ID1020: Mergesort

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kap 2.2

Slides adapted from Algoritms 4th Edition, Sedgewick.

Two classical algorithms: mergesort och quicksort

- Key components in the world’s IT-infrastructure.
  - Full scientific understanding of their properties has enabled their development into practical system sorts
  - Quicksort honored as one the top 10 algorithms of the 20th century.

- Mergesort.

…

- Quicksort.

…

Divide-and-Conquer algorithms

- Principle Divide-and-Conquer in algorithm design:
  - **Divide**: divide up the problem into smaller, independent subproblems. Subproblems are of the same type as the original problem. Continue dividing until the problem is broken down into many trivial mini-problems.
  - **Conquer**: solve the trivial subproblems. Then construct partial solutions from solved subproblems until full solution can be constructed
- Often done by recursion
- The order-of-growth can usually be solved by analysis of the recursion relation.

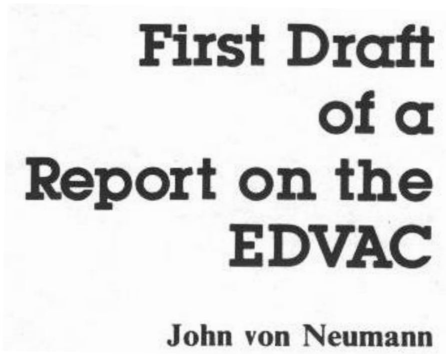
Mergesort

Mergesort

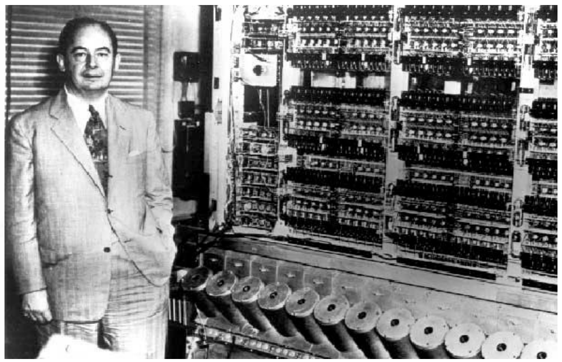
- Method.
  - Divide the array into two halves.
  - Sort the two halves recursively.
  - Merge the two halves.

input	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sort left half	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
sort right half	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge results	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Mergesort overview


First Draft of a Report on the EDVAC

John von Neumann



Abstract in-place merge demo

- Goal. Given two sorted subarrayer `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace this with a sorted subarray `a[lo]` till `a[hi]`.

	lo				mid	mid+1				hi
a[]	E	E	G	M	R	A	C	E	R	T

$\underbrace{\text{E E G M R}}_{\text{sorted}}$ $\underbrace{\text{A C E R T}}_{\text{sorted}}$

Abstract in-place merge demo

- **Goal.** Given two sorted subarrays `a[lo]` to `a[mid]` and `a[mid+1]` to `a[hi]`, replace this with a sorted subarray `a[lo]` till `a[hi]`.

	lo									hi
a[]	A	C	E	E	E	G	M	R	R	T

sorted

Merging

- How can we merge two sorted subarrays into one sorted array?
- Use an auxiliary array.

		a[]												aux[]									
	k	0	1	2	3	4	5	6	7	8	9	i	j	0	1	2	3	4	5	6	7	8	9
input		E	E	G	M	R	A	C	E	R	T			-	-	-	-	-	-	-	-	-	-
copy		E	E	G	M	R	A	C	E	R	T			E	E	G	M	R	A	C	E	R	T
												0	5										
	0	A										0	6	E	E	G	M	R	A	C	E	R	T
	1	A	C									0	7	E	E	G	M	R		C	E	R	T
	2	A	C	E								1	7	E	E	G	M	R			E	R	T
	3	A	C	E	E							2	7		E	G	M	R			E	R	T
	4	A	C	E	E	E						2	8			G	M	R			E	R	T
	5	A	C	E	E	E	G					3	8			G	M	R				R	T
	6	A	C	E	E	E	G	M				4	8				M	R				R	T
	7	A	C	E	E	E	G	M	R			5	8					R				R	T
	8	A	C	E	E	E	G	M	R	R		5	9									R	T
	9	A	C	E	E	E	G	M	R	R	T	6	10										T
merged result		A	C	E	E	E	G	M	R	R	T												

Abstract in-place merge trace

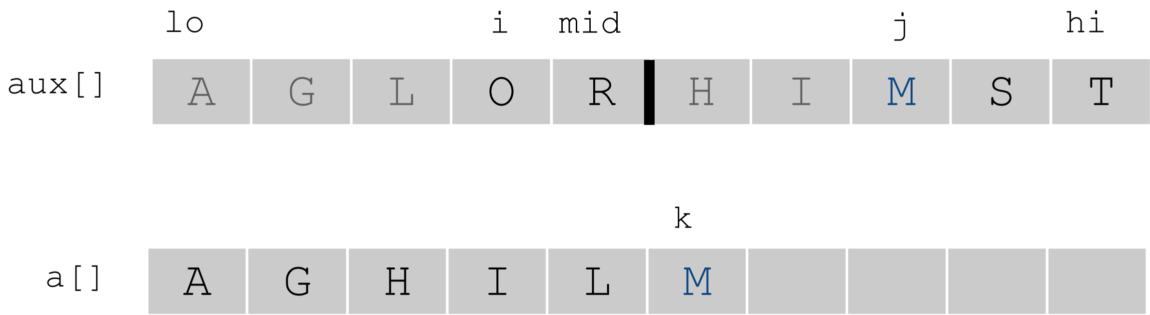
Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid, int hi)
{

   for (int k = lo; k <= hi; k++) {
      aux[k] = a[k];
   }
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)    {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

copy

merge



Merging: Java implementation

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int mid,
  int hi) {
   assert isSorted(a, lo, mid);     // precondition: a[lo..mid]   sorted
   assert isSorted(a, mid+1, hi);   // precondition: a[mid+1..hi] sorted

   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];
                                                                  copy
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];              merge
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }

   assert isSorted(a, lo, hi);     // postcondition: a[lo..hi] sorted
}
```

	lo			i	mid			j		hi
aux[]	A	G	L	O	R	H	I	M	S	T
						k				
a[]	A	G	H	I	L	M				

Assertions

- Assertion. A statement to check assumptions about program.
  - Assertions can be used to find bugs.
  - Assertions also serve as good documentation.

- Java assert statement. Throws an exception if the condition is false.

```
assert isSorted(a, lo, hi);
```

- Can be enabled or disabled during execution. ⇒ No extra overhead in production code.

