Homework 3

Blake Hamilton

February 13, 2024

Question 1

(a) We rearrange the equation to get $f(x) = 2x - \sin x - 1$. We will then chose an interval $[a, b] = [0, 2]$. We can see that for $x = 0$, $2(0) - \sin(0) - 1 = -1 < 0$ and for $x = 2$, $2(2) - \sin 2 - 1 > 0$. giving one positive value and one negative value for the equation. Since we can find the derivative $f'(x) = 2 - \cos x$, we know that the function is differentiable and therefore continuous on the interval $[-\infty, \infty]$ so therefore continuous on $[0, 2]$. Therefore, we can use the Intermediate Value Theorem to prove that since $f(a) < 0$ and $f(b) > 0$, there must be a root r such that $f(r) = 0$.

(b) To show that there is only one root $\mathbb{R}$, we will show that $f(x)$ is strictly increasing. Since we showed that there is one root r in $[a, b]$, we know the one root in $\mathbb{R}$ is the root in $[a, b]$. As mentioned above, we have $f'(x) = 2 - \cos(x)$. As $\cos x \leq 1$ for all x, we know $f'(x) \geq 2 - (1) > 0$, so it is strictly increasing. As $f(a) < 0$ and $f(b) > 0$, the root r such that $f(r) = 0$ is the only root of the equation.

(c) The resulting final approximation was 0.8878622120246291 after 30 iterations.

Question 2

(a) We get a result of $x = 5.000073242187501$ when using $f(x) = (x - 5)^9$.

(b) We can use Pascal's Triangle to find the expanded expression:

$$f(x) = (1)(-5)^0x^9+(9)(-5)^1x^8+(36)(-5)^2x^7+(84)(-5)^3x^6+(126)(-5)^4x^5+(126)(-5)^5x^4+(84)(-5)^6x^3+$$

$$f(x) = x^9-45x^2+900x^3-10500x^4+78750x^5-393750x^4+1312500x^3-2812500x^2+3515625x-1953125$$

Using this expanded method, we get $x = 5.12875$.

(c) The expanded version in part (b) does a much worse job approximating the root because of cancellation. Subtraction operations are vulnerable to cancellation, and this is what is causing the discrepancy between the two versions of the expressions when using the bisection algorithm.

Question 3

1. Theorem 2.1: "Suppose that $f \in C[a,b]$ and $f(a) \cdot f(b) < 0$. The Bisection method generates a sequence $\{p_n\}_{n=1}^{\infty}$ approximating a zero p of f with $|p_n - p| \leq \frac{b-a}{2^n}$, when $n \geq 1$."

   We want our accuracy to be 10^{-3}, so we find $10^{-3} \geq \frac{b-a}{2^n}$ then solve for n with $[a,b] = [1,4]$:

$$\begin{aligned} \text{Error} &\leq \frac{b-a}{2^n} \\ 2^n &\leq \frac{4-1}{10^{-3}} \\ n &\leq \log_2 3000 \\ &\leq 11.55 \end{aligned}$$

   We must round up to the nearest integer, so the upper bound is 12 iterations.

(b) The approximate root calculated was 1.378662109375 after 11 iterations. We can see that the number of iterations taken was indeed less than our upper bound.

Question 4

(a) For this expression, we have $g(x) = -16 + 6x + \frac{12}{x}$ and the derivative $g'(x) = 6 - \frac{12}{x^2}$. At $x_* = 2$, we see the derivative is $g'(2) = 6 - \frac{12}{(2)^2} = 3 > 1$, so it does not converge.

(b) For this expression, we have $g(x) = \frac{2}{3}x + \frac{1}{x^2}$ and $g'(x) = \frac{2}{3} - \frac{2}{x^3}$, so at $x_* = 3^{1/3}$, $g'(3^{1/3}) = \frac{2}{3} - \frac{2}{(3^{1/3})^3} = 0$. We will use Taylor Series for the first three terms about fixed point x_* for some $\alpha \in (x, x_*)$:

$$\begin{aligned} g(x) &= \frac{g(x_*)}{0!} + \frac{g'(x_*)}{1!}(x - x_*) + \frac{g''(\alpha)}{2!}(x - x_*)^2 \\ &= \frac{g(x_*)}{0!} + \frac{g''(\alpha)}{2!}(x - x_*)^2 && \because g'(x_*) = 0 \\ g(x) - x_* &= \frac{g''(\alpha)}{2}(x - x_*)^2 && \because g(x_*) = x_* \\ \frac{g(x) - x_*}{(x - x_*)^2} &= \frac{g''(\alpha)}{2} \\ \lim_{k\to\infty} \frac{x_{k+1} - x_*}{(x_k - x)^2} &= \lim_{k\to\infty} \frac{g''(x_k)}{2} \\ &= \frac{g''(x_*)}{2} && \because \lim_{k\to\infty} g''(\alpha) = g''(\lim_{k\to\infty} \alpha) \end{aligned}$$

