t\equiv0,4$ mod 8. The table also includes the real dimension of the minimal spinor.

d		$t\equiv0$		1		2		3 mod 4	
1	(D 2)	M	1	M	1				
2	(W 2)	M$^-$	2	MW	1	M$^+$	2		
3	(D 4)	s	4	M	2	M	2	s	4
4	(W 4)	sW	4	M$^+$	4	MW	2	M$^-$	4
5	(D 8)	s	8	s	8	M	4	M	4
6	(W 8)	M$^+$	8	sW	8	M$^-$	8	MW	4
7	(D 16)	M	8	s	16	s	16	M	8
8	(W16)	MW	8	M$^-$	16	sW	16	M$^+$	16
9	(D 32)	M	16	M	16	s	32	s	32
10	(W32)	M$^-$	32	MW	16	M$^+$	32	sW	32
11	(D 64)	s	64	M	32	M	32	s	64
12	(W64)	sW	64	M$^+$	64	MW	32	M$^-$	64

Flavour symmetries of N minimal spinors. This is also the R-symmetry of the N-extended superalgebra. For (symplectic) Majorana Weyl spinors, specify $N=(N_L,N_R)$ left/right-handed.

$$\text{M}\ \begin{cases}\mathfrak{u}(N) & \text{if } d \text{ even}\\ \mathfrak{so}(N) & \text{if } d \text{ odd}\end{cases}$$

MW: $\mathfrak{so}(N_L)\times\mathfrak{so}(N_R)$
s : $\mathfrak{usp}(2N)$
sW : $\mathfrak{usp}(2N_L)\times\mathfrak{usp}(2N_R)$

E.g., Lorentzian 6d $(2,0)$ has $\mathfrak{usp}(4)\times\mathfrak{usp}(0)$ R-symmetry.

Products of spinor representations. For odd $d=2m+1$, let $\mathcal{S}$ be a spinor representation of complex dimension 2^m. The symmetric product $S^2\mathcal{S}$ consists of k-forms with $k\equiv m$ mod 4. Since k-forms and $(d-k)$-forms are the same representation, other descriptions can be given. For the antisymmetric product $\bigwedge^2\mathcal{S}$, take $k\equiv m-1$ mod 4. See the list of forms in the table.

d	1	3	5	7	9	11
$\dim_{\mathbb{C}}\mathcal{S}$	1	2	4	8	16	32
$S^2\mathcal{S}$	0	1	2	0, 3	0, 1, 4	1, 2, 5
$\bigwedge^2\mathcal{S}$	.	0	0, 1	1, 2	2, 3	0, 3, 4

For even $d=2m$, let $\mathcal{S}_\pm$ be the Weyl spinor representations of complex dimension 2^{m-1}. The tensor product $\mathcal{S}_+\otimes\mathcal{S}_-$ consists of $(m-1-2j)$-forms for $0\le j\le(m-1)/2$. The symmetric products $S^2\mathcal{S}_\pm$ decompose into the (anti-)self-dual m-forms and $(m-4j)$-forms for $0<j\le m/4$. The antisymmetric products $\bigwedge^2\mathcal{S}_\pm$ decompose into $(m-2-4j)$-forms for $0\le j\le(m-2)/4$.

d	2	4	6	8	10	12
$\dim_{\mathbb{C}}\mathcal{S}_\pm$	1	2	4	8	16	32
$S^2\mathcal{S}_\pm$	$1^\dagger$	$2^\dagger$	$3^\dagger$	$0,4^\dagger$	$1,5^\dagger$	$2,6^\dagger$
$\bigwedge^2\mathcal{S}_\pm$	.	0	1	2	3	0, 4
$\mathcal{S}_+\otimes\mathcal{S}_-$	0	1	0, 2	1, 3	0, 2, 4	1, 3, 5

Note that $S^2(\mathcal{S}_+\oplus\mathcal{S}_-)=S^2\mathcal{S}_+\oplus(\mathcal{S}_+\otimes\mathcal{S}_-)\oplus S^2\mathcal{S}_-$

$$\bigwedge^2(\mathcal{S}_+\oplus\mathcal{S}_-)=\bigwedge^2\mathcal{S}_+\oplus(\mathcal{S}_+\otimes\mathcal{S}_-)\oplus\bigwedge^2\mathcal{S}_-$$

§4 Supersymmetry algebras

The Poincaré algebra is $\mathbb{R}^{s,t} \rtimes \mathfrak{so}(s,t)$, the semi-direct product of translations by rotations. Namely, $[P_a, P_b] = 0$, $[M_{ab}, P_c] = 2ih_{c[a}P_{b]}$, and $[M_{ab}, M^{cd}] = 4ih^{[c}_{[a}M^{d]}_{b]}$.

Super-Poincaré algebra. Add supercharges in some spinor representation Q of the Poincaré algebra (so $[P_a, Q] = 0$). Their anticommutator transforms in the representation S^2Q and should include the one-form P. Depending on s, t they can include other k-forms Z, called central charges because $[P, Z] = [Z, Z] = 0$. The super-Poincaré algebra is $((\mathbb{R}^{s,t} \times Z).Q) \rtimes (\mathfrak{so}(s,t) \times R)$, where the R-symmetry acts on Q. This Lie superalgebra is graded: $\mathrm{gr}(\mathbb{R}^{s,t} \times Z) = -2$, $\mathrm{gr}(Q) = -1$, and $\mathrm{gr}(\mathfrak{so}(s,t) \times R) = 0$. The supertranslations consist of $(\mathbb{R}^{s,t} \times Z).Q$.

Example: M-theory algebra. $d = 10 + 1$ super-Poincaré algebra with $Q =$ Majorana. Since S^2Q has 1, 2, and 5-forms, there are 2-form and 5-form central charges $Z_{(2)}$ and $Z_{(5)}$ (under which M2 and M5 branes are charged):

$$\{Q_\alpha, Q_\beta\} = (\gamma^M C)_{\alpha\beta} P_M + \frac{1}{2}(\gamma_{MN}C)_{\alpha\beta} Z^{MN}_{(2)} + \frac{1}{5!}(\gamma_{MNPQR}C)_{\alpha\beta} Z^{MNPQR}_{(5)}$$

Altogether the M-theory algebra is $\mathfrak{osp}(1|32)$.