```
% java -ea MyProgram   // enable assertions
% java -da MyProgram   // disable assertions
(default)
```

- Best practice. Use assertions to check *invariants*, *preconditions* and *postconditons,* for example, to an API or method.
  Assume that assertions are disabled in produktion code.

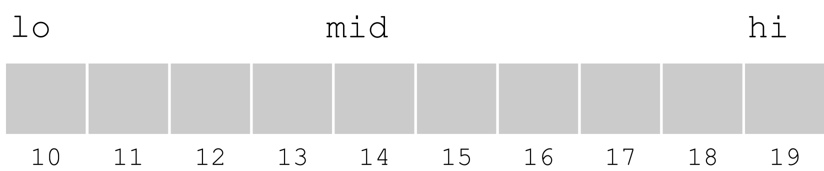
- Unit tests on the other hand check the *external behavior* of an API or method

Mergesort: Java implementation

```
public class Merge {
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int
hi)     {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a) {
      Comparable[] aux = new Comparable[a.length];
      sort(a, aux, 0, a.length - 1);
   }
}
```



Mergesort: trace

lo hi

a[]

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 0, 1, 3)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 4, 5)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 6, 6, 7)	E	G	M	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, 4, 5, 7)	E	G	M	R	E	O	R	S	T	E	X	A	M	P	L	E
merge(a, 0, 3, 7)	E	E	G	M	O	R	R	S	T	E	X	A	M	P	L	E
merge(a, 8, 8, 9)	E	E	G	M	O	R	R	S	E	T	X	A	M	P	L	E
merge(a, 10, 10, 11)	E	E	G	M	O	R	R	S	E	T	A	X	M	P	L	E
merge(a, 8, 9, 11)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, 12, 12, 13)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	L	E
merge(a, 14, 14, 15)	E	E	G	M	O	R	R	S	A	E	T	X	M	P	E	L
merge(a, 12, 13, 15)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
merge(a, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Trace of merge results for top-down mergesort

Mergesort: empirical analysis

- Running times:
  - A laptop executes 10^8 comparisons/second.
  - A supercomputer executes 10^{12} comparisons/second.

	insertion sort (N^2)			mergesort (N log N)		
computer	thousand	million	billion	thousand	million	billion
home	instant	2.8 hours	317 years	instant	1 second	18 min
super	instant	1 second	1 week	instant	instant	instant

- Conclusion. Good algorithms are better than supercomputers.

Mergesort: comparisons

- **Theorem.** Mergesort compares ≤ $N \lg N$ to sort an array of length N.

- **Proof.** The number of comparisons $C(N)$ to run mergesort on an array of length N satisfies the recurrence relation:

$$C(N) \leq C(\lceil N/2 \rceil) + C(\lfloor N/2 \rfloor) + N \quad \text{där } N > 1, \text{ med } C(1) = 0.$$

Left half Right half merge

$$A(N) \leq A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \quad \text{där } N > 1, \text{ med } A(1) = 0.$$

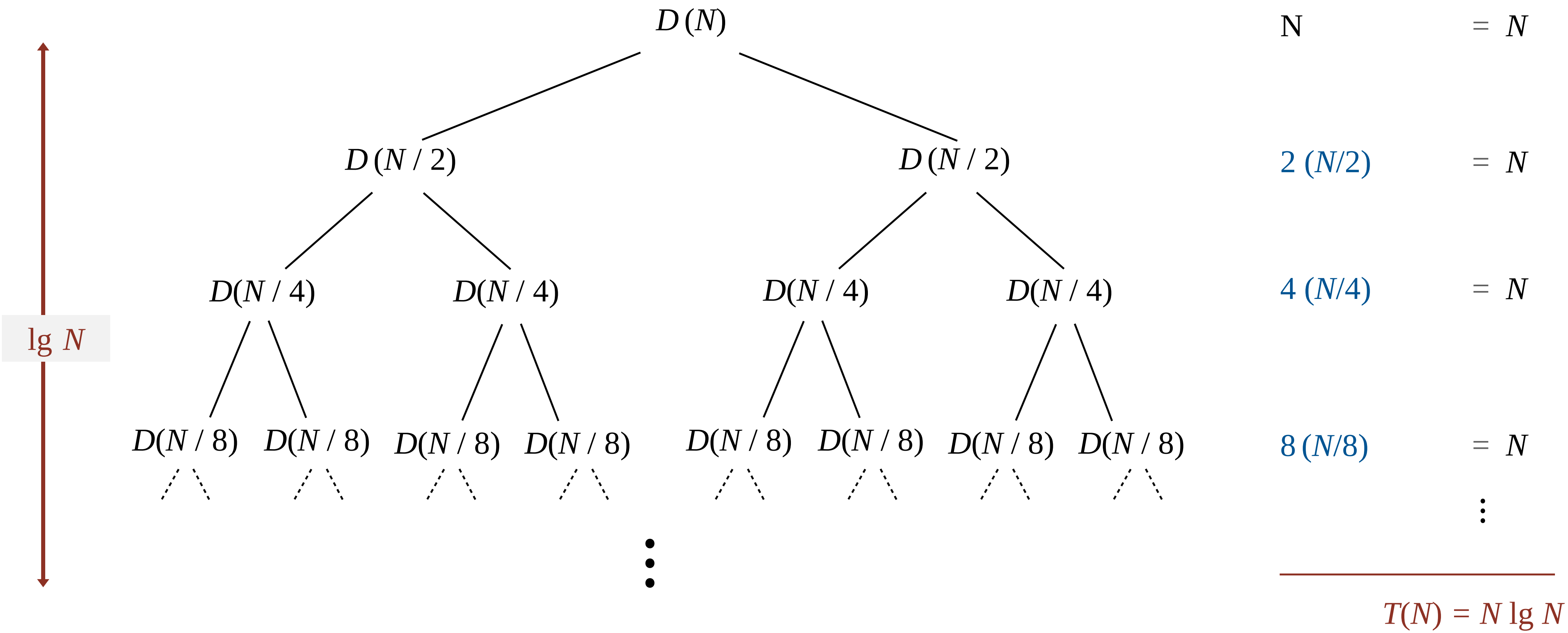
- We solve the recurrence relation when N is a power of two:

$D(N) = 2\,D(N/2) + N$, för $N > 1$, med $D(1) = 0$. Result valid for all N

Divide-och-conquer recurrence: picture proof

- **Theorem.** IF $D(N)$ satisfies $D(N) = 2\,D(N/2) + N$ for $N > 1$, with $D(1) = 0$,
  THEN $D(N) = N \lg N$.

- **Proof 1.**



Divide-och-conquer recurrence : proof with expansion

- Theorem. IF $D(N)$ satisfierar $D(N) = 2\,D(N/2) + N$ för $N > 1$, med $D(1) = 0$,
  THEN $D(N) = N \lg N$.

- Proof 2

$$
\begin{aligned}
D(N) &= 2\,D(N/2) + N && \text{given} \\
D(N)/N &= 2\,D(N/2)/N + 1 && \text{divide both sides by N} \\
&= D(N/2)/(N/2) + 1 && \text{algebra} \\
&= D(N/4)/(N/4) + 1 + 1 && \text{apply to first term} \\
&= D(N/8)/(N/8) + 1 + 1 + 1 && \text{apply to first term again} \\
&\;\;\cdots \\
&= D(N/N)/(N/N) + 1 + 1 + \ldots + 1 && \text{stop applying, D(1) = 0} \\
&= \lg N
\end{aligned}
$$

Divide-och-conquer recurrence: proof by induction

- Sats. IF $D(N)$ satisfierar $D(N) = 2\,D(N/2) + N$ för $N > 1$, med $D(1) = 0$,
  THEN $D(N) = N \lg N$.

- Proof 3.
  - Base case: $N = 1$.
  - Induktion hypothesis : $D(N) = N \lg N$.
  - Goal: show that $D(2N) = (2N) \lg (2N)$.

$$D(2N) = 2\,D(N) + 2N \qquad \text{given}$$

$$= 2\,N \lg N + 2N \qquad \text{inductive hypothesis}$$

$$= 2\,N\,(\lg(2N) - 1) + 2N \qquad \text{algebra}$$

$$= 2\,N \lg(2N) \qquad \text{QED}$$

Mergesort: array accesses

- **Theorem.** Mergesort access the array $\leq 6 N \lg N$ times to sort an array of length N.

- **Proof.** Array accesses $A(N)$ satisfies the recurrence-relation:

$$A(N) \leq A(\lceil N/2 \rceil) + A(\lfloor N/2 \rfloor) + 6N \text{ } \mathit{för} \text{ } N > 1, \text{ with } A(1) = 0$$

- **Note.** All algorithms with the following struture take $N \log N$ time:

