Sum-of-Squares Programming in Julia with JuMP

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Sum-of-Squares (SOS) Programming

Nonnegative quadratic forms into sum of squares

(x_1, x_2, x_3) unique

$$p(x) = x^\top Q x$$

$$x_1^2 + 2x_1x_2 + 5x_2^2 + 4x_2x_3 + x_3^2 = x^\top \begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix} x$$

$$p(x) \geq 0\ \forall x \iff Q \succeq 0$$

cholesky

$$(x_1 + x_2)^2 + (2x_2 + x_3)^2 \longleftarrow x^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} x$$

Nonnegative polynomial into sum of squares

(x_1, x_2, x_3) (x_1, x_1x_2, x_2) *not* unique

$$p(x) = X^\top Q X$$

$$\begin{matrix} x_1^2 + 2x_1^2x_2 + 5x_1^2x_2^2 \\ +4x_1x_2^2 + x_2^2 \end{matrix} = X^\top \begin{pmatrix} 1 & 1 & 0 \\ 1 & 5 & 2 \\ 0 & 2 & 1 \end{pmatrix} X$$

$$p(x) \geq 0\ \forall x \Longleftarrow Q \succeq 0$$

cholesky

$$\begin{matrix} (x_1 + x_1x_2)^2 + \\ (2x_1x_2 + x_2)^2 \end{matrix} \longleftarrow X^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}^\top \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} X$$

When is nonnegativity equivalent to sum of squares ?

Determining whether a polynomial is nonnegative is NP-hard.

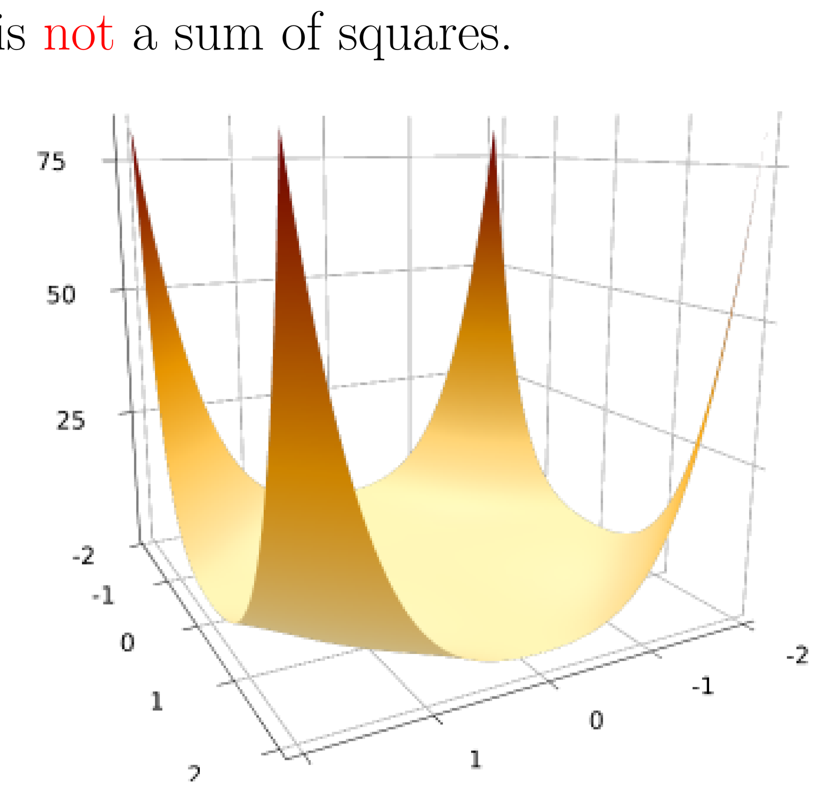
Hilbert 1888 Nonnegativity of $p(x)$ of n variables and degree $2d$ is equivalent to sum of squares in the following three cases:

- $n = 1$: Univariate polynomials
- $2d = 2$: Quadratic polynomials
- $n = 2$, $2d = 4$: Bivariate quartics

Motzkin 1967 First explicit example:

$$x_1^4x_2^2 + x_1^2x_2^4 + 1 - 3x_1^2x_2^2 \geq 0 \quad \forall x$$

but is not a sum of squares.