Superconformal algebras are the same as super AdS_{d+1}. The bosonic part is $\mathfrak{so}(d,2)$ and R-symmetries. As a supermatrix: $\begin{pmatrix} \mathfrak{so}(d,2) & Q+S \\ Q-S & R \end{pmatrix}$ or $\mathfrak{so}(d,2) \leftrightarrow R$. Note that $\{Q, S\}$ contains R. For $d = 2$, the finite conformal algebra is $\mathfrak{so}(2,2) = \mathfrak{so}(2,1) \oplus \mathfrak{so}(2,1)$, sum of two $d = 1$ algebras, so the superalgebra is sum of two $d = 1$ superalgebras.

d	Superalgebra	R-symmetries	#Q+#S
1	$\mathfrak{osp}(N\|2)$	$\mathfrak{o}(N)$	$2N$
	$\mathfrak{su}(N\|1,1)$	$\mathfrak{su}(N) \oplus \mathfrak{u}(1)$ for $N \neq 2$	$4N$
	$\mathfrak{su}(2\|1,1)$	$\mathfrak{su}(2)$	8
	$\mathfrak{osp}(4^*\|2N)$	$\mathfrak{su}(2) \oplus \mathfrak{usp}(2N)$	$8N$
	$\mathfrak{g}(3)$	$\mathfrak{g}_2$	14
	$\mathfrak{f}^0(4)$	$\mathfrak{so}(7)$	16
	$\mathfrak{d}^0(2,1,\alpha)$	$\mathfrak{su}(2) \oplus \mathfrak{su}(2)$	8
3	$\mathfrak{osp}(N\|4)$	$\mathfrak{so}(N)$	$4N$
4	$\mathfrak{su}(2,2\|N)$	$\mathfrak{su}(N) \oplus \mathfrak{u}(1)$ for $N \neq 4$	$8N$
	$\mathfrak{su}(2,2\|4)$	$\mathfrak{su}(4)$	32
5	$\mathfrak{f}^2(4)$	$\mathfrak{su}(2)$	16
6	$\mathfrak{osp}(8^*\|N)$	$\mathfrak{usp}(N)$ (N even)	$8N$

Dimensional reduction of Lorentzian supersymmetry algebras. The 1d column gives the number of real supercharges.

10d	6d	5d	4d	3d	2d	1d
$\mathcal{N} = (1,0)$	$(1,1)$	2	4	8	$(8,8)$	16
	$(1,0)$	1	2	4	$(4,4)$	8
			1	2	$(2,2)$	4

Explicit supersymmetry algebras 4d $\mathcal{N} = 2$ $\{Q^A_\alpha, \overline{Q}^B_{\dot\alpha}\} = \epsilon^{AB} P_{\alpha\dot\alpha}$

Supersymmetry on symmetric curved spaces 4d $\mathcal{N} = 2$ supersymmetry on S^4 is $\mathfrak{osp}(2|4)$. 2d $\mathcal{N} = (2,2)$ supersymmetry on S^2 is $\mathfrak{osp}(2|2)$.

§5 Supermultiplets

§5.1 Spin ≤ 1 supermultiplets

For 16 **supercharges**, there is only the vector multiplet.

For 8 **supercharges**, vector multiplet and hypermultiplet; in 3d and lower also twisted vector multiplet and twisted hypermultiplet.

For 4 **supercharges**, vector, chiral, linear multiplets; in 2d $\mathcal{N} = (2,2)$ also twisted vector, twisted chirals, semichirals, ...

For 2 **supercharges**, vector, chiral, linear, Fermi, ...

§5.2 Other supermultiplets

6d $\mathcal{N} = (2,0)$ tensor multiplet with self-dual two-form gauge field B (namely $\mathrm{d}B = \star\,\mathrm{d}B$), four spinors, five scalars.

6d $\mathcal{N} = (1,0)$ tensor multiplet (contains one scalar), reduces to 4d $\mathcal{N} = 2$ vector.

6d $\mathcal{N} = (1,0)$ supergravity multiplet, reduces to 4d $\mathcal{N} = 2$ supergravity multiplet and two vectors.

§6 Supersymmetric gauge theories

§6.1 Generalities

Super Yang–Mills (SYM). A gauge group is a compact reductive Lie group G such as $(\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1))/\mathbb{Z}_6$. The SYM term is $\mathcal{L}_{\mathrm{SYM}} = g^{-2} \operatorname{Tr} F \wedge \star F +$ superpartners, with one real gauge coupling g per simple factor.

Theta term in even dimension: $\theta \operatorname{Tr} F^{\wedge(d/2)}$ with θ periodic. In 4d, θ and g combine to $\tau = \theta/(2\pi) + 4\pi i/g^2$.

Chern–Simons term in 3d: $k \operatorname{Tr}\big(A \wedge \mathrm{d}A + \frac{2}{3} A \wedge A \wedge A\big)$ with k quantized (normalization missing).

Matter. For 16 supercharges, none. For 8 supercharges, symplectic representation $V \simeq \mathbb{H}^n$ namely $G \to F = \mathrm{USp}(2n)$. For 4 supercharges, unitary representation $V \simeq \mathbb{C}^n$ namely $G \to F = \mathrm{U}(n)$. Canonical kinetic term for bosons: $D_\mu \phi_i D^\mu \phi_i$.

Superpotential term. For 4 supercharges, $\int \mathrm{d}^2\theta\, W$ gives a potential for scalars and Yukawa-type interactions. W is holomorphic in chiral fields and in couplings seen as background fields. Example: the kinetic term $\operatorname{Im} \int \mathrm{d}^2\theta [\tau W^2_\alpha]$ of an abelian gauge field: W^2_α is a chiral field so τ is the background value of a chiral field.

An accidental symmetry is a flavour symmetry of the IR but not of the UV.

R-symmetry mixing. In 2d and higher, the superconformal algebra contains the IR R-symmetry. The UV R-symmetry can be a mixture of the IR R-symmetry and of a flavour symmetry: $R_{\mathrm{UV}} \subset R_{\mathrm{IR}} \times F$. For nonabelian R-symmetry that flavour symmetry must be accidental as it does not commute with R_{UV}. For abelian R-symmetry the mixing is continuous; assuming no accidental flavour symmetries it is fixed in 4d $\mathcal{N} = 1$ by a-extremization, in 3d $\mathcal{N} = 2$ by Z_{S^3}-extremization, in 2d $\mathcal{N} = (0,2)$ by c-extremization.

Classical vacua: Coulomb, Higgs and mixed branches. Coulomb branch ($\mathfrak{g}$ modulo conjugation by G) parametrized by vector multiplet scalars, can be lifted by quantum effects, larger in 3d due to monopoles. For 4 supercharges, Higgs branch parametrized by chiral multiplet scalars: Kähler quotient $R//G$. For 8 supercharges, Higgs branch parametrized by hypermultiplet scalars: hyper-Kähler quotient $\widetilde{R}///G$. The Higgs branch has flavour symmetry $\{x \in F \mid xG = Gx\}/G$ normalizer of G in $F = U(R)$ or $F = USp(\widetilde{R})$ modulo G. Background vector multiplet scalars (real/twisted masses) reduce the Higgs and mixed branches to fixed points of corresponding flavour symmetries.