```
public static void linearithmic(int N) {
    if (N == 0) {
        return;
    }
    linearithmic(N/2);
    linearithmic(N/2);
    linear(N);
}
```

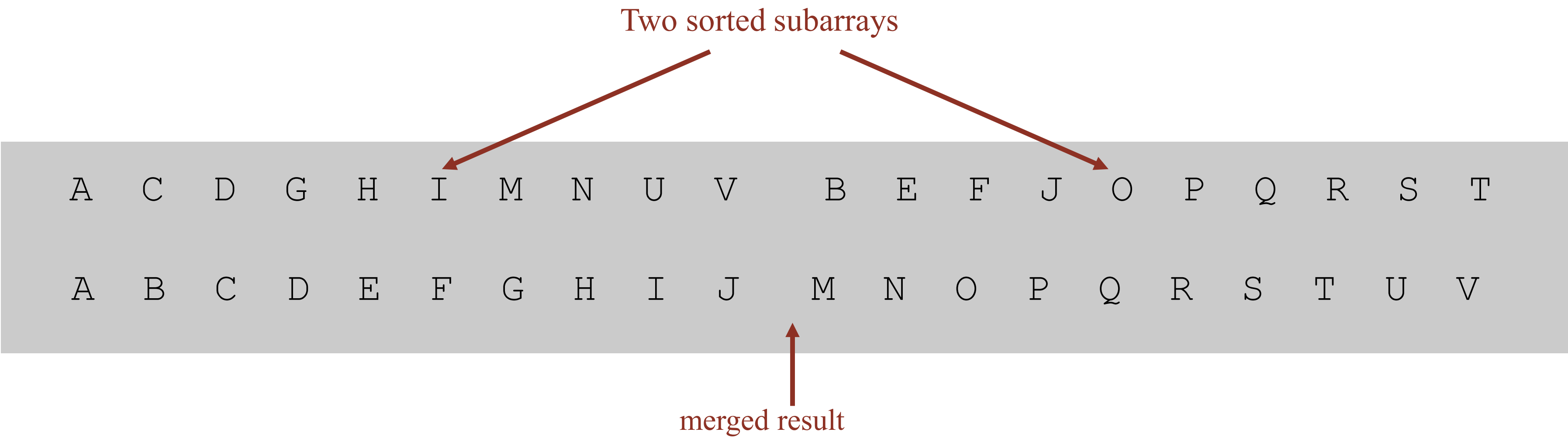
Solve two subproblems of half the size

Linear amount of work

- **Exceptions.** FFT, m.m., …

Mergesort analysis: memory usage

- Theorem: Mergesort uses extra memory proportional to N.
- Proof . The array `aux[]` needs to be of length N for the last merge.



- Def. A sorting algorithm is "in-place" if it uses $\leq c \log N$ extra memory
- T.ex. Insertion sort, selection sort, shellsort. Not this mergesort !!

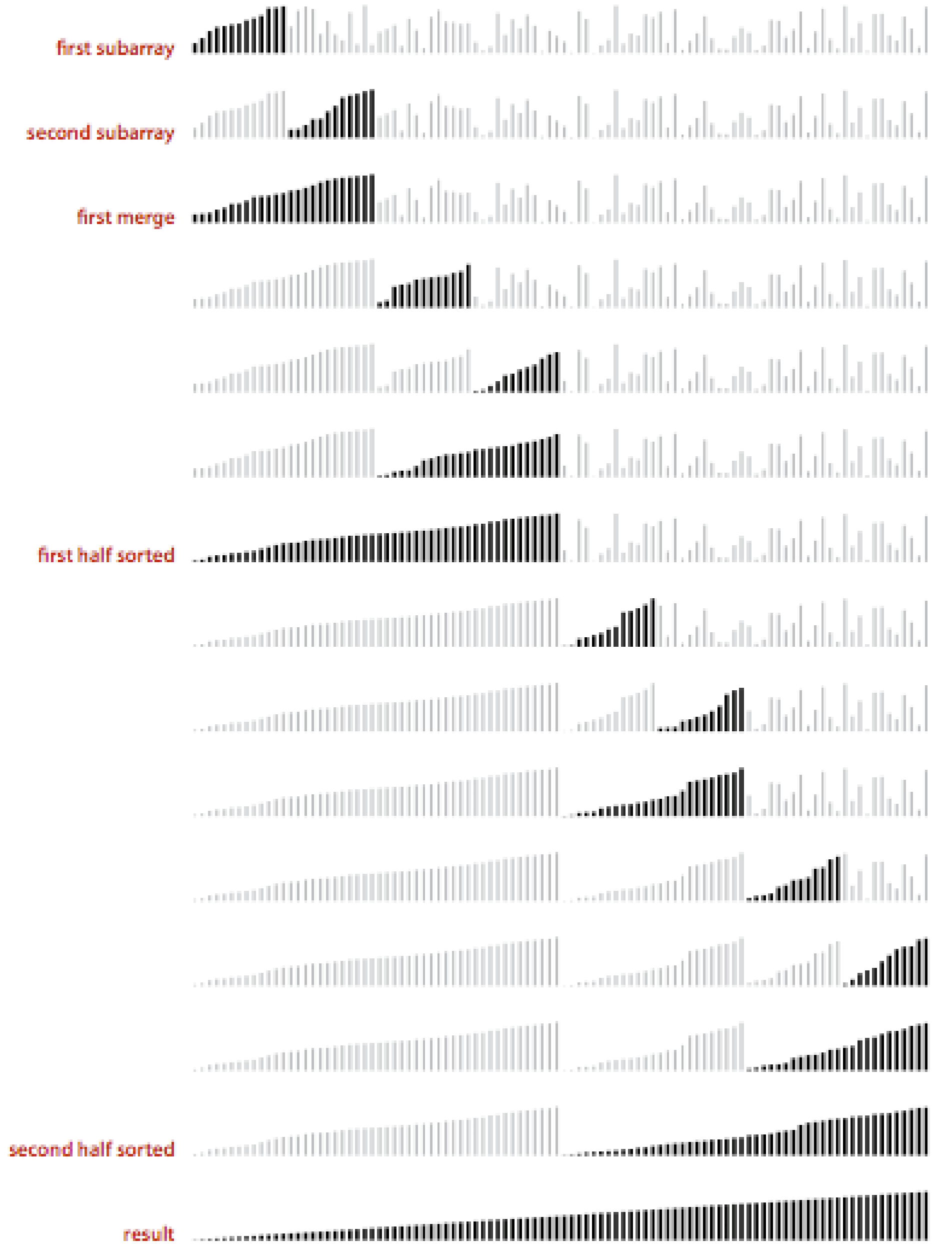
- Challenge 1 (not difficult). Use `aux[]` array of length ~ ½ N instead of N.
- Challenge 2 (extremely difficult). *In-place merge*. [Kronrod 1969]

Mergesort: improvements

- Use insertion sort for small subarrays.
  - Mergesort has too much overhead for small subarrays.
  - Cutoff – less than approx. 10 elements – use insertion sort.

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi){
   if (hi <= lo + CUTOFF - 1)   {
      Insertion.sort(a, lo, hi);
      return;
   }
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

Mergesort cutoff for insertion sort: visualization



Visual trace of top-down mergesort for with cutoff for small subarrays

Mergesort: further improvement

- Stop if already sorted.
  - Stop if the largest element in first half ≤ smallest element in second half?
  - Helpful for partially ordered arrays.

A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V
A	B	C	D	E	F	G	H	I	J	M	N	O	P	Q	R	S	T	U	V

