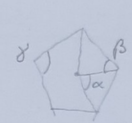
a = Kantenlänge

R = Radius Umkugel

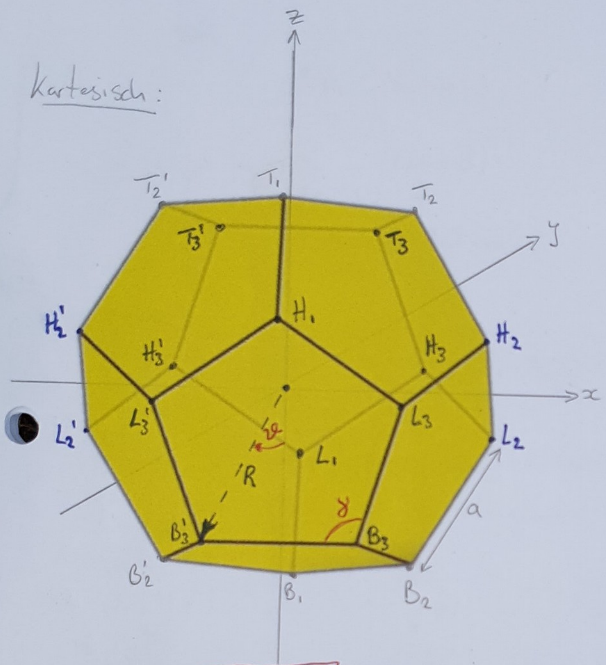


$\alpha = 72° = \frac{2}{5}\pi$

$\beta = 54° = \frac{3}{10}\pi$

$\gamma = 108° = \frac{3}{5}\pi$

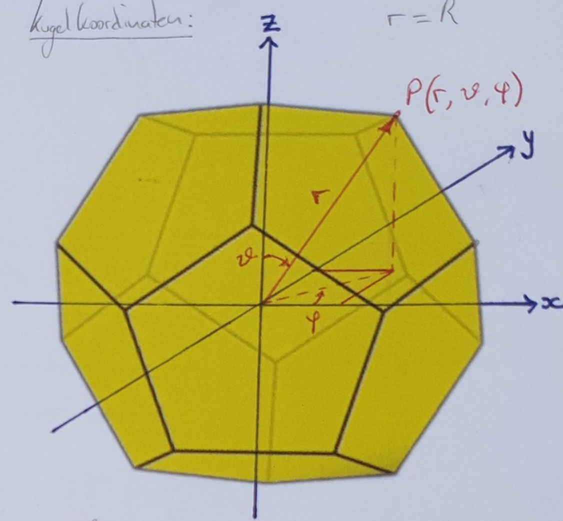
Kartesisch:



$$R = \frac{\sqrt{18 + 6\sqrt{5}}}{4} \cdot a$$

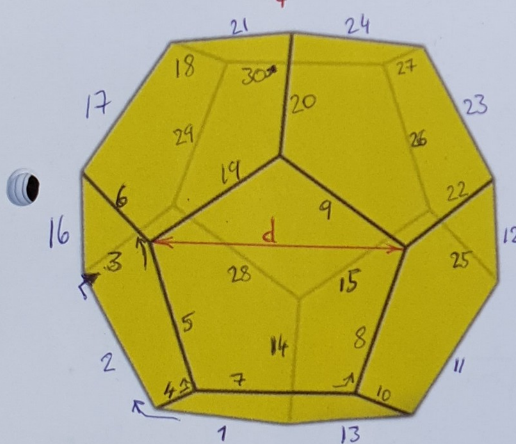
Kugelkoordinaten:

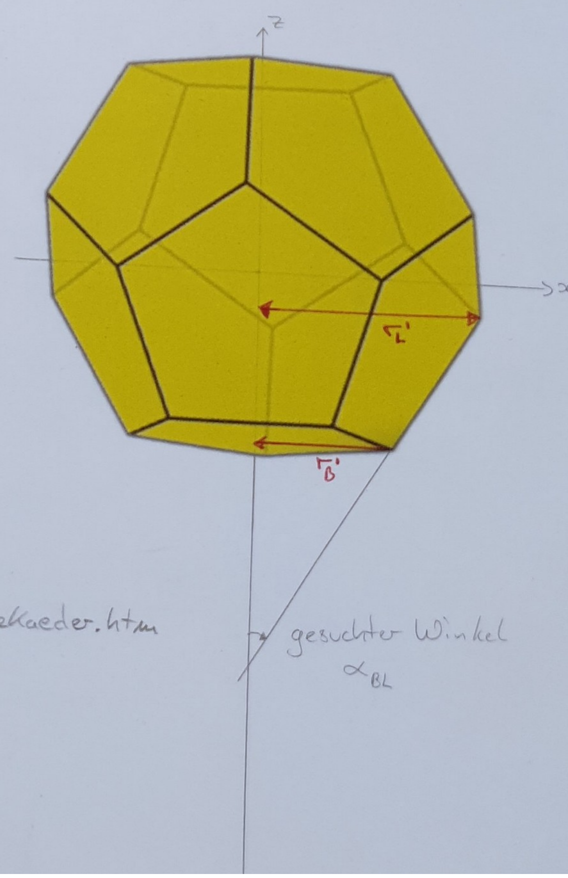
$r = R$



$\vartheta_B = 180° - \vartheta_T$

$\vartheta_L = 180° - \vartheta_H$



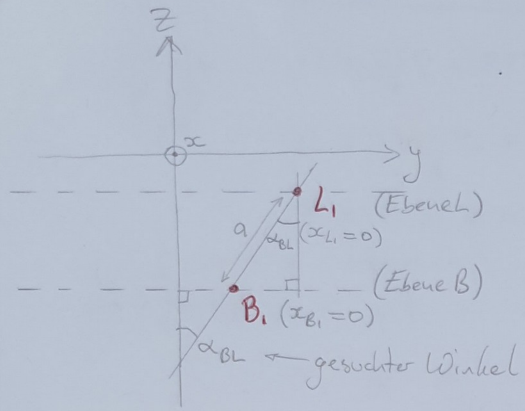


$$d = \frac{(1 + \sqrt{5})}{2} \cdot a$$

mathematische-basteleien.de/pentagondodekaeder.htm

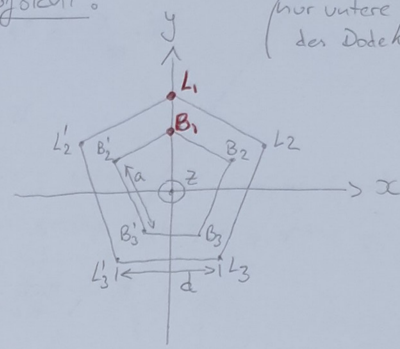
Dodekaeder – vereinfachter Lösungsweg

Seitenansicht:



Draufsicht:

(nur untere Hälfte des Dodekaeders)

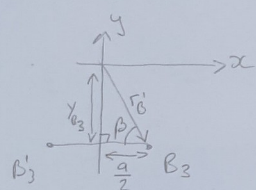


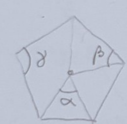
gegeben:

$$\boxed{d = \frac{1+\sqrt{5}}{2} \cdot a} \approx 1.61803399 \cdot a$$

Idee: Δy oder Δz zwischen Eckpunkten L_1 und B_1 berechnen um α_{BL} zu bestimmen

↳ y-Koordinaten sind einfacher zu bestimmen!





$$\alpha = \frac{360°}{5} = 72° \mathrel{\hat=} \frac{2}{5} \cdot \pi$$

$$\beta = \frac{180° - \alpha}{2} = \frac{108°}{2} = 54° \mathrel{\hat=} \frac{3}{10} \cdot \pi$$

$$\gamma = 2 \cdot \beta = 108° \mathrel{\hat=} \frac{3}{5} \cdot \pi$$

Der projezierte Radius r'_B entspricht der y-Komponente von Punkt B_1.

$$\cos(\beta) = \frac{a}{2} \cdot \frac{1}{r'_B} \longrightarrow r'_B = y_{B_1} = \frac{a}{2} \cdot \frac{1}{\cos(\beta)} \longrightarrow y_{B_1} \approx 0.85065081 \cdot a$$

Das gleiche gilt für Punkt L_1 aber mit Kantenlänge d statt a.

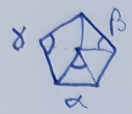
Somit: $r'_L = y_{L_1} = \frac{d}{2} \cdot \frac{1}{\cos(\beta)} = \frac{1+\sqrt{5}}{4} \cdot a \cdot \frac{1}{\cos(\beta)} \longrightarrow y_{L_1} \approx 1.37638192 \cdot a$

Somit $\Delta y_{BL} = y_{L_1} - y_{B_1} \approx 0.52573111 \cdot a$

↳ $\sin(\alpha_{BL}) = \frac{\Delta y_{BL}}{a} \longrightarrow \underline{\underline{\alpha_{BL} \approx 31.717474°}}$