Boundaries and gauge redundancies. On a non-compact spacetime one can consider the group of gauge redundancies with various boundary conditions. Let $H \subset F$ be the constant transformations included as gauge redundancies (including constant gauge transformations by $H \cap G$). The Higgs branch flavour symmetry is then $\{x \in F \mid xG = Gx, xH = Hx\}/H$.

§6.2 Maximal super Yang–Mills

Data: gauge group.

Lorentzian 10d $\mathcal{N} = 1$ SYM is anomalous unless the gauge group is abelian. Its dimensional reductions are anomaly-free and have one gauge field, $10 - d$ scalars and $\mathcal{N}$ (symplectic or Majorana, and Weyl or not) spinors. The Lagrangian's R-symmetry $\mathrm{Spin}(10 - d)$ is contained in the automorphism group of the superalgebra (they coincide for $d \geq 5$).

dim.	$\mathcal{N}$ spinors	autom. $\supset$ R-sym.
10d	$(1, 0)$ MW	
9d	1 M	
8d	1 M	$\mathrm{U}(1) = \mathrm{Spin}(2)$
7d	1 s	$\mathrm{USp}(2) = \mathrm{Spin}(3)$
6d	$(1, 1)$ sW	$\mathrm{USp}(2)^2 = \mathrm{Spin}(4)$
5d	2 s	$\mathrm{USp}(4) = \mathrm{Spin}(5)$
4d	4 M	$\mathrm{U}(4) \supset \mathrm{Spin}(6)$
3d	8 M	$\mathrm{Spin}(8) \supset \mathrm{Spin}(7)$
2d	$(8, 8)$ MW	$\mathrm{Spin}(8)^2 \supset \mathrm{Spin}(8)$
1d	16 M	$\mathrm{Spin}(16) \supset \mathrm{Spin}(9)$

4d $\mathcal{N} = 4$ has exactly marginal parameter $\tau = \theta/(2\pi) + 4\pi i/g^2$.

§6.3 Theories with 8 supercharges

5d $\mathcal{N} = 1$ **SCFTs** built from 5-brane diagrams or UV fixed point of gauge theory.

$\mathrm{SU}(2N)$ SYM with $N_f \leq 7$ fundamental hypermultiplets has $\mathrm{SO}(2N_f) \times \mathrm{U}(1)_T \subset \mathrm{E}_{N_f+1}$ flavour symmetry enhancement. For $\mathrm{SU}(2)$ and $N_f = 0$, non-trivial "θ" in $\pi_4(\mathrm{SU}(2)) = \mathbb{Z}_2$ gives the $\widetilde{E}_1$ theory with $\mathrm{U}(1)_T$ symmetry only.

4d $\mathcal{N} = 2$ **generalities**

There can be no continuous flavour symmetry enhancement.

The theory on $\mathbb{R}^4_{\epsilon_1,0}$ (Nekrasov–Shatashvili limit) $\leftrightarrow$ quantum integrable system with Planck constant ϵ_1.

Coulomb moduli $\leftrightarrow$ action variables.

Supersymmetric vacua $\leftrightarrow$ eigenstates.

Lift to $\mathbb{R}^4 \times S^1$ gives K-theoretic Nekrasov partition function. The 5d theory $\leftrightarrow$ relativistic version of the integrable system.

4d $\mathcal{N} = 2$ (G, G') **Argyres–Douglas theories** (with G and G' among A_k, D_k, $\mathrm{E}_{6,7,8}$) are engineered as IIB strings on three-fold singularity $f_G(x_1, x_2) + f_{G'}(x_3, x_4) = 0$ where $f_{A_k}(x, y) = x^2 + y^{k+1}$ etc. (see page 2).

3d $\mathcal{N} = 4$ has $\mathrm{SU}(2)_C \times \mathrm{SU}(2)_H$ R-symmetry acting on the Coulomb and Higgs branch. Both branches are hyper-Kähler and the $\mathrm{SU}(2)$ rotates their $\mathbb{CP}^1$ worth of complex structures. Denote $\mathrm{T} \subset \mathrm{G}$ the Cartan torus. The Coulomb branch is a holomorphic Lagrangian fibration $\mathcal{M}_C \to \mathfrak{t}_{\mathbb{C}}/\mathrm{Weyl}$ with generic fiber $\mathrm{T}^\vee_{\mathbb{C}} \simeq (\mathbb{C}^*)^{\mathrm{rank}\,\mathrm{G}}$. Its classical $\mathrm{Hom}(\pi_1(\mathrm{G}), \mathrm{U}(1))$ topological flavour symmetry can be enhanced quantum mechanically.

2d $\mathcal{N} = (4, 4)$ **gauge theories.** Typically get in the IR a direct sum of 2d $\mathcal{N} = (4, 4)$ SCFTs (from the Coulomb and Higgs branches) whose central charges are different. Their $\mathrm{SU}(2) \times \mathrm{SU}(2)$ left/right-moving R-symmetries are different.

§6.4 Theories with 4 supercharges

4d $\mathcal{N} = 1$ **pure SYM** classically has $\mathrm{U}(1)_R$ symmetry, broken by instantons to $\mathbb{Z}_{2h}$ with $h = C_2(\mathrm{adj})$. It confines, is mass-gapped, and has $C_2(A)$ vacua associated to breaking $\mathbb{Z}_{2h}$ to $\mathbb{Z}_2$ by gaugino condensation $\langle\lambda\lambda\rangle$. Witten index $\mathrm{Tr}(-1)^F = h$.

Wess-Zumino model: chiral multiplet ϕ with superpotential $W = m\phi^2 + g\phi^3$.

3d $\mathcal{N} = 2$

2d $\mathcal{N} = (2, 2)$**.** Classical $\mathrm{U}(1) \times \mathrm{U}(1)$ R-symmetry. The axial $\mathrm{U}(1)$ R-symmetry has an anomaly with $\mathrm{U}(1)$ gauge symmetry proportional to the total charge under that gauge symmetry.

The gauge field strength is a twisted chiral multiplet Σ.

Integrating out massive chirals gives a twisted superpotential $-\mathrm{Tr}_R\big(\Sigma \log(\Sigma/\mu) - \Sigma\big)$ where Σ combines gauge field strength and twisted masses. FI parameters (twisted superpotentials linear in Σ) thus run as $\log(\mu)$ times the sum of charges.

Twisted chiral ring relations: $\partial W/\partial\Sigma_j \in 2\pi \mathrm{i}\mathbb{Z}$.

1d $\mathcal{N} = 4$

Data: gauge group G, representation V of G for chiral multiplets. Gauge couplings, FI parameters, superpotential W. Flavour Wilson line, twisted and real masses $v, m_1 + im_2, m_3 \in \mathfrak{g}_F$ that commute.