```
private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (a, aux, lo, mid);
   sort (a, aux, mid+1, hi);
   if (!less(a[mid+1], a[mid])) return;
   merge(a, aux, lo, mid, hi);
}
```

Mergesort: further improvement

- Eliminate copying to auxiliary array. Save time but not memory.
- Exchange the role of the input and auxiliary array in each recursive call.

```
private static void merge(Comparable[] a, Comparable[] aux, int lo, int
mid, int hi)
{
   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++) {
      if      (i > mid)          aux[k] = a[j++];
      else if (j > hi)           aux[k] = a[i++];
      else if (less(a[j], a[i])) aux[k] = a[j++];
      else                       aux[k] = a[i++];
   }
}

private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
{
   if (hi <= lo) return;
   int mid = lo + (hi - lo) / 2;
   sort (aux, a, lo, mid);
   sort (aux, a, mid+1, hi);
   merge(a, aux, lo, mid, hi);
}
```

merge from `a[]` to `aux[]`

assumes `aux[]` is initialized to `a[]` once, before recursive calls

switch roles of `aux[]` och `a[]`

Java 6 systemsort

- The chosen algorithm for sorting objects = mergesort.
  - Cutoff to insertion sort = 7.
  - Stop if already sorted test.
  - Use the no copying to auxillary array trick.

Arrays.sort(a)

http://www.java2s.com/Open-Source/Java/6.0-JDK-Modules/j2me/java/util/Arrays.java.html

Bottom-up Mergesort

- Basic method
  - Pass through the array, merging subarrays of size 1.
  - Repeat for subarrays av size 2, 4, 8, ...

	a[i]															
	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	M	E	R	G	E	S	O	R	T	E	X	A	M	P	L	E
sz = 1																
merge(a, aux, 0, 0, 1)	E	M	R	G	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 2, 2, 3)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 4, 4, 5)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 6, 6, 7)	E	M	G	R	E	S	O	R	T	E	X	A	M	P	L	E
merge(a, aux, 8, 8, 9)	E	M	G	R	E	S	O	R	E	T	X	A	M	P	L	E
merge(a, aux, 10, 10, 11)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 12, 12, 13)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	L	E
merge(a, aux, 14, 14, 15)	E	M	G	R	E	S	O	R	E	T	A	X	M	P	E	L
sz = 2																
merge(a, aux, 0, 1, 3)	E	G	M	R	E	S	O	R	E	T	A	X	M	P	E	L
merge(a, aux, 4, 5, 7)	E	G	M	R	E	O	R	S	E	T	A	X	M	P	E	L
merge(a, aux, 8, 9, 11)	E	G	M	R	E	O	R	S	A	E	T	X	M	P	E	L
merge(a, aux, 12, 13, 15)	E	G	M	R	E	O	R	S	A	E	T	X	E	L	M	P
sz = 4																
merge(a, aux, 0, 3, 7)	E	E	G	M	O	R	R	S	A	E	T	X	E	L	M	P
merge(a, aux, 8, 11, 15)	E	E	G	M	O	R	R	S	A	E	E	L	M	P	T	X
sz = 8																
merge(a, aux, 0, 7, 15)	A	E	E	E	E	G	L	M	M	O	P	R	R	S	T	X

Bottom-up mergesort: Java implementation

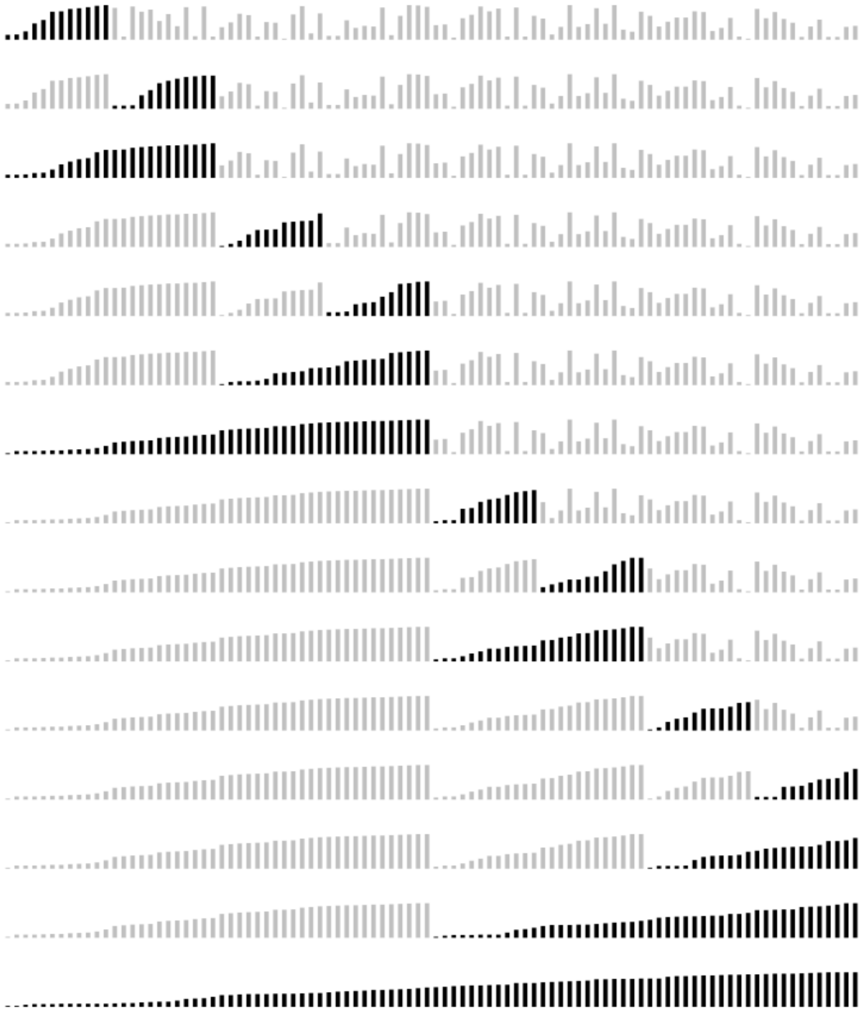
```
public class MergeBU
{
   private static void merge(...)
   {  /* as before */  }

