Sum-of-Squares Programming in Julia with JuMP

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Sum-of-Squares Programming

Nonnegative quadratic forms into sum of squares

(x_1, x_2, x_3) unique

$$p(x) = x^\top Q x$$

$$x_1^2 + 2x_1x_2 + 5x_2^2 + 4x_2x_3 + x_3^2 = x^\top \begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix} x$$

$$p(x) \geq 0 \ \forall x \Longleftrightarrow Q \succeq 0$$

cholesky

$$(x_1 + x_2)^2 + (2x_2 + x_3)^2 \longleftarrow x^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} x$$

Nonnegative polynomial into sum of squares

(x_1, x_2, x_3) (x_1, x_1x_2, x_2) *not* unique

$$p(x) = X^\top Q X$$

$$x_1^2 + 2x_1^2x_2 + 5x_1^2x_2^2 + 4x_1x_2^2 + x_2^2 = X^\top \begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix} X$$

$$p(x) \geq 0 \ \forall x \Longleftarrow Q \succeq 0$$

cholesky

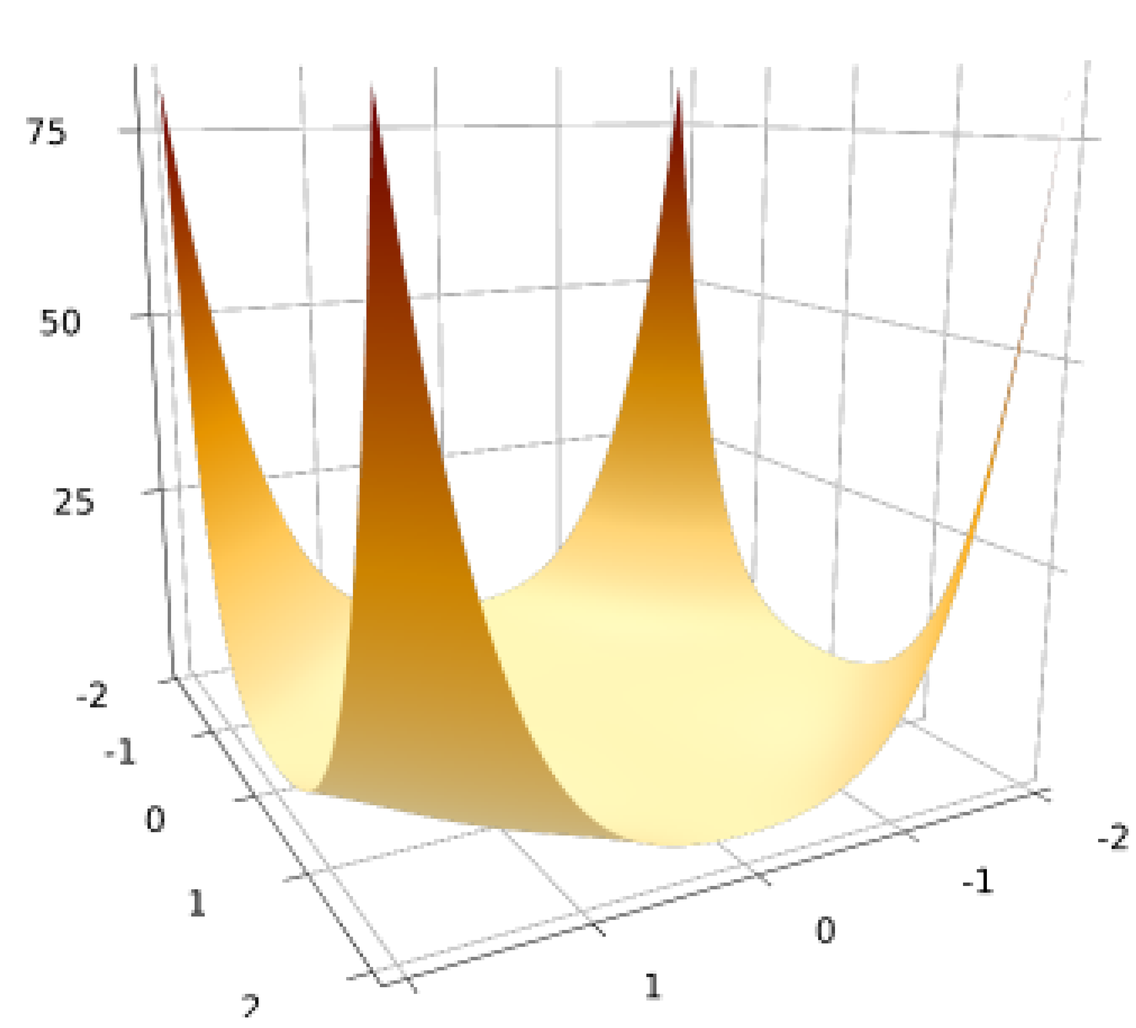
$$(x_1 + x_1x_2)^2 + (2x_1x_2 + x_2)^2 \longleftarrow X^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} X$$

