Double Integrator Control: Cascaded P Controller Gains vs Damping Ratio

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Let's say we have a system where we command an acceleration $\ddot{x}$ with the goal of controlling position x to a desired value u.

We use a cascaded proportional controller: an inner one for velocity, with gain K_v and an outer one with gain K_p. A cascaded P controller may be used instead of a PD controller instead of a for various reasons, including being able to explicitly constrain the desired velocity.

How does the ratio between K_v and K_p relate to system stability / overshoot / undershoot behavior? How can we make tuning these parameters simpler?

Start by writing out the math for a double proportional controller:

$$\ddot{x} = K_v(K_p(u - x) - \dot{x}) \tag{1}$$

where $\ddot{x}$ is the commanded acceleration, u is the desired position, K_p is the position proportional gain, K_v is the velocity proportional gain, and x and $\dot{x}$ are the position and velocity.

The above equation can be written out as

$$\ddot{x} + K_v\dot{x} + K_vK_px = K_vK_pu \tag{2}$$

which means the transfer function is

$$G(s) = \frac{y(s)}{u(s)} = \frac{K_vK_p}{s^2 + K_vs + K_vK_p} \tag{3}$$

Compare this to the canonical ω_n (natural frequency) and ζ (damping ratio) formulation for the transfer function of a 2nd order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_ns + \omega_n^2} \tag{4}$$

By doing some algebra we find

$$K_p = \frac{\omega_n}{2\zeta}, \quad K_v = 2\zeta\omega_n \tag{5}$$

which finally leads us to the ratio of the inner gain to outer gain:

$$\frac{K_v}{K_p} = 4\zeta^2 \tag{6}$$

For a critically damped system, $\zeta = 1$, meaning $K_v/K_p = 4$. For an underdamped system, e.g. $\zeta = 0.7$, $K_v/K_p = 1.96$. In many situations, we can achieve reasonable tuning by assuming that ζ should be somewhere between 0.7 and 1, meaning that the ratio K_v/K_p should be around 2-4.

NOTES

- Please note that this is NOT a PD controller, which typically has gains such as K_p and K_d, and which we are not talking about here.
- There are often additional complexities such as underlying dynamics to $\ddot{x}$, saturation, etc, that you should consider in tuning your system.