   public static void sort(Comparable[] a)
   {
      int N = a.length;
      Comparable[] aux = new Comparable[N];
      for (int sz = 1; sz < N; sz = sz+sz)
         for (int lo = 0; lo < N-sz; lo += sz+sz)
            merge(a, aux, lo, lo+sz-1, Math.min(lo+sz+sz-1, N-1));
   }
}
```

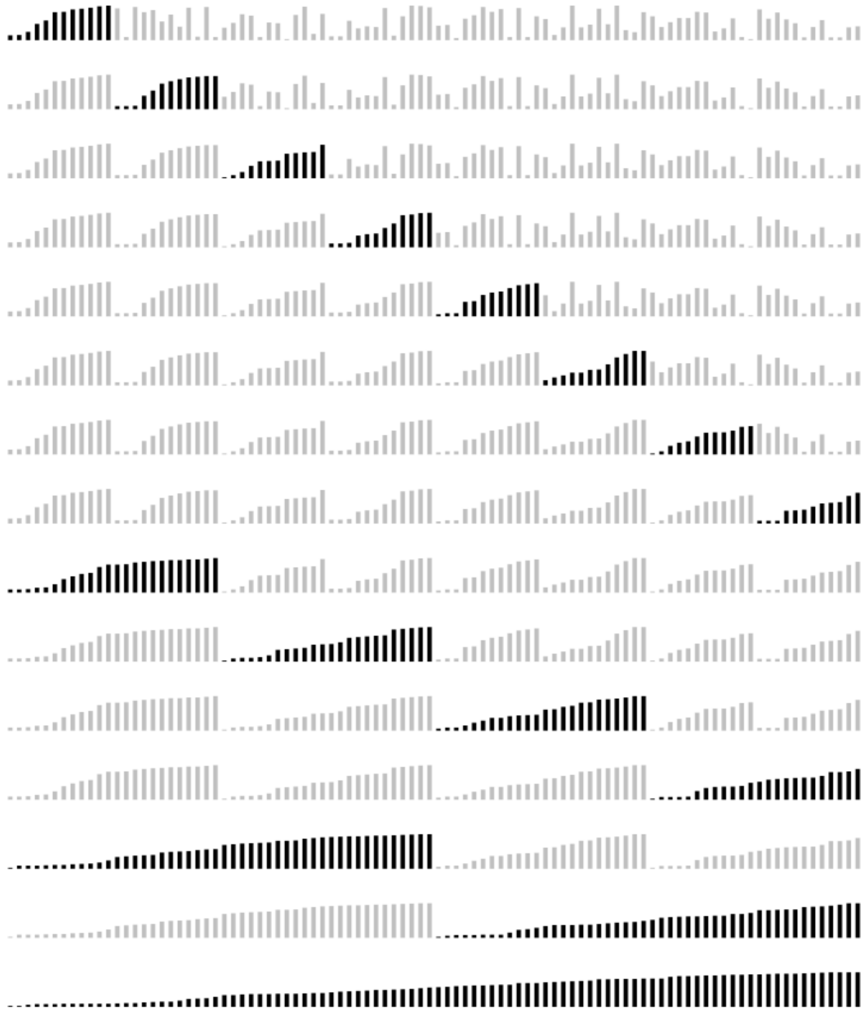
approximately 10% slower than,
top-down mergesort on most systems

Conclusion. Simple non recursive version of mergesort.

Mergesort: visualization



top-down mergesort (cutoff = 12)



bottom-up mergesort (cutoff = 12)

Computational complexity of mergesort

Computational complexity

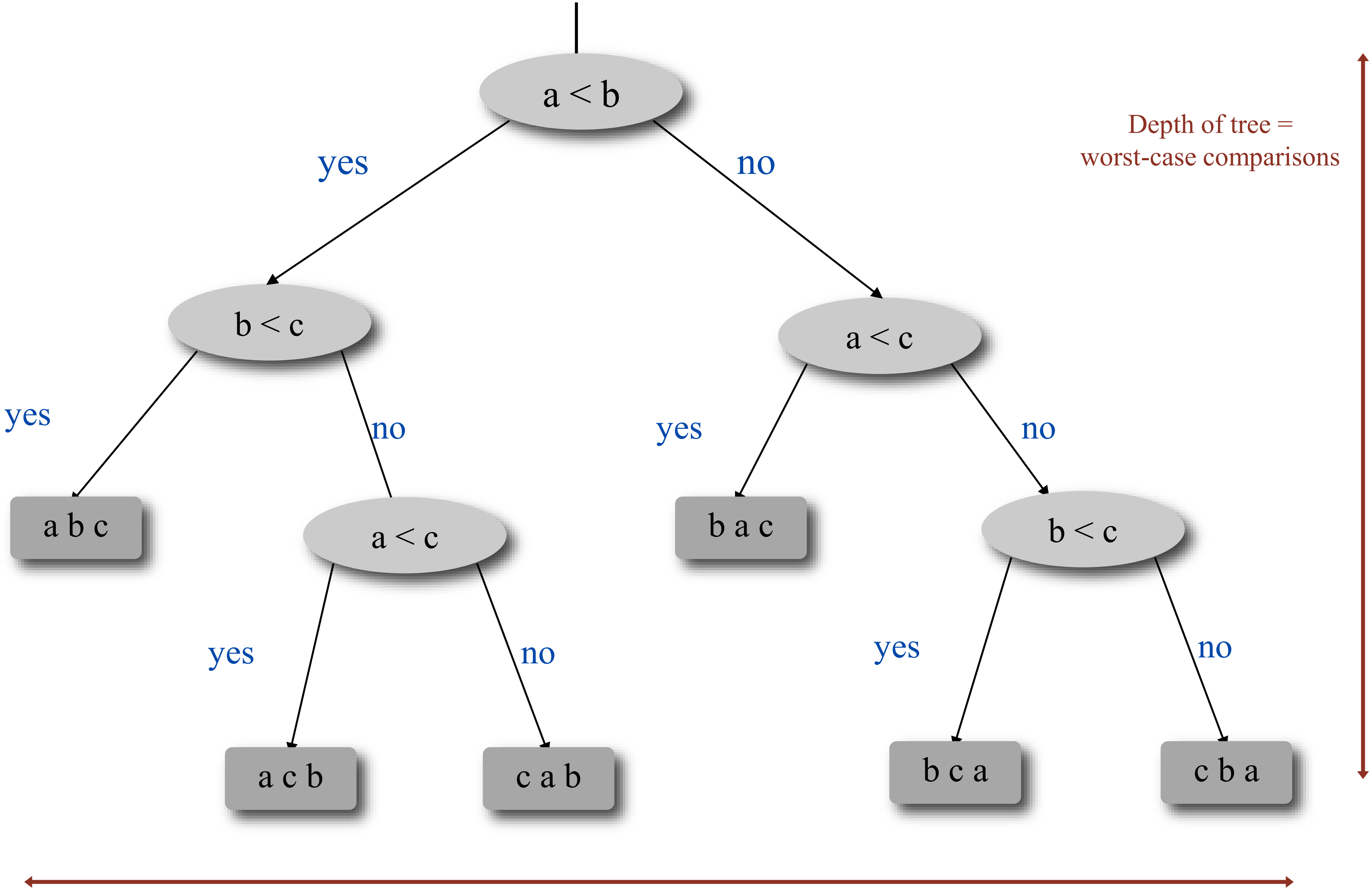
- Cost model. Number of operations
- Upped bound. Cost guarantee given by some algorithm X.
- Lower bound. No algorithm can cost less than the lower bound.
- Optimal algoritm. Algorithm with best cost guarantee för X.

Lower bound ~ upper bound

- Exempel: sortng.
  - Model of computation: decision tree.
  - Cost model: # compares
  - Upper bound: $\sim N \lg N$ från mergesort.
  - Lower bound: ?
  - Optimal algoritm: ?

Can only access by comparisons
(e.g. Java Comparable framework)

Decision tree (for 3 distinct keys a, b, och c)

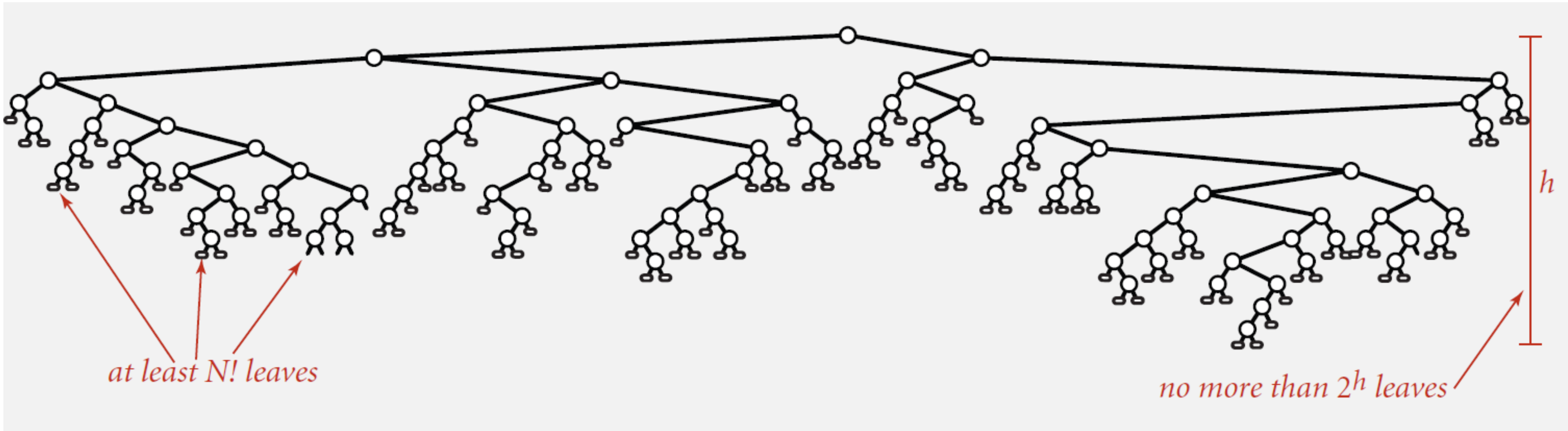


Every leaf corresponds to one distinct order,

need a leaf for every possible order.

Comparison - based lower bound for sorting

- Theorem. Alla comparison-based sorting algorithmer must compare at least $\lg(N!) \sim N \lg N$ times in the worst-case
- Proof.
  - We assume that the array consists of N distinct values a_1 to a_N.
  - Worst case is determined by the depth (height) h of the decision tree..
  - A binary tree of depth h has at most 2^h leaves.
  - $N!$ different orders $\Rightarrow$ at least $N!$ leaves.



Comparison - based lower bound for sorting

- Theorem. All comparison-based sorting algorithms must compare $\lg(N!) \sim N \lg N$ times in the worst-case.

- Proof.
  - We assume that tha array consists of N distinct values a_1 to a_N.
  - Worst case is determined by the height h of the decision tree.
  - A binary tree av height h has at most 2^h leaves.
  - $N!$ Different possible orderings $\Rightarrow$ at least $N!$ leaves.

$$2^h \geq \#\,\text{leaves} \geq N!$$
$$\Rightarrow \quad h \geq \underbrace{\lg(N!) \sim}_{\text{Stirlings formel}} N \lg N$$

Time complexity for sorting

- Cost model . Number of operations.
- Upper bound. Guarantee provided by some algorithm.
- Lower bound. Proof that no algorithm X can have a lower cost than the limit.
- Optimal algorithm. Upper bound ~lower bound.

- Example: sorting
  - Model: decision tree.
  - Cost model : # comparisons.
  - Upper bound ~ $N \lg N$ fom mergesort.
  - Lower bound: ~ $N \lg N$.
  - Optimal algorithm = mergesort.

- Primary goal of algorithm design: optimal algorithms.

Context

- Comparisons? Mergesort is optimal with respect to comparisons
  - Time complexity
- Space? Mergesort is not optimal with respect to memory
  - Auxillary array

Lesson. Theory helps us find solutions - and stops us from attempting the impossible.

Question: Is there an optimal sorting algorithm with respect to both time and space?

Time complexity in context

- Faster sorting might be achieved if the algorithmen can take advantage of:

  - Invariants in the input.
    Ex.: insertion sort need only a linear number of comparions on partially sorted array.
    Covered in previous lecture

  - Key distribution.
    Ex.: When the array contains many duplicates (i.e. if the number of distinct keys is limited). More on this later

Mergesort comparators

Need for comparators

- Forst step to generalize sorting
  - Comparable and compareTo
  - To be able to compare and sort user-defined