R-symmetry: $\mathrm{SU}(2)$, times $\mathrm{U}(1)$ if W has charge 2.

1d $\mathcal{N} = 2$

Discrete data: gauge group G, chiral multiplets in a representation V of G, Wilson line in a unitary representation $M = M_0 \oplus M_1$ of $\mathfrak{g}$, flavour symmetry group $G_F \subseteq \mathrm{U}(V) \times \mathrm{U}(M_0) \times \mathrm{U}(M_1)$ commuting with G. Gauge anomaly cancellation: $M \otimes \det^{1/2} V$ must be a representation of G.

Continuous data: gauge couplings, FI parameters, flavour Wilson line and real mass $v, \sigma \in \mathfrak{g}_F$ that commute, $\mathfrak{g}$-equivariant holomorphic odd map $Q\colon V \to \mathrm{End}\, M$ with $Q^2 = 0$ describing how supercharges act on M.

Special case: Fermi multiplets in representation V_{f} of G with G-equivariant holomorphic maps $E\colon V \to V_{\mathrm{f}}$ and $J\colon V \to V_{\mathrm{f}}^\vee$ obeying $J \cdot E = 0$ are equivalent to Wilson line in $M = \wedge V_{\mathrm{f}} \otimes \det^{-1/2} V_{\mathrm{f}}$ with $Q = E\wedge + J\lrcorner$.

R-symmetry: $\mathrm{U}(1)$ if $Q\colon V \to \mathrm{End}\, M$ has charge 1. Mixing with flavour symmetries not fixed by superconformal algebra.

NLSM Chiral multiplet: scalar ϕ in a Kähler target X and fermion in holomorphic bundle $\phi^* T_X$. Wilson line depends on a complex of vector bundles $\mathcal{F}$. Fermi multiplet takes values in a holomorphic vector bundle $\mathcal{E}$ with hermitian metric, equivalent to Wilson line with $\mathcal{F} = \det^{-1/2} \mathcal{E} \otimes \wedge\mathcal{E}$. Anomaly cancellation: $\sqrt{K_X} \otimes \wedge T_X \otimes \det^{-1/2} \mathcal{E} \otimes \wedge\mathcal{E} \otimes \mathcal{F}$ is a well-defined vector bundle on X.

§7 Other theories

§7.1 2d conformal field theories

Virasoro algebra $[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3-m)\delta_{m+n,0}$ where $m \in \mathbb{Z}$. Adjoint $L_n^\dagger = L_{-n}$ and $c^\dagger = c$.

$\mathcal{N} = 1$ **super-Virasoro algebra** additionally $[L_m, G_r] = (m/2 - r)G_{m+r}$ and $\{G_r, G_s\} = 2L_{r+s} + \frac{c}{3}(r^2 - 1/4)\delta_{r+s,0}$ where either $r \in \mathbb{Z}$ (Ramond algebra) or $r \in \mathbb{Z} + 1/2$ (Neveu–Schwarz algebra). Adjoint $G_r^\dagger = G_{-r}$.

$\mathcal{N} = 2$ **super-Virasoro algebra** $[L_m, J_n] = -nJ_{m+n}$, $[J_m, J_n] = \frac{c}{3}m\delta_{m+n,0}$, $[L_m, G_r^\pm] = (m/2 - r)G_{m+r}^\pm$, $[J_m, G_r^\pm] = \pm G_{m+r}^\pm$, $\{G_r^+, G_s^+\} = \{G_r^-, G_s^-\} = 0$, $\{G_r^+, G_s^-\} = L_{r+s} + \frac{1}{2}(r-s)J_{r+s} + \frac{c}{6}(r^2 - 1/4)\delta_{r+s,0}$. Adjoint $L_m^\dagger = L_{-m}$, $J_m^\dagger = J_{-m}$, $(G_r^\pm)^\dagger = G_{-r}^\mp$, $c^\dagger = c$. The algebras with $r \in \mathbb{Z}$ (Ramond) or $r \in \mathbb{Z} + 1/2$ (Neveu–Schwarz) are isomorphic under spectral shift $\alpha_{\pm 1/2}$ where $\alpha_\eta(L_n) = L_n + \eta J_n + \frac{c}{6}\eta^2\delta_{n,0}$, $\alpha_\eta(J_n) = J_n + \frac{c}{3}\eta\delta_n$, $\alpha_\eta(G_r^\pm) = G_{r\pm\eta}^\pm$. Another automorphism is $G_r^+ \leftrightarrow G_r^-$, $J_m \mapsto -J_m - \frac{c}{3}\delta_{m,0}$. We get a $\mathbb{Z} \rtimes \mathbb{Z}_2$ automorphism group.

$SW(3/2, 2)$ **super-Virasoro algebra** has L, G, W, U

bc **system,** $\beta\gamma$ **system**

Liouville CFT has $c = 1 + 6(b + 1/b)^2$ and primary operators with $h(\alpha) = \alpha(b + 1/b - \alpha)$ for "momentum" $\alpha \in \frac{1}{2}(b + 1/b) + i\mathbb{R}$.

Minimal model $\mathcal{M}_{p,q}$ for $p > q$ coprime is a quotient of $b = i\sqrt{p/q}$ Liouville CFT. It has $c = 1 - \frac{6(p-q)^2}{pq}$ and primary operators with $h_{r,s} = \frac{(ps-qr)^2-(p-q)^2}{4pq}$ for $0 < r < p$ and $0 < s < q$; no degeneracy besides $h_{r,s} = h_{p-r,q-s}$. Example: Ising model $\mathcal{M}_{4,3}$, tricritical Ising model $\mathcal{M}_{5,4}$, Yang–Lee singularity $\mathcal{M}_{5,2}$.

Unitary minimal model $\mathcal{M}_{k+2,k+1}$ is coset $\frac{\hat{\mathfrak{su}}(2)_{k-1} \times \hat{\mathfrak{su}}(2)_1}{\hat{\mathfrak{su}}(2)_k}$

§7.2 Chern–Simons

Chern–Simons $(2m-1)$-form $m \operatorname{Tr}\bigl(A \int_0^1 \mathrm{d}t (tdA + t^2A^2)^{m-1}\bigr)$.

§7.3 Supergravity and strings

String actions Polyakov action $L_{\mathrm{P}} = \lambda^{mn}[(\partial_m X)(\partial_n X) - g_{mn}] + \frac{1}{\alpha'}\sqrt{-g}$. Using equations of motion get Nambu–Goto action $L_{\mathrm{NG}} = \frac{1}{\alpha'}\sqrt{-\det[(\partial_m X)(\partial_n X)]}$ or Brink–di Veccia–Howe–Deser–Zumino action $L_{\mathrm{BdVHDZ}} = \frac{1}{2\alpha'}\sqrt{-g}[g^{mn}(\partial_m X)(\partial_n X) - (d-2)]$ with $d = 2$ the world-sheet dimension.