Dodekaeder / Dodekahedron : Langer Rechenweg

↳ alle Koordinaten bestimmen



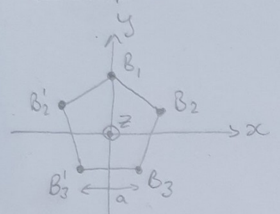
$$\alpha = \frac{360}{5} = 72° \mathrel{\hat=} \frac{2}{5}\cdot\pi$$

$$\beta = \frac{180-\alpha}{2} = \frac{108°}{2} = 54° \mathrel{\hat=} \frac{3}{10}\cdot\pi$$

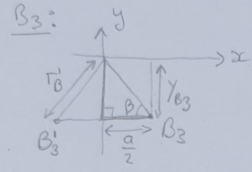
$$\gamma = 2\cdot\beta = 108° \mathrel{\hat=} \frac{3}{5}\cdot\pi$$

Baseplate:

$z_B = \text{const}$



B3:



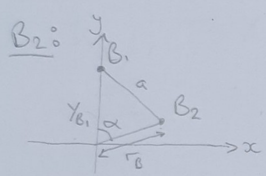
$$\tan(\beta) = y_{B3}\cdot\frac{2}{a} \rightarrow y_{B3} = \frac{a}{2}\cdot\tan(\beta) \approx 0.688a$$

$$\cos(\beta) = \frac{a}{2}\cdot\frac{1}{r'_B} \rightarrow r'_B = \frac{a}{2}\cdot\frac{1}{\cos\beta}$$

$y_{B1} = r'_B$ ↳ 2D-Projektion $\neq R$!

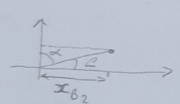
$y_{B1} \approx 0.85a$

B2:



$$\sin\alpha = \frac{x_{B2}}{r_B} \rightarrow x_{B2} = \frac{a}{2}\cdot\frac{\sin\alpha}{\cos\beta} \approx 0.809a$$

$$\cos\alpha = \frac{y_{B2}}{r_B} \rightarrow y_{B2} = \frac{a\cdot\cos\alpha}{2\cdot\cos\beta} \approx 0.263a$$



$$\varepsilon = 90° - \alpha = 18°$$

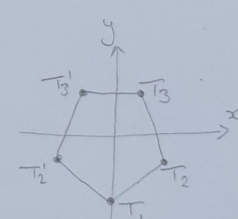
Kartesisch: (x, y, z)

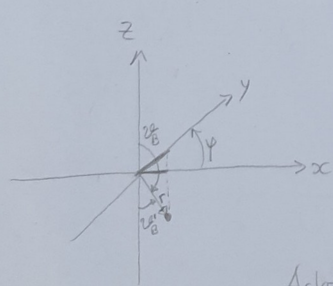
$B_1 = (0, 0.85a, z_B)$
$B_2 = (0.809a, 0.263a, z_B)$
$B'_2 = (-0.809a, 0.263a, z_B)$
$B_3 = (a/2, -0.688a, z_B)$
$B'_3 = (-a/2, -0.688a, z_B)$

Analog Topplate T: (x, y, z) (Symmetrie)

$T_1 = (0, -0.85a, z_T)$
$T_2 = (0.809a, -0.263a, z_T)$
$T'_2 = (-0.809a, -0.263a, z_T)$
$T_3 = (a/2, 0.688a, z_T)$
$T'_3 = (-a/2, 0.688a, z_T)$

mit $z_B = -z_T$





Kartesisch ⟷ Kugelkoordinaten

$x \longleftrightarrow r\cdot\sin\vartheta\cdot\cos\varphi$

$y \longleftrightarrow r\cdot\sin\vartheta\cdot\sin\varphi$

$z \longleftrightarrow r\cdot\cos\vartheta$

$r = R$ (Radius Umkugel)

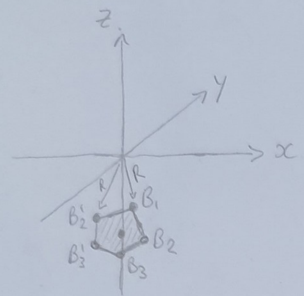
$$\boxed{R = \frac{\sqrt{18+6\cdot\sqrt{5}}}{4}\cdot a}$$

(gegeben)

Achtung! Definition ϑ immer relativ zur positiven z-Achse!