  - To implement generic sorting functions

- Second step
  - Work with multiple order relations on the same objects
  - Very common
  - Complex records

Sorting countries by number of gold medals

NOC	Gold	Silver	Bronze	Total
United States (USA)	46	29	29	104
China (CHN)§	38	28	22	88
Great Britain (GBR)*	29	17	19	65
Russia (RUS)§	24	25	32	81
South Korea (KOR)	13	8	7	28
Germany (GER)	11	19	14	44
France (FRA)	11	11	12	34
Italy (ITA)	8	9	11	28
Hungary (HUN)§	8	4	6	18
Australia (AUS)	7	16	12	35

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Australia (AUS)	7	16	12	35
France (FRA)	11	11	12	34
South Korea (KOR)	13	8	7	28
Italy (ITA)	8	9	11	28

Sorting music by the name of the artist

	Name	Artist ▲	Time	Album
12	☑ Let It Be	The Beatles	4:03	Let It Be
13	☑ Take My Breath Away	BERLIN	4:13	Top Gun - Soundtrack
14	☑ Circle Of Friends	Better Than Ezra	3:27	Empire Records
15	☑ Dancing With Myself	Billy Idol	4:43	Don't Stop
16	☑ Rebel Yell	Billy Idol	4:49	Rebel Yell
17	☑ Piano Man	Billy Joel	5:36	Greatest Hits Vol. 1
18	☑ Pressure	Billy Joel	3:16	Greatest Hits, Vol. II (1978 - 1985) (Disc 2)
19	☑ The Longest Time	Billy Joel	3:36	Greatest Hits, Vol. II (1978 - 1985) (Disc 2)
20	☑ Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
21	☑ Sunday Girl	Blondie	3:15	Atomic: The Very Best Of Blondie
22	☑ Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
23	☑ Dreaming	Blondie	3:06	Atomic: The Very Best Of Blondie
24	☑ Hurricane	Bob Dylan	8:32	Desire
25	☑ The Times They Are A-Changin'	Bob Dylan	3:17	Greatest Hits
26	☑ Livin' On A Prayer	Bon Jovi	4:11	Cross Road
27	☑ Beds Of Roses	Bon Jovi	6:35	Cross Road
28	☑ Runaway	Bon Jovi	3:53	Cross Road
29	☑ Rasputin (Extended Mix)	Boney M	5:50	Greatest Hits
30	☑ Have You Ever Seen The Rain	Bonnie Tyler	4:10	Faster Than The Speed Of Night
31	☑ Total Eclipse Of The Heart	Bonnie Tyler	7:02	Faster Than The Speed Of Night
32	☑ Straight From The Heart	Bonnie Tyler	3:41	Faster Than The Speed Of Night
33	☑ Holding Out For A Hero	Bonny Tyler	5:49	Meat Loaf And Friends
34	☑ Dancing In The Dark	Bruce Springsteen	4:05	Born In The U.S.A.
35	☑ Thunder Road	Bruce Springsteen	4:51	Born To Run
36	☑ Born To Run	Bruce Springsteen	4:30	Born To Run
37	☑ Jungleland	Bruce Springsteen	9:34	Born To Run

Sorting music by the name of the song

	Name	Artist	Time	Album
1	Alive	Pearl Jam	5:41	Ten
2	All Over The World	Pixies	5:27	Bossanova
3	All Through The Night	Cyndi Lauper	4:30	She's So Unusual
4	Allison Road	Gin Blossoms	3:19	New Miserable Experience
5	Ama, Ama, Ama Y Ensancha El …	Extremoduro	2:34	Deltoya (1992)
6	And We Danced	Hooters	3:50	Nervous Night
7	As I Lay Me Down	Sophie B. Hawkins	4:09	Whaler
8	Atomic	Blondie	3:50	Atomic: The Very Best Of Blondie
9	Automatic Lover	Jay-Jay Johanson	4:19	Antenna
10	Baba O'Riley	The Who	5:01	Who's Better, Who's Best
11	Beautiful Life	Ace Of Base	3:40	The Bridge
12	Beds Of Roses	Bon Jovi	6:35	Cross Road
13	Black	Pearl Jam	5:44	Ten
14	Bleed American	Jimmy Eat World	3:04	Bleed American
15	Borderline	Madonna	4:00	The Immaculate Collection
16	Born To Run	Bruce Springsteen	4:30	Born To Run
17	Both Sides Of The Story	Phil Collins	6:43	Both Sides
18	Bouncing Around The Room	Phish	4:09	A Live One (Disc 1)
19	Boys Don't Cry	The Cure	2:35	Staring At The Sea: The Singles 1979-1985
20	Brat	Green Day	1:43	Insomniac
21	Breakdown	Deerheart	3:40	Deerheart
22	Bring Me To Life (Kevin Roen Mix)	Evanescence Vs. Pa…	9:48	
23	Californication	Red Hot Chili Pepp…	1:40	
24	Call Me	Blondie	3:33	Atomic: The Very Best Of Blondie
25	Can't Get You Out Of My Head	Kylie Minogue	3:50	Fever
26	Celebration	Kool & The Gang	3:45	Time Life Music Sounds Of The Seventies - C

Comparable interface: repetition

- `Comparable` interface: sorting with the natural order of a data type.