Pure supergravities in $4 \le d \le 11$. Gravity is topological in $d = 3$. The maximum number of supercharges $Q = 32$ forbids $d > 11$. A priori, all $Q = 4k$ are possible. Focus on $32, 16, 8, 4$.

d	$Q = 32$	16	8	4
11	✓			
10	$\overset{IIB}{(2,0)}$ $\overset{IIA}{(1,1)}$	$\overset{I}{(1,0)}$		
9	✓	✓		
8	✓	✓		
7	✓	✓		
6	$(2,2)$	$(2,0)$ $(1,1)$	$(1,0)$	
5	✓	✓	✓	
4	$N = 8$	$N = 4$	$N = 2$	$N = 1$

M-theory has as its low-energy limit 11d supergravity, which has two $\frac{1}{2}$-BPS membrane solutions (with 16 Killing spinors): M2-brane $\mathrm{d}s^2 = \Lambda^4\,\mathrm{d}x^2 + \frac{\mathrm{d}y^2}{\Lambda^2}$ with $\Lambda = (1 + \frac{c_2 N_2 l^6}{|y|^6})^{-1/6}$, and M5-brane $\mathrm{d}s^2 = \Lambda\,\mathrm{d}x^2 + \mathrm{d}y^2/\Lambda^2$ with $\Lambda = (1 + \frac{c_5 N_5 l^3}{|y|^3})^{-1/3}$, where $x \in \mathbb{R}^{p,1}$ and $y \in \mathbb{R}^{10-p}$. In the near horizon $y \to 0$ these become $\mathrm{AdS}_4 \times S^7$ and $\mathrm{AdS}_7 \times S^4$ with 32 Killing spinors.

Branes IIA strings: D0, F1 (strings), D2, D4, O4$^\pm$, $\widetilde{\mathrm{O4}}^+$, NS5, D6, D8 (wall), O8 (wall), etc.. IIB strings: D(-1), F1 (strings), D1, D3, (p,q) 5-branes (includes D5 and NS5), O5$^\pm$, $\widetilde{\mathrm{O5}}^+$, D7, O7$^\pm$, ON^0, etc.. M-theory: M2, M5, OM5, M9.

Flat space brane configurations Flat space preserves 32 supercharges. One stack of parallel branes breaks half; two stacks break all unless: Dp and Dq have 0, 4 or 8 directions that one brane spans and not the other; Dp branes have 1 or 3 directions transverse to any NS5; any pair of NS5 branes has 2, 4, 6 common directions. Kappa-projection: for Dp is $\Gamma_{01\ldots p}\epsilon_L = \epsilon_R$, for NS5 is $\Gamma_{01\ldots 5}\epsilon_L = \epsilon_L$. Then at least $32/2^{\#\mathrm{stacks}}$ supercharges preserved.

S-rule, brane creation Let a Dp have 3 directions transverse to an NS5. Zero or one D$(p-2)$ can stretch between the two (spanning common directions and directions where neither Dp nor NS5 stretch). Moving Dp through NS5 toggles between zero and one.

Little string theory (LST) Decoupled $g_s \to 0$, fixed α description of k coincident NS5 branes on transverse T^5 gives $(1,1)$ LST for IIB and $(2,0)$ LST for IIA. Has AdS/CFT dual with linear dilaton background.

§7.4 Integrable models

Relativistic quantum Toda chain. $H = \sum_{n=1}^N \bigl(\cos(2\eta\hat{p}_n) + g^2\cos(\eta\hat{p}_n + \eta\hat{p}_{n+1})\mathrm{e}^{x_{n+1}-x_n}\bigr)$. Its non-relativistic limit is $\eta \to 0$ imaginary with $g/(i\eta\sqrt{2}) = c$ fixed.

§7.5 Localization results

3d $\mathcal{N} = 2$: $Z = \int_{\mathfrak{t}} \mathrm{d}u\, \frac{\prod_{\alpha\ \mathrm{root}}(2\sinh(\alpha u/2))^2}{\prod_{w\in\mathcal{R}}\cosh(wu/2)} \mathrm{e}^{ik\operatorname{Tr} u^2/(4\pi)}$.

§8 Manifolds

§8.1 Riemannian geometry

§8.2 G-structures, holonomy

Structure group. A G-structure on a manifold X (with $n = \dim_{\mathbb{R}} X$) is a G-subbundle of the $\mathrm{GL}(n,\mathbb{R})$-principal bundle $\mathrm{GL}(TX)$ of tangent frames, namely a global section of $\mathrm{GL}(TX)/G$.

A manifold is oriented if it has a $\mathrm{GL}_+(n,\mathbb{R}) = \{\det > 0\}$ structure. Similar definitions for Riemannian manifolds etc.:

G-structure	Manifold type	Other characterization‡
$\mathrm{O}(n)$	Riemannian	metric $g > 0$
$\mathrm{SO}(n)$	oriented, Riemannian	
$\mathrm{O}(p,q)$	pseudo-Riemannian	metric of signature (p,q)
$\mathrm{SO}_+(p,q)$	pseudo-Riemannian, oriented, time-oriented	
$\mathrm{Pin}_\pm$ or Spin	(pseudo)-Riemannian $\mathrm{pin}_\pm$ or spin manifold	
$\mathrm{GL}(n/2,\mathbb{C})$	Almost complex	$\mathbb{C} \circlearrowright TX$ (i.e., $J^2 = -1$)
$\mathrm{Sp}(2n/2,\mathbb{R})$	Almost symplectic	Non-degenerate $\omega \in \Omega^2 X$
$\mathrm{U}(n/2)$	Almost Hermitian	Two compatible (g, J, ω)§
$\mathrm{U}^*(n/2)$	Almost hypercomplex¶	$J_1, J_2, J_3 \circlearrowright TX$
$\mathrm{USp}(n/2)$	Almost hyperHermitian	$(g, J_{1,2,3}, \omega_{1,2,3})$
$\mathrm{U}^*(n/2)\mathrm{USp}(2)$	Almost quaternionic¶	$\mathbb{H} \circlearrowright TX$
$\mathrm{USp}(n/2)\mathrm{USp}(2)$	Almost quaternion-Hermitian	$(g, \mathbb{H}, \omega_{1,2,3})$

‡ All sections are global. For instance, an almost complex structure is a global section J of $\operatorname{End} TX$ with $J^2 = -1$. A metric is a global section g of $S^2(T^*X)$.

