Homework 2

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Question 1

We know the following:

- $f(x) = O(g(x))$ iff $\lim_{x\ to0} \frac{f(x)}{g(x)} = M$ for some constant M that can be bounded.
- $f(x) = o(g(x))$ iff $\lim_{x\to 0} \frac{f(x)}{g(x)} = 0$

(a) To prove $(1+x)^n = 1 + nx + o(x)$ as $x \to \text{inf}$, we must prove $\lim_{x\to 0}(1+x)^n - (1+nx) = o(x)$. For $x = 0$, we see we get an indeterminant fraction using $f(x) = (1+x)^n - nx - 1$ and $g(x) = x$:

$$\frac{f(0)}{g(0)} = \frac{(1+0)^n - (1+n(0))}{0} = \frac{1^n - 1}{0} = \frac{0}{0}$$

Using L'Hospital's rule, we can prove that the limit approaches 0, so we can prove $(1+x)^n = 1 + nx + 0(x)$:

$$\frac{f'(x)}{g'(x)} = \frac{n(1+x)^{n-1} - n}{1}$$

For $x = 0$:

$$\frac{n(1+(0))^{n-1} - n}{1} = \frac{n-n}{1} = 0$$

Since the limit approaches 0, we know that $\lim_{x\to 0}(1+x)^n = 1+nx+o(x)$ as $x \to \text{inf}$.

(b) To prove $x\sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$, we must show that $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{x\sin(\sqrt{x})}{x^{3/2}} = M$:

$$\frac{f(x)}{g(x)} = \frac{x\sin\sqrt{x}}{x^{3/2}} = \frac{\sin\sqrt{x}}{\sqrt{x}} = \frac{\sin u}{u} \qquad \text{For } u = \sqrt{x} \tag{1}$$

$$= 1 \qquad \frac{\sin u}{u} = 1 \text{ as } x\ to0 \tag{2}$$

Therefore, for $M = 1$, we can show that $\lim_{x\ to0} \frac{f(x)}{g(x)} = M$ and $x\sin(\sqrt{x}) = O(x^{3/2})$ as $x \to 0$.

(c) We will show that $\lim_{x\to\infty} \frac{f(t)}{g(t)} = \lim_{t\to\inf} \frac{e^{-t}}{1/t^2} = \lim_{t\to\inf} \frac{t^2}{e^t} = 0$ as $t \to$ inf:

$$\begin{aligned}
\lim_{t\to\infty} \frac{f(\infty)}{g(\infty)} &= \frac{\infty}{\infty} \\
\frac{f'(t)}{g'(t)} &= \frac{2t}{e^t} && \text{Apply L'Hospital} \\
\lim_{t\to\infty} \frac{f'(\infty)}{g'(\infty)} &= \frac{\infty}{\infty} \\
\frac{f''(t)}{g''(t)} &= \frac{2}{e^t} && \text{Apply L'Hospital} \\
\lim_{t\to\infty} \frac{f''(\infty)}{g''(\infty)} &= 0
\end{aligned}$$

Since $\lim_{x\to\infty} \frac{f(t)}{g(t)} = 0$, we know $e^{-t} = o(\frac{1}{t^2})$ as $t \to \infty$.

(d) We will show that $\int_0^\epsilon e^{-x^2} dx = O(\epsilon)$ by showing that $\lim_{\epsilon\to\infty} \frac{f(\epsilon)}{g(\epsilon)} = M$ for some constant M.

$$\lim_{\epsilon\to 0} \frac{f(\epsilon)}{g(\epsilon)} = \lim_{\epsilon\to 0} \frac{\int_0^\epsilon e^{-x^2} dx}{O(\epsilon)} \tag{3}$$

$$\frac{f(0)}{g(0)} = \frac{\int_0^{(0)} e^{-x^2} dx}{0} = \frac{0}{0} \qquad \text{Definition of Integrals} \tag{4}$$

$$\lim_{\epsilon\to 0} \frac{f'(\epsilon)}{g'(\epsilon)} = \frac{e^{-x^2}}{1} \qquad \text{L'Hospital} \tag{5}$$

We can see that we found $M = e^{-x^2}$. As we are investigating when $\epsilon \to 0$, e^{-x^2} is a constant term, therefore showing that $\int_0^\epsilon e^{-x^2} dx = O(\epsilon)$.