```
public class Date implements Comparable<Date> {
   private final int month, day, year;

   public Date(int m, int d, int y) {
      month = m;
      day   = d;
      year  = y;
   }
      …
   public int compareTo(Date that)   {
      if (this.year  < that.year ) return -1;
      if (this.year  > that.year ) return +1;
      if (this.month < that.month) return -1;
      if (this.month > that.month) return +1;
      if (this.day   < that.day  ) return -1;
      if (this.day   > that.day  ) return +1;
      return 0;
   }
}
```

natural order

Comparator interface

- `Comparator` interface: sorting with an alternative order.

public interface Comparator<Key>	
`int compare(Key v, Key w)`	*jämför nycklar v och w*

- Requirement. Must define a total order.

natural order	`Now is the time`
case insensitive	`is Now the time`
Spanish language	`café cafetero cuarto churro nube ñoño`
British phone book	`McKinley Mackintosh`

pre-1994 order for digraphs ch och ll och rr

Decouple comparator interface from the data type

- Can be used with Java system sort:
  - Create a `Comparator` object.
  - Send as a second argument to `Arrays.sort()`.

uses natural order

Uses and alternative order

given by Comparator<String> objekt

```
String[] a;
...
Arrays.sort(a);
...
Arrays.sort(a, String.CASE_INSENSITIVE_ORDER);
...
Arrays.sort(a, Collator.getInstance(new Locale("es")));
...
Arrays.sort(a, new BritishPhoneBookOrder());
...
```

Comparator interface:

- To used comparators in sorting:
  - Use `Object` instead of `Comparable`.
  - Send the `Comparator` to `sort()` and `less()` and use it in `less()`.

insertion sort with a Comparator

```
public static void sort(Object[] a, Comparator comparator)
{
   int N = a.length;
   for (int i = 0; i < N; i++)
      for (int j = i; j > 0 && less(comparator, a[j], a[j-1]); j--)
         exch(a, j, j-1);
}

private static boolean less(Comparator c, Object v, Object w) {
   return c.compare(v, w) < 0;
}

private static void exch(Object[] a, int i, int j){
   Object swap = a[i]; a[i] = a[j]; a[j] = swap;
}
```

To implement a Comparator interface

- Define a nested class that implements the `Comparator` interface.
- Implement the `compare()` method.

```
public class Student {
private final String name;
   private final int section;
   ...

   public static class ByName implements Comparator<Student>
   {
     public int compare(Student v, Student w)
      {  return v.name.compareTo(w.name);  }
   }

   public static class BySection implements Comparator<Student>
   {
      public int compare(Student v, Student w)
      {  return v.section - w.section;  }
   }
}
```

To implement a Comparator interface

- Define a nested class with implement the `Comparator` interface.
- Implement the `compare()` method.

`Arrays.sort(a, new Student.ByName());`

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

`Arrays.sort(a, new Student.BySection());`

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Andrews	3	A	664-480-0023	097 Little
Chen	3	A	991-878-4944	308 Blair
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To implement a Comparator interface

- Definie a nested class with implement the `Comparator` interface.
- Implement the `compare()` method.