§ Any two of (g, J, ω) fix the third by $\omega_{ik} = J_i{}^j g_{jk}$ if they are compatible: $J_i{}^j J_l{}^k \omega_{jk} = \omega_{il}$ or $J_i{}^j J_l{}^k g_{jk} = g_{il}$ namely ω or g is J-invariant, or $\omega_{ij} g^{jk} \omega_{kl} = -g_{il}$. In a basis $e^\beta, \bar{e}^{\bar\gamma}$ ($= \mathrm{d}z^\beta, \mathrm{d}\bar{z}^{\bar\gamma}$ for Hermitian manifolds) of $(1,0)$ and $(0,1)$ forms, $\omega = \frac{\mathrm{i}}{2} h_{\beta\bar\gamma}\, e^\beta \wedge \bar{e}^{\bar\gamma}$ and $g = \frac{1}{2} h_{\beta\bar\gamma}(e^\beta \otimes \bar{e}^{\bar\gamma} + \bar{e}^{\bar\gamma} \otimes e^\beta)$.

On an almost complex manifold, (p,q)-forms are wedge products $\Omega^{(p,q)} X = \bigwedge^p \big(\Omega^{(1,0)} X\big) \wedge \bigwedge^q \big(\Omega^{(0,1)} X\big)$ where J acts by $\pm\mathrm{i}$ on $\Omega^1 X = \Omega^{(1,0)} X \oplus \Omega^{(0,1)} X$. The exterior derivative is $\mathrm{d} = \mathrm{d}^{2,-1} + \mathrm{d}^{1,0} + \mathrm{d}^{0,1} + \mathrm{d}^{-1,2}$ with $\mathrm{d}^{i,j} : \Omega^{(p,q)} \to \Omega^{(p+i,q+j)}$. Dolbeault differential operators are $\partial = \mathrm{d}^{1,0}$ and $\overline{\partial} = \mathrm{d}^{0,1}$.

An almost symplectic $2m$-manifold admits the volume form $\omega^m/m!$. On an almost Hermitian manifold X it is equal to the Riemannian volume form and belongs to $\Omega^{(m,m)} X$.

¶ While almost quaternionic manifolds have a 3d subbundle of $\operatorname{End} TX$ locally spanned by J_1, J_2, J_3 with $J_i^2 = J_1 J_2 J_3 = -1$, almost hypercomplex manifolds require J_1, J_2, J_3 to be global.

Integrability. A G-structure is k-integrable (resp. integrable) near $x \in X$ if it can be trivialized to order k (resp. all orders) in a neighborhood of x. We automatically have 0-integrability.

Any Riemannian structure is 1-integrable thanks to Riemann normal coordinates. Integrability is equivalent to the Riemann curvature vanishing.

An almost complex structure is complex if (equivalently) it is integrable; it is 1-integrable; it has a vanishing Nijenhuis tensor $N_J : \bigwedge^2 X \to TX$ defined on vector fields u, v by the Lie brackets $N_J(u,v) = -J^2[u,v] + J[Ju,v] + J[u,Jv] - [Ju,Jv]$; the Lie bracket of $(1,0)$ vector fields is a $(1,0)$ vector field; $\mathrm{d} = \partial + \overline{\partial}$ namely $\mathrm{d}^{2,-1} = 0 = \mathrm{d}^{-1,2}$; or $\overline{\partial}^2 = 0$.

A symplectic structure is an integrable almost symplectic structure. Equivalently, it is 1-integrable: $\mathrm{d}\omega = 0$. Altogether,

$$\begin{tikzcd}
\text{general} \arrow[rr, "\omega"] \arrow[dr, "g\ddagger"] \arrow[dd, "J"'] & & \text{almost symplectic} \arrow[r, "\mathrm{d}\omega=0"] \arrow[dd, "J \text{ or } g"] & \text{symplectic} \arrow[dd, "J"] \\
& \text{Riemannian} \arrow[dr, "\omega \text{ or } J"'] & & \\
\text{almost complex} \arrow[rr, "\omega \text{ or } g\ddagger"] \arrow[d, "N_J=0"] & & \text{almost Hermitian} \arrow[r, "\mathrm{d}\omega=0"] \arrow[d, "N_J=0"] \arrow[dr, "\S"] & \text{almost Kähler} \arrow[d, "N_J=0"] \\
\text{complex} \arrow[rr, "\omega \text{ or } g\ddagger"] & & \text{Hermitian} \arrow[r, "\mathrm{d}\omega=0"] & \text{Kähler}
\end{tikzcd}$$

(Almost) quaternionic/quaternionHermitian/quaternionKähler and (almost) hypercomplex/hyperHermitian/hyperKähler manifolds are defined by replacing J by a 3d subbundle of $\operatorname{End} TX$ or by global sections J_1, J_2, J_3 as in the table of G-structures.

‡ Since $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$ is contractible, any manifold admits (non-canonically) an $\mathrm{O}(n)$-structure, namely a smooth choice of which frames are orthonormal, i.e., a Riemannian metric g. Similarly $\mathrm{GL}(n/2,\mathbb{C})/\mathrm{U}(n/2)$ is contractible so almost complex manifolds admit almost Hermitian structures.

§ An almost Hermitian manifold is Kähler if (equivalently) its $\mathrm{U}(n/2)$-structure is 1-integrable; $\mathrm{d}\omega = 0$ and $N_J = 0$; $\nabla\omega = 0$; $\nabla J = 0$; or the holonomy group is in $\mathrm{U}(n/2)$. Locally, $\omega = \mathrm{i}\partial\overline{\partial}\rho$ for some real-valued Kähler potentials ρ, and ω is invariant under Kähler transformations $\rho \to \rho + f(z) + \bar{f}(\bar{z})$.

The holonomy group at $x \in X$ of a connection ∇ on a bundle $E \to X$ is the group of symmetries of E_x arising from parallel transport along closed curves based at x.

For Riemannian manifolds X the holonomy group is defined as that of the Levi-Civita connection on the tangent bundle. It is a subgroup of $\mathrm{O}(n)$ $\big($or $\mathrm{SO}(n)$ for X orientable$\big)$ since parallel transport preserves orthogonality ($\nabla g = 0$).

If the holonomy group acts reducibly on the tangent space then X is locally (globally if X is geodesically complete) a product. Simply connected X that are locally neither products nor symmetric spaces (we give a list later) can have the following special holonomy subgroups of $\mathrm{SO}(n)$ (Berger's theorem)

Holonomy	Manifold type	$\dim_{\mathbb{R}}$
$\mathrm{U}(m)$	Kähler	$2m$
$\mathrm{SU}(m)$	Calabi–Yau CY_m	$2m$
$\big(\mathrm{USp}(2k) \times \mathrm{USp}(2)\big)/\mathbb{Z}_2$	quaternionic Kähler	$4k$
$\mathrm{USp}(2k)$	hyperKähler	$4k$
Spin(7)	Spin(7) manifold	8
G_2	G_2 manifold	7

Note that hyperKähler $\implies$ Calabi–Yau $\implies$ Kähler since $\mathrm{USp}(m) \subset \mathrm{SU}(m) \subset \mathrm{U}(m)$. In contrast, quaternionic-Kähler manifolds are not Kähler.