(φ von positiv x in Richtung positiv y)

Baseplate (3D):



B_1 liegt auf y-Achse $(x_{B_1}=0)$ → $\varphi_{B_1} = 90° \hat{=} \pi/2$

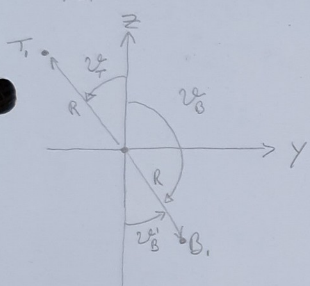
$r_{B_1} = R = \frac{\sqrt{18+6\sqrt{5}}}{4}\cdot a \approx 1.401a$

$\vartheta_B = ?$ (gesucht)

Koordinatenvergleich:

$$B_1 = \begin{pmatrix} 0 \\ 0.85a \\ z_B \end{pmatrix} = \begin{pmatrix} r\cdot\sin\vartheta\cdot\cos\varphi \\ r\cdot\sin\vartheta\cdot\sin\varphi \\ r\cdot\cos\vartheta \end{pmatrix} = \begin{pmatrix} 1.401a\cdot\cancelto{0}{\cos(\tfrac{\pi}{2})}\cdot\sin\vartheta_B \\ 1.401a\cdot\cancelto{1}{\sin(\tfrac{\pi}{2})}\cdot\sin\vartheta_B \\ 1.401a\cdot\cos\vartheta_B \end{pmatrix}$$

$y_{B_1} = 0.85a = 1.401a\cdot\sin\vartheta_B' \rightarrow \sin(\vartheta_B') \approx \frac{0.85}{1.401} \approx 0.6067 \rightarrow \vartheta_B' \approx 37.35°$

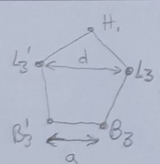


$\vartheta_B = 180° - \vartheta_B' \approx \underline{\underline{142.65°}}$

$\vartheta_T = \vartheta_B' \approx \underline{\underline{37.35°}}$ ← wegen Symmetrie

$\underline{\underline{z_B}} \approx 1.401a\cdot\cos\vartheta_B \approx \underline{\underline{-1.1137a}}$, $\underline{\underline{z_T}} \approx 1.401a\cdot\cos\vartheta_T \approx \underline{\underline{1.1137a}}$

⇒ Topplate und Baseplate vollständig bestimmt!

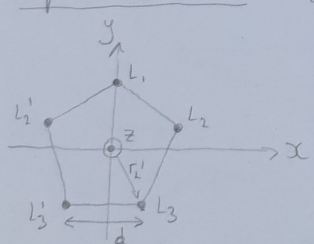


Frontplate → analog zur vorigen Rechnung, nur mit neuer Kantenlänge d!

$$\boxed{d = \frac{(1+\sqrt{5})\cdot a}{2}} \approx 1.618a$$

(gegeben)

Topview Ebene L:



$z_L = \text{const}$

$y_{L_3} = \frac{d}{2}\cdot\tan(\beta) \approx 0.809\cdot 1.3763a \approx 1.1134a$

(2D Projektion $\neq R$!) → $r_L' = y_{L_1} = \frac{d}{2}\cdot\frac{1}{\cos\beta} \approx 1.376a$

$x_{L_2} = r_L'\cdot\sin\alpha \approx 1.3087a$

$y_{L_2} = r_L'\cdot\cos\alpha \approx 0.4252a$

$x_{L_3} = d/2 \approx 0.809a$

Kartesisch:

$L_1 = (0,\ 1.376a,\ z_L)$	$H_1 = (0,\ -1.376a,\ z_H)$
$L_2 = (1.3087a,\ 0.4252a,\ z_L)$	$H_2 = (1.3087a,\ -0.4252a,\ z_H)$
$L_2' = (-1.3087a,\ 0.4252a,\ z_L)$	$H_2' = (-1.3087a,\ -0.4252a,\ z_H)$
$L_3 = (0.809a,\ -1.1134a,\ z_L)$	$H_3 = (0.809a,\ 1.1134a,\ z_H)$
$L_3' = (-0.809a,\ -1.1134a,\ z_L)$	$H_3' = (-0.809a,\ 1.1134a,\ z_H)$

Schnittebene L: für $L_1: x_{L_1} = 0$, somit $\varphi_{L_1} = 90° \hat{=} \pi/2$, $r_{L_1} = R$, $\vartheta_L = ?$