```
public class Student {
   public static final Comparator<Student> BY_NAME = new ByName();
   public static final Comparator<Student> BY_SECTION = new BySection();
   private final String name;
   private final int section;
   ...

   public static class ByName implements Comparator<Student>
   {
     public int compare(Student v, Student w)
      {  return v.name.compareTo(w.name);  }
   }

   public static class BySection implements Comparator<Student>
   {
      public int compare(Student v, Student w)
      {  return v.section - w.section;  }
   }
}
```

Mergesort and stability

A problem ?

- Common application. First, sort by namn; thereafter by ”Section”.

```
Selection.sort(a, new Student.ByName());
```

Andrews	3	A	664-480-0023	097 Little
Battle	4	C	874-088-1212	121 Whitman
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Furia	1	A	766-093-9873	101 Brown
Gazsi	4	B	766-093-9873	101 Brown
Kanaga	3	B	898-122-9643	22 Brown
Rohde	2	A	232-343-5555	343 Forbes

```
Selection.sort(a, new Student.BySection());
```

Furia	1	A	766-093-9873	101 Brown
Rohde	2	A	232-343-5555	343 Forbes
Chen	3	A	991-878-4944	308 Blair
Fox	3	A	884-232-5341	11 Dickinson
Andrews	3	A	664-480-0023	097 Little
Kanaga	3	B	898-122-9643	22 Brown
Gazsi	4	B	766-093-9873	101 Brown
Battle	4	C	874-088-1212	121 Whitman

- @#%&@! Students in section 3 are no longer sorted by name.

- A stable sorting maintains the relative order of records with equal keys
- The above sorting not stable.

Stability

- Which sorting algorithms are stable?
  Need to check algorithm (and implementation) to check for stability.

sorted by time		sorted by location (not stable)		sorted by location (stable)	
Chicago	09:00:00	Chicago	09:25:52	Chicago	09:00:00
Phoenix	09:00:03	Chicago	09:03:13	Chicago	09:00:59
Houston	09:00:13	Chicago	09:21:05	Chicago	09:03:13
Chicago	09:00:59	Chicago	09:19:46	Chicago	09:19:32
Houston	09:01:10	Chicago	09:19:32	Chicago	09:19:46
Chicago	09:03:13	Chicago	09:00:00	Chicago	09:21:05
Seattle	09:10:11	Chicago	09:35:21	Chicago	09:25:52
Seattle	09:10:25	Chicago	09:00:59	Chicago	09:35:21
Phoenix	09:14:25	Houston	09:01:10	Houston	09:00:13
Chicago	09:19:32	Houston	09:00:13	Houston	09:01:10
Chicago	09:19:46	Phoenix	09:37:44	Phoenix	09:00:03
Chicago	09:21:05	Phoenix	09:00:03	Phoenix	09:14:25
Seattle	09:22:43	Phoenix	09:14:25	Phoenix	09:37:44
Seattle	09:22:54	Seattle	09:10:25	Seattle	09:10:11
Chicago	09:25:52	Seattle	09:36:14	Seattle	09:10:25
Chicago	09:35:21	Seattle	09:22:43	Seattle	09:22:43
Seattle	09:36:14	Seattle	09:10:11	Seattle	09:22:54
Phoenix	09:37:44	Seattle	09:22:54	Seattle	09:36:14

no longer sorted by time

still sorted by time

Stability: insertion sort

- Theorem. Insertion sort is stable.

```
public class Insertion
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
         for (int j = i; j > 0 && less(a[j], a[j-1]); j--)
            exch(a, j, j-1);
   }
}
```

i	j	0	1	2	3	4
0	0	B_1	A_1	A_2	A_3	B_2
1	0	A_1	B_1	A_2	A_3	B_2
2	1	A_1	A_2	B_1	A_3	B_2
3	2	A_1	A_2	A_3	B_1	B_2
4	4	A_1	A_2	A_3	B_1	B_2
		A_1	A_2	A_3	B_1	B_2

Proof. If keys are equal, then insertion sort will never move one key past the other.

Stability: selection sort

- Theorem. Selection sort is not stable.

```
public class Selection
{
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      for (int i = 0; i < N; i++)
      {
         int min = i;
         for (int j = i+1; j < N; j++)
            if (less(a[j], a[min]))
               min = j;
         exch(a, i, min);
      }
   }
}
```

i	min	0	1	2
0	2	B_1	B_2	A
1	1	A	B_2	B_1
2	2	A	B_2	B_1
		A	B_2	B_1

- Proof by counter-example: Long distance swapping kan move an item past an item with the same key.

Stability: shellsort

- Sats. Shellsort sort is not stabile.

```
public class Shell {
   public static void sort(Comparable[] a)
   {
      int N = a.length;
      int h = 1;
      while (h < N/3) h = 3*h + 1;
      while (h >= 1)
      {
         for (int i = h; i < N; i++)
         {
            for (int j = i; j > h && less(a[j], a[j-h]); j -= h)
               exch(a, j, j-h);
         }
         h = h/3;
      }
   }
}
```

h	0	1	2	3	4
	B_1	B_2	B_3	B_4	A_1
4	A_1	B_2	B_3	B_4	B_1
1	A_1	B_2	B_3	B_4	B_1
	A_1	B_2	B_3	B_4	B_1

- Proof by counter-example. Long distance swapping can move an element past another with the same key.

Stability: mergesort

- Theorm. Mergesort is stable!!

```
public class Merge
{
   private static void merge(...)
   {  /* as before */  }

   private static void sort(Comparable[] a, Comparable[] aux, int lo, int hi)
   {
      if (hi <= lo) return;
      int mid = lo + (hi - lo) / 2;
      sort(a, aux, lo, mid);
      sort(a, aux, mid+1, hi);
      merge(a, aux, lo, mid, hi);
   }

   public static void sort(Comparable[] a)
   {  /* as before */  }
}
```

- Proof. It is enough to show that the merge operation is stable. Why?

Stability: mergesort

- Theorem. Merging is stable

```
private static void merge(...)
{
   for (int k = lo; k <= hi; k++)
      aux[k] = a[k];

   int i = lo, j = mid+1;
   for (int k = lo; k <= hi; k++)
   {
      if      (i > mid)              a[k] = aux[j++];
      else if (j > hi)               a[k] = aux[i++];
      else if (less(aux[j], aux[i])) a[k] = aux[j++];
      else                           a[k] = aux[i++];
   }
}
```

0	1	2	3	4	5	6	7	8	9	10
A_1	A_2	A_3	B	D	A_4	A_5	C	E	F	G

- Proof. Take the value from the left subarray if the keys are equal.

Sorting summary

	inplace?	stable?	best	average	worst	remarks
selection	✓		$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	$\frac{1}{2} N^2$	N exchanges
insertion	✓	✓	N	$\frac{1}{4} N^2$	$\frac{1}{2} N^2$	use for small N or partially ordered
shell	✓		$N \log_3 N$	?	$c N^{3/2}$	tight code; subquadratic
mergesort		✓	$\frac{1}{2} N \lg N$	$N \lg N$	$N \lg N$	$N \log N$ guarantee; stable
?	✓	✓	N	$N \lg N$	$N \lg N$	holy sorting grail