We know above that $\lim_{k\to\infty} x_k = x_*$ because as the interval $\alpha in(x_k, x_*)$ gets smaller, it converges on a single point, x_*. We now solve for the above expression at x_*:

$$\frac{g''(x_*)}{2} = \frac{6}{2x^4} \implies \frac{g''(3^{1/3})}{2} = \frac{6}{2(3^{1/3})^4} \approx 0.69$$

Since we see that the equation of the form

$$\lim_{n\to\infty} \frac{|p_{n+1} - p|}{|p_n - p|^\alpha} = \lambda$$

has $0 < \lambda < 1$ with an $\alpha = 2$, we see that the equation is quadratic convergence with order 2.

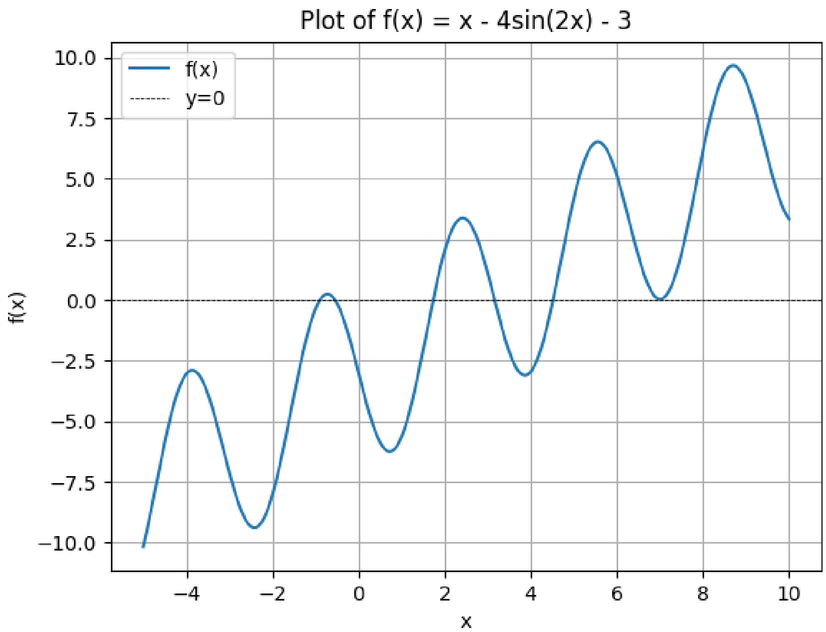
(c) For this expression, we have $g(x) = \frac{12}{1+x}$ and $g'(x) = -\frac{12}{(1+x)^2}$, so $|g'(3)| = |-\frac{12}{(1+(3))^2}| = 0.75 < 1$, so we know there is convergence around the root. As above, we will use Taylor Series for the first two terms (two since $g'(3) \neq 0$) about fixed point x_* for some $\alpha \in (x, x_*)$:

$$\begin{aligned}
g(x) &= \frac{g(x_*)}{0!} + \frac{g'(\alpha)}{1!}(x - x_*) \\
g(x) - x_* &= g'(\alpha)(x - x_*) && \because g(x_*) = x_* \\
\frac{g(x) - x_*}{x - x_*} &= g'(\alpha) \\
\lim_{k\to\infty} \frac{x_{k+1} - x_*}{x_k - x} &= \lim_{k\to\infty} g'(x) \\
&= g'(x_*) && \because \lim_{k\to\infty} g'(\alpha) = g'(\lim_{k\to\infty} \alpha)
\end{aligned}$$

We know from earlier that $|g'(3)| = 0.75$, we we know that the rate of linear convergence is 0.75.

Question 5

(a) There are five "zero crossings".



(b) It was found empirically that only two of the roots can be found using fixed point iteration: $x = -0.5444424007, 3.1618264866$. We can give a theoretical rationale for this by looking for where $|g'(x)| < 1$ in the interval $(-2, 5)$ (where $f(x)$ crosses zero on the graph). $g'(x) = -4\cos(2x) + \frac{5}{4}$, so to find where the derivative fits our criteria:

$$\begin{aligned} g'(x) &= -4\cos(2x) + \frac{5}{4} \\ 1 &> -4\cos(2x) + \frac{5}{4} \\ x &> \frac{1}{2}\arccos(\frac{1}{16}) \end{aligned}$$

Knowing the behavior of arccos, we know that the points described are the only ones that fit the requirement.