A Calabi–Yau manifold is a Kähler manifold such that (equivalently) some Kähler metric has global holonomy group in $\mathrm{SU}(m)$; the structure group can be reduced to $\mathrm{SU}(m)$; or the holomorphic canonical bundle is trivial i.e., there exists a nowhere vanishing holomorphic top-form. A weaker set of equivalent conditions

todo: here

For simply connected manifolds, the conditions above are equivalent to the following (always equivalent) conditions on X: some Kähler metric has local holonomy group in $\mathrm{SU}(m)$; some Kähler metric has vanishing Ricci curvature; the first real Chern class vanishes; a positive power of the holomorphic canonical bundle is trivial; X has a finite cover with trivial holomorphic canonical bundle; X has a finite cover equal to the product of a torus and a simply connected manifold with trivial holomorphic canonical bundle.

Spin structures *todo: see* `http://mathoverflow.net/questions/220502/`

Symmetric spaces *todo: list missing*

K3 surfaces are the only CY_2: they have holonomy $\mathrm{SU}(2)$.

Yau's theorem. Fix a complex structure on a compact complex manifold X of $\dim_{\mathbb{C}} X > 1$ and vanishing real first Chern class. Any real class $H^{1,1}(X,\mathbb{C})$ of positive norm contains a unique Kähler form whose metric is Ricci flat.

(from Wikipedia on Calabi conjecture: "The Calabi conjecture states that a compact Khler manifold has a unique Khler metric in the same class whose Ricci form is any given 2-form representing the first Chern class.")

§9 Dualities

§9.1 Field theory dualities

2d $\mathcal{N}=(0,2)$ Gadde–Gukov–Putrov triality (IR).

2d $\mathcal{N}=(2,2)$ mirror symmetry of Calabi–Yau sigma models (exact).

2d $\mathcal{N}=(2,2)$ Hori–Tong (SU), Hori (Sp, SO groups), plus adjoint (ADE-type and $(2,2)^*$-like) dualities (IR).

2d $\mathcal{N}=(2,2)$ Hori–Vafa/Hori–Kapustin duality of gauged linear sigma models and Landau–Ginzburg models (IR).

3d Chern–Simons level-rank duality.

3d $\mathcal{N}=2$ Aharony, Giveon–Kutasov, Aharony–Fleischer dualities (IR).

3d $\mathcal{N}=2$ and $\mathcal{N}=4$ mirror symmetry exchanging Coulomb and Higgs branches (IR).

4d $\mathcal{N}=1$ Seiberg, Kutasov–Schwimmer, Brodie, Intriligator–Pouliot, Argyres–Intriligator–Leigh–Strassler, Klebanov cascade, Intriligator–Leigh–Strassler, duality (IR).

S-duality of 4d $\mathcal{N}=2$ gauge theories (exact).

S-duality of 4d $\mathcal{N}=4$ SYM (exact).

§9.2 4d $\mathcal{N}=1$ dualities

Seiberg: $\mathrm{SU}(N_c)$, $N_f\square$, $N_f\overline{\square}$ $\Leftrightarrow$ $\mathrm{SU}(N_f-N_c)$, $N_f\square$, $N_f\overline{\square}$, N_f^2 free, with $W=M\tilde{Q}Q$.

Seiberg: $\mathrm{SO}(N_c)$, $N_f\square \Leftrightarrow \mathrm{SO}(N_f-N_c+4)$, $N_f\square$, #? free, $W=$?

Seiberg: $\mathrm{USp}(2N_c)$, $2N_f\square \Leftrightarrow \mathrm{USp}(2N_f-2N_c-4)$, $2N_f\square$, #? free, $W=$?

These three cases are self-dual when $C(R_{\text{chirals}})=2C(\text{adj})$, namely $N_f=2N_c$, $N_f=2(N_c-2)$ and $N_f=2(N_c+1)$ respectively; adding an adjoint gives $\mathcal{N}=2$ SCFTs.

§9.3 String theory dualities

In this table "type IIA" etc. refer to string theories not supergravities

F-theory on K3	$\Leftrightarrow$ $E_8\times E_8$ heterotic on T^2
M-theory on K3	$\Leftrightarrow$ heterotic or type I on T^3
Type IIA on K3	$\Leftrightarrow$ heterotic or type I on T^4
M-theory on G_2-manifolds[1]	$\Leftrightarrow$ heterotic or type I on CY_3
M-theory on K3[2]	$\Leftrightarrow$ type IIA on $T^3/\mathbb{Z}_2$

§10 Misc

§10.1 Physics of gauge theories

Phases characterized by potential $V(R)$ (up to a constant) between quarks at distance R: Coulomb $1/R$, free electric $1/\big(R\log(R\Lambda)\big)$, free magnetic $\log(R\Lambda)/R$, Higgs (constant), confining σR.

§10.2 Homology and cohomology

$H_k(\mathbb{CP}^n,M)=M$ for $0\le k\le 2n$ even, else 0.

§10.3 Homotopy groups π_n

Basic properties. $\pi_0(X,x)$ is the set of connected components. $\pi_1(X,x)$ is the fundamental group. For $k\ge 1$, $\pi_k(X,x)$ only depends on the connected component of x. $\pi_k(X\times Y,(x,y))=\pi_k(X,x)\times\pi_k(Y,y)$.

Quotient. If G acts on connected simply-connected X then $\pi_1(X/G)=\pi_0(G)$ ($=G$ for G discrete).

Long exact sequence for a fiber bundle $F\hookrightarrow E\twoheadrightarrow B$: for base-points $b_0\in B$ and $e_0=f_0\in F=p^{-1}(b_0)\subset E$, $\cdots\to\pi_{i+1}(B)\to\pi_i(F)\to\pi_i(E)\to\pi_i(B)\to\cdots\to\pi_0(E)$ is exact, namely each image equals the next kernel (inverse image of the constant map).

Homotopy groups of spheres are finite except $\pi_n(S^n)=\mathbb{Z}$ and $\pi_{4n-1}(S^{2n})=\mathbb{Z}\times\text{finite}$. For $k<n$, $\pi_k(S^n)=0$, and $\pi_{n+k}(S^n)$ is independent of n for $n\ge k+2$. All $\pi_k(S^0)=0$, $\pi_k(S^1)=0$ for $k\ne 1$, and $\pi_k(S^3)=\pi_k(S^2)$ for $k\ne 2$.