When is nonnegativity equivalent to sum of squares ?

Determining whether a polynomial is nonnegative is NP-hard.

Hilbert 1888 Nonnegativity of $p(x)$ of n variables and degree $2d$ is equivalent to sum of squares in the following three cases:
- $n = 1$: Univariate polynomials
- $2d = 2$: Quadratic polynomials
- $n = 2$, $2d = 4$: Bivariate quartics

Motzkin 1967 First explicit example:
$$x_1^4x_2^2 + x_1^2x_2^4 + 1 - 3x_1^2x_2^2 \geq 0 \quad \forall x$$
but is not a sum of squares.



Manipulating Polynomials

Two implementations: `TypedPolynomials` and `DynamicPolynomials`.
One common independent interface: `MultivariatePolynomials`.

```
@polyvar y # one variable
@polyvar x[1:2] # tuple/vector
```

Build a vector of monomials:

```
X = monomials(x, 2) # [x1^2, x1*x2, x2^2]
X = monomials(x, 0:2) # [x1^2, x1*x2, x2^2, x1, x2, 1]
```

Polynomial variables

By hand, with an integer variable `a`:

```
@variable(model, a, Int)
@variable(model, b)
p = a*x^2 + (a+b)*y^2*x + b*y^3
```

From a polynomial basis, e.g. the *scaled monomial* basis, with integer coefficients:

```
@variable(model, Poly(ScaledMonomialBasis(X)),
          Int)
```

Polynomial constraints

Constrain $p(x, y) \geq q(x, y) \ \forall x, y$ such that $x \geq 0, y \geq 0, x + y \geq 1$ using the scaled monomial basis.

```
S = @set x >= 0 && y >= 0 && x + y >= 1
@constraint(model, p >= q, domain = S,
            basis = ScaledMonomialBasis)
```

Interpreted as:

```
@constraint(model, p - q in SOSCone(), domai
            basis = ScaledMonomialBasis)
```

To use DSOS or SDSOS [1]:

```
@constraint(model, p - q in DSOSCone())
@constraint(model, p - q in SDSOSCone())
```