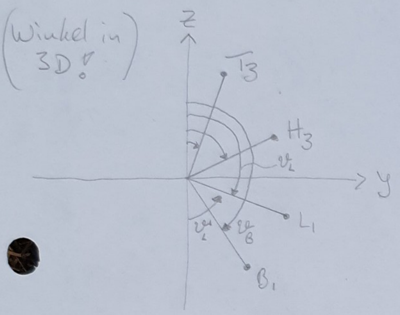
Koordinatenvergleich: $y_{L_1} = 1.376a = r_L \cdot \sin\vartheta_L \cdot \sin\varphi_{L_1} = R \cdot \overset{1}{\cancel{\sin(\pi/2)}} \cdot \sin\vartheta_L$

(in 3D $= R$)

$$\sin\vartheta_L' \approx \frac{1.376a}{1.401a} \approx 0.9822$$

↳ $\vartheta_L' \approx 79.16°$ → Vorsicht wg. Def. ϑ! $\quad \underline{\vartheta_H} = \vartheta_L' \approx \underline{79.16°}$ (Symmetrie)

(Winkel in 3D!)



$$\underline{\vartheta_L} = 180° - \vartheta_L' \approx \underline{100.84°}$$

$$z_L = R \cdot \cos\vartheta_L = \underline{-0.2635a}$$

$$z_H = R \cdot \cos\vartheta_H = \underline{0.2635a}$$

$\varphi_{B_1} = \varphi_{L_1} = 90° \hat{=} \pi/2$

$\varphi_{B_2} = \varphi_{L_2} = 90° - \alpha = 18° \hat{=} \pi/10$

$\varphi_{B_2}' = \varphi_{L_2}' = 90° + \alpha = 162° \hat{=} 9/10 \cdot \pi$

$\varphi_{B_3} = \varphi_{L_3} = 90° + 3\cdot\alpha = 306° \hat{=} 17/10 \cdot \pi$

$\varphi_{B_3}' = \varphi_{L_3}' = 90° + 2\cdot\alpha = 234° \hat{=} 13/10 \cdot \pi$

$\varphi_{T_1} = \varphi_{H_1} = 270° \hat{=} 3/2 \cdot \pi$

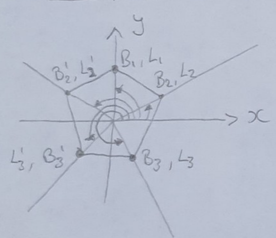
$\varphi_{T_2} = \varphi_{H_2} = 270° + \alpha = 342° \hat{=} 19/10 \cdot \pi$

$\varphi_{T_2}' = \varphi_{H_2}' = 270° - \alpha = 198° \hat{=} 11/10 \cdot \pi$

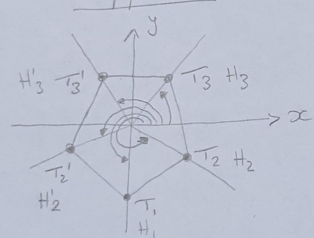
$\varphi_{T_3} = \varphi_{H_3} = 270° - 3\cdot\alpha = 54° \hat{=} 3/10 \cdot \pi$

$\varphi_{T_3}' = \varphi_{H_3}' = 270° - 2\cdot\alpha = 126° \hat{=} 7/10 \cdot \pi$

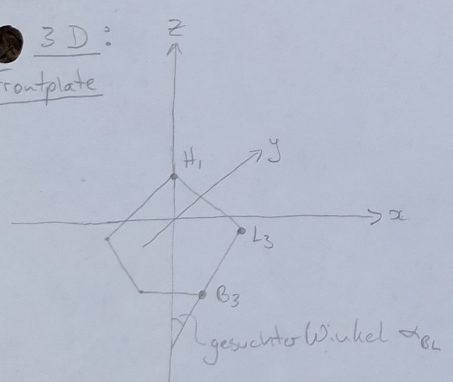
Baseplate:



Topplate:

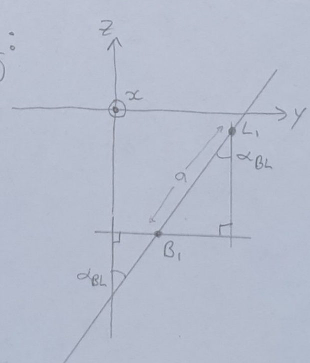


3D: Frontplate



⟹

2D: ($x = 0$)



oder:

$$\sin\alpha_{BL} = \frac{\Delta y}{a} = 0.526$$

$$\alpha_{BL} = 31.74°$$

$$\tan(\alpha_{BL}) = \frac{\Delta y}{\Delta z} = \frac{y_{L_1} - y_{B_1}}{z_L - z_B} \approx \frac{1.376 - 0.85}{-0.2635 + 1.137} \approx 0.6187$$

↳ $\underline{\underline{\alpha_{BL} = 31.744°}}$