	π_1	π_2	π_3	π_4	π_5	π_6	π_7	π_8
S^0	0	0	0	0	0	0	0	0
S^1	$\mathbb{Z}$	0	0	0	0	0	0	0
S^2	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^3	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{12}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$
S^4	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}\times\mathbb{Z}_{12}$	$\mathbb{Z}_2^2$
S^5	0	0	0	0	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{24}$

$\pi_1(\mathbb{RP}^n)=\mathbb{Z}_2$ for $n\ge 2$ and $\pi_k(\mathbb{RP}^n)=\pi_k(S^n)$ for $k\ge 2$. $\pi_1(\mathbb{CP}^n)=0$, $\pi_2(\mathbb{CP}^n)=\mathbb{Z}$, $\pi_k(\mathbb{CP}^n)=\pi_k(S^{2n+1})$ for $k\ge 3$.

Topological groups have abelian $\pi_1(G)$. Proofs. 1. The multiplication in G (point-wise) and concatenation of loops are two compatible group structures, hence (by Eckmann–Hilton theorem) coincide and are commutative. 2. Explicitly, for $\alpha_1,\alpha_2\in\pi_1(G)$ loops, $(t_1,t_2)\mapsto\alpha_1(t_1)\alpha_2(t_2)\in G$ is a homotopy between $\alpha_1\star\alpha_2$ (concatenation) along bottom and right edges, $\alpha_1\cdot\alpha_2$ (point-wise multiplication) along the diagonal, and $\alpha_2\star\alpha_1$ along left and top edges.

§10.4 Kähler 4-manifolds

K3 surfaces are (the only besides T^4) compact complex surfaces of trivial canonical bundle. They have $h^{1,0}=0$ (in contrast to T^4 which has *todo: value*). Their first Chern class $c_1\in H^2(X,\mathbb{Z})$ thus vanishes. By Yau's theorem there exists a Ricci flat metric, whose holonomy is then $\mathrm{SU}(2)=\mathrm{USp}(2)$ by Berger's classification. K3 surfaces are thus Calabi–Yau (CY_2) and hyperKähler (hK_4). Their moduli space is connected and they are all diffeomorphic.

Examples of K3 surfaces. Quartic hypersurface in $\mathbb{P}^4$. Kummer surface namely resolution of $T^4/\mathbb{Z}_2$.

Non-simply connected Ricci-flat Kähler manifolds may fail to be CY_n when the restricted holonomy group is $\mathrm{SU}(n)$ but the global holonomy group is disconnected. For example an Enriques surface $K3/\mathbb{Z}_2$ has a non-trivial canonical bundle.

A gravitational instanton is a metric with (anti-)self-dual curvature. A simply-connected Riemannian 4-manifold is hyperKähler if and only if it is a gravitational instanton. Compact hK_4 are K3 and T^4. Non-compact hK_4 are ALE (asymptotically locally Euclidean), ALF (asymptotically locally flat), ALG, ALH if their volume growth rate is of order 4, 3, 2, 1. ALE spaces are resolutions of $\mathbb{H}/\Gamma$ for a finite subgroup $\Gamma<\mathrm{USp}(2)$. The quotient $\mathbb{H}/\Gamma$ can appear as a local model of an orbifold singularity in a K3 surface.

ALE hyperKähler 4-manifolds X are diffeomorphic to the minimal resolution of $\mathbb{H}/\Gamma$ for some finite $\Gamma \subset \mathrm{SU}(2)$. The metric is fixed (up to isometry) by cohomology classes $\alpha_1, \alpha_2, \alpha_3 \in H^2(X, \mathbb{R})$ such that there is no two-cycle Σ such that $\Sigma \cdot \Sigma = -2$ and all $\alpha_i(\Sigma) = 0$.

todo: Taub-NUT spaces, multi-Taub-NUT spaces, Eguchi–Hanson spaces, Gibbons–Hawking multicenter spaces. Write metric. todo: Non-explicitly: Atiyah–Hitchin space (moduli space of two $\mathrm{SU}(2)$ *'t Hooft-Polyakov monopoles in 4d).*

todo: The only compact CY_2 *are* T^4 *and K3 surfaces.*

todo: The only compact hypercomplex 4*-manifolds are* T^4*, K3 surfaces, and the Hopf surface* $((\mathbb{H}\backslash 0)/(q^{\mathbb{Z}})$ *for a quaternion* $|q| > 1$*; it is diffeomorphic to* $S^3 \times S^1$*).*

§10.5 Some algebraic constructions

Reduction of a Lie (super)algebra $\mathfrak{g}$. If $\mathfrak{g} = V_1 \oplus V_2$ with $[V_1, V_2] \subseteq V_2$ then the bracket of $\mathfrak{g}$ restricted and projected to V_1 defines a Lie (super)algebra.

S**-expansion of a Lie (super)algebra** $\mathfrak{g}$ by an abelian multiplicative semigroup S: Lie (super)algebra $\mathfrak{g} \times S$ with bracket $[(x, \alpha), (y, \beta)] = ([x, y], \alpha\beta)$. If $S = S_1 \cup S_2$ with $S_1 S_2 \subseteq S_2$ (in particular if there is a zero element $0_S = 0_S\alpha = \alpha 0_S$) then by reduction we get a Lie (super)algebra structure on $\mathfrak{g} \times S_1$.

A color (super)algebra is a graded vector space with a bracket such that (for X, Y, Z with definite grading) $\mathrm{gr}[X, Y] = \mathrm{gr}\, X + \mathrm{gr}\, Y$ and $[X, Y] = -(-1)^{(\mathrm{gr}\, X, \mathrm{gr}\, Y)}[Y, X]$ and $[X, [Y, Z]](-1)^{(\mathrm{gr}\, Z, \mathrm{gr}\, X)} + [Y, [Z, X]](-1)^{(\mathrm{gr}\, X, \mathrm{gr}\, Y)} + [Z, [X, Y]](-1)^{(\mathrm{gr}\, Y, \mathrm{gr}\, Z)} = 0$, where $(\bullet, \bullet)$ is some bilinear mapping into $\mathbb{C}/(2\mathbb{Z})$.

§10.6 Other

A fuzzy space is d Hermitian matrices X^a ("coordinates") acting on some Hilbert space H. The dispersion of $\psi \in H$ is $\delta_\psi = \sum_a \big(\langle\psi|(X^a)^2|\psi\rangle - \langle\psi|X^a|\psi\rangle^2\big)$.

[1] Tools for supersymmetry by Antoine Van Proeyen

[2] Various Wikipedia articles.

[3] Various ncatlab.org articles.