Manipulating Polynomials

Two implementations: `TypedPolynomials.jl` and `DynamicPolynomials.jl`.
One common independent interface: `MultivariatePolynomials.jl`.

```
@polyvar y # one variable
@polyvar x[1:2] # tuple/vector
```

Build a vector of monomials:

- (x_1^2, x_1x_2, x_2^2):
  `X = monomials(x, 2)`
- $(x_1^2, x_1x_2, x_2^2, x_1, x_2, 1)$:
  `X = monomials(x, 0:2)`

Polynomial variables

By hand, with an integer decision variable `a` and real decision variable `b`:

```
@variable(model, a, Int)
@variable(model, b)
p = a*x^2 + (a+b)*y^2*x + b*y^3
```

From a polynomial basis, e.g. the *scaled monomial* basis, with integer decision variables as coefficients:

```
@variable(model,
          Poly(ScaledMonomialBasis(X)),
          Int)
```

Polynomial constraints

Constrain $p(x, y) \geq q(x, y)\ \forall x, y$ such that $x \geq 0, y \geq 0, x + y \geq 1$ using the scaled monomial basis:

```
S = @set x >= 0 && y >= 0 && x + y >= 1
@constraint(model, p >= q, domain = S,
            basis = ScaledMonomialBasis)
```

Interpreted as:

```
@constraint(model, p - q in SOSCone(),
            domain = S,
            basis = ScaledMonomialBasis)
```

To use DSOS or SDSOS (Ahmadi, Majumdar 2017):

```
@constraint(model, p - q in DSOSCone())
@constraint(model, p - q in SDSOSCone())
```

SOS on algebraic domain

The domain S is defined by equalities forming an *algebraic variety* V and inequalities q_i. We search for Sum-of-Squares polynomials s_i such that

$$p(x) - q(x) \equiv s_0(x) + s_1(x)q_1(x) + \cdots \pmod{V}$$

The Gröbner basis of V is computed the equation is reduced modulo V.

Dual value

The dual of the constraint is a positive semidefinite (PSD) matrix of moments μ. The `extractatoms` function attempts to find an *atomic* measure with these moments by solving an algebraic system.

Sum-of-Squares extension

`MathOptInterface.jl` (MOI)