SOS on algebraic domain

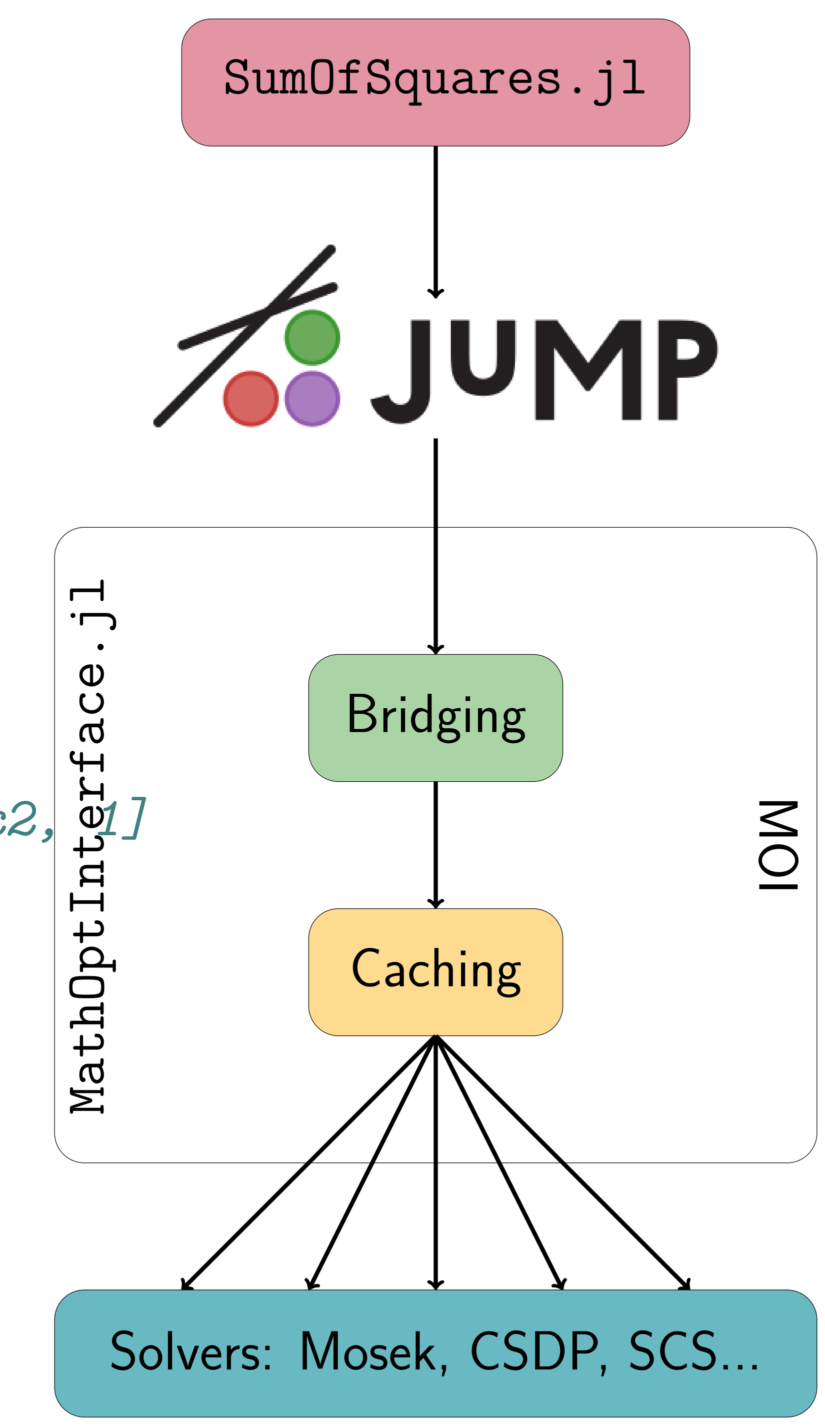
The domain S is defined by equalities and inequalities q_i. The equalities form an *algebraic variety* V. We then search for Sum-of-Squares polynomials s_i such that

$$p(x) - q(x) \equiv s_0(x) + s_1(x)q_1(x) + \cdots \pmod V$$

Groebner basis of V is computed to do the division.

Dual value

The dual of the constraint is a PSD matrix of moments μ. The `extractatoms` function attempts to find an *atomic* measure with these moments by solving an algebraic system.



Bridging Automatic reformulation of a constraint into an equivalent form supported by the solver. In particular, reformulates SOS constraints into PSD constraints (except `Alfonso.jl` implementing [2]).

Caching Cache of the problem data in case the solver do not support a modification (can be disabled).

References

[1] Amir Ali Ahmadi and Anirudha Majumdar. "DSOS and SDSOS optimization: more tractable alternatives to sum of squares and semidefinite optimization". In: *arXiv preprint arXiv:1706.02586* (2017).

[2] D. Papp and S. Yıldız. "Sum-of-squares optimization without semidefinite programming". In: *ArXiv e-prints* (Dec. 2017). arXiv: `1712.01792 [math.OC]`.