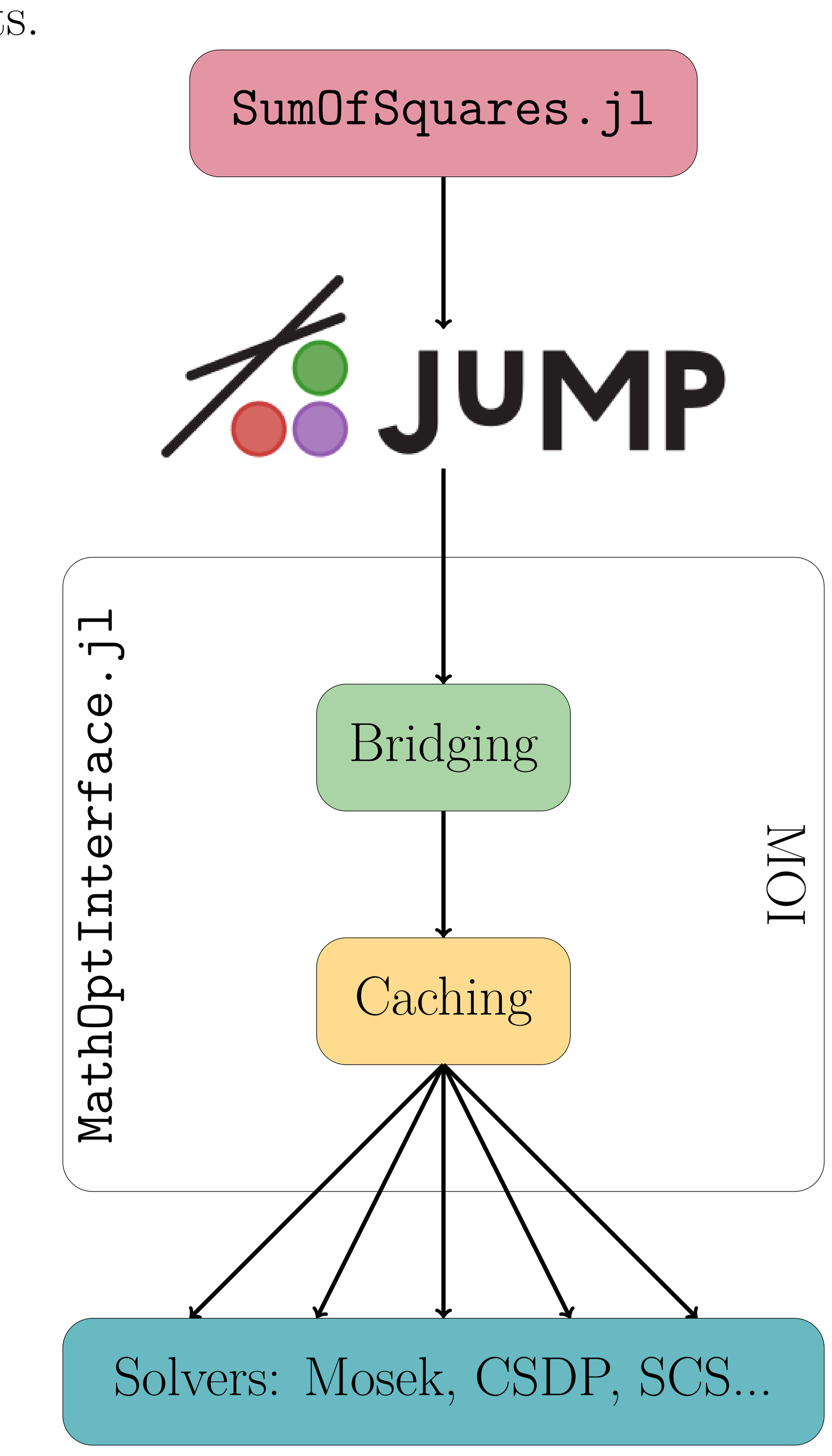
MOI is an abstraction layer for mathematical optimization solvers. A constraint is defined by a "function" ∈ "set" pair.

MOI extension: `AbstractVectorFunction` ∈ `SOS(X)` (resp. `WSOS(X)`): SOS constraint without (resp. with) `domain` equipped with a bridge to `AbstractVectorFunction` ∈ `PSD` (resp. `SOS(X)`).

JuMP

JuMP is a domain-specific modeling language for mathematical optimization. It stores the problem directly (a cache can optionally be used) in the solver using MOI.

JuMP extension: $p(x) \geq q(x)$ and $p(x) \in$ `SOS()` are rewritten into MOI SOS or WSOS constraints, e.g. $x^2 + y^2 \geq 2xy$ is rewritten into $[1, -2, 1] \in \mathrm{SOS}(x^2, xy, y^2)$. $p(x) \in$ `DSOS()` (resp. `SDSOS()`) is rewritten into linear (resp. second-order cone) constraints.



Bridging Automatic reformulation of a constraint into an equivalent form supported by the solver, e.g. quadratic constraint into second-order cone constraint. In particular, reformulates SOS/WSOS constraints into PSD constraints. An interior-point solver that natively supports SOS and WSOS without reformulation to SDP using the approach of (Papp, Yıldız 2017) is under development.

Caching Cache of the problem data in case the solver do not support a modification (can be disabled). For instance, Mosek provides many modification capabilities in the API but CSDP only support pre-allocating and then loading the whole problem at once.