Question 2

(a) We know that $\mathbf{Ax} = \mathbf{b}$ and A is invertible, so we can do the following to find Δx:

$$\begin{aligned}
A^{-1}Ax &= A^{-1}b \\
x &= A^{-1}b \\
x + \Delta x &= A^{-1}\hat{b} && \hat{b} \text{ is perturbed } b \\
&= A^{-1}b + A^{-1}\delta b && \text{where } \delta b = \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} \\
\Delta x &= A^{-1}\delta b
\end{aligned}$$

Therefore, we find the exact perturbation for the change in the solution below:

$$\Delta x = \begin{bmatrix} 1-10^{10} & 10^{10} \\ 1+10^{10} & -10^{10} \end{bmatrix} \begin{bmatrix} \Delta b_1 \\ \Delta b_2 \end{bmatrix} = \begin{bmatrix} (1-10^{10})\Delta b_1 + 10^{10}\Delta b_2 \\ (1+10^{10})\Delta b_1 - 10^{10}\Delta b_2 \end{bmatrix}$$

(b) Condition number $\kappa(A) = \|A\|\|A^{-1}\|$.

$$||A||_\infty = \max(|1| + |1|, |1+10^{-10}| + |1-10^{-10}|) = 2$$

$$||A^{-1}||_\infty = \max(|1-10^{10}| + |10^{10}|, |1+10^{10}| + |-10^{10}|) \approx 2 \cdot 10^{11}$$

Therefore, we have $\kappa(A) = 2 \cdot 2 \cdot 10^{11} = 4 \cdot 10^{11}$, which would be considered a very large conditioning number, meaning the matrix is ill-conditioned.

(c) The relative error in the solution is $\frac{|\Delta x - x|}{|x|}$. We are told that $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and will use the equation above and $\Delta b_1 = 2 \cdot 10^{-5}$ and $\Delta b_2 = 8 \cdot 10^{-5}$:

$$\Delta x = \begin{bmatrix} (1-10^{10})(2 \cdot 10^{-5}) + 10^{10}(8 \cdot 10^{-5}) \\ (1+10^{10})(2 \cdot 10^{-5}) - 10^{10}(8 \cdot 10^{-5}) \end{bmatrix} = \begin{bmatrix} 600000.0000200002 \\ -599999.9999800001 \end{bmatrix}$$

Therefore, the relative error is $\frac{|\Delta x - x|}{|x|} = \begin{bmatrix} \frac{|600000-1|}{|1|} \\ \frac{|-600000-1|}{|1|} \end{bmatrix} = \begin{bmatrix} 6 \cdot 10^5 \\ 6 \cdot 10^5 \end{bmatrix}$ The relative error is based on the condition number and perturbation. It is intuitive that as perturbations get larger, so does the relative error. Also, as the condition number $(K(A))$increases, this means our matrix is more ill-conditioned, so it will render larger relative errors. If the perturbations are the same in this scenario, they actually cancel each other out for the terms that make errors explode. Let β be the perturbation for both:

$$\Delta x = \begin{bmatrix} (1-10^{10})\beta + 10^{10}\beta \\ (1+10^{10})\beta - 10^{10}\beta \end{bmatrix} = \begin{bmatrix} \beta \\ \beta \end{bmatrix}$$

We see that since the error β is of magnitude 10^{-5}, the relative error is of magnitude 10^{-5}, which is a significantly smaller error than when Δb_1 and Δb_2 are different from one another. It is much more realistic that the two errors are different from one another in the real world, which means that this matrix will likely be ill-conditioned in a real world application.

Question 3

(a) The relative condition number is $\kappa_f(x) = |\frac{xf'(x)}{f(x)}|$. With $f(x) = e^x - 1$ and $f'(x) = e^x$, we have:

$$\kappa_f(x) = |\frac{xe^x}{e^x - 1}|$$

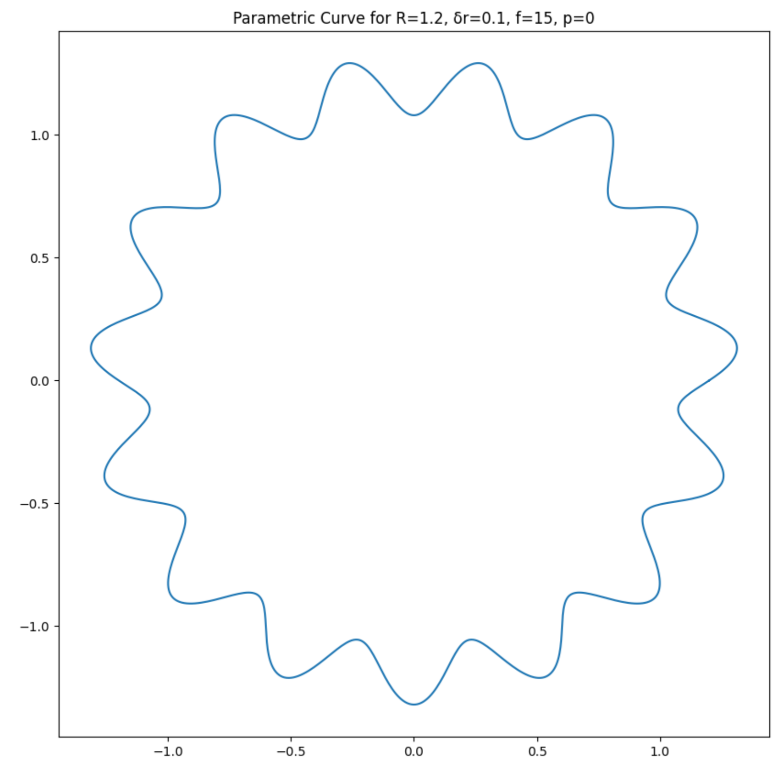
The above is ill-conditioned for any large values of x, as it directly increases as x increases.

(b) No, using the mentioned algorithm is not stable. Since it uses subtraction, any value of x near zero will make the expression e^x close to 1, therefore making it susceptible to cancellation when subtracting 1.

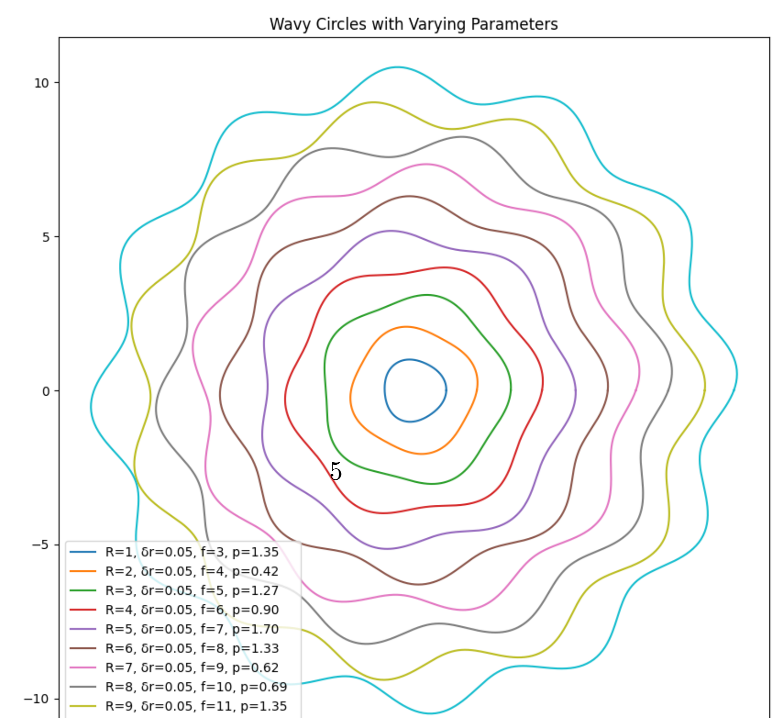
(c) For the given x, we have $y = 1.000000001$ and return 0.000000001, giving only $16 - 9 = 7$ correct digits. This is because the first exponential step keeps full precision but when returning $y - 1$ we have 9 digits of cancellation. This is expected because, as mentioned in (a), when x is near 0, there will be large cancellation errors.

(d) We will use a Taylor Series approximation. In order to make the approximation accurate to a certain degree, we find the remainder term in the Taylor series to be less than our acceptable error. Since we are working with e^x and $x_0 = 0$, this makes the approximation simple, as each term is $\frac{x^n}{n!}$. We can find the correct remainder term by solving for n in $R_n \leq \frac{x^{n+1}}{(n+1)!}$ to limit the error to be less than 16 digits. This was done in Python, and it was found that the error term that gives an acceptable error is at $n = 3$, so we will include the first three terms $(n = 0, 1, 2)$. Therefore, we have the expression $e^x - 1 \approx \frac{x^0}{0!} + \frac{x}{1!} + \frac{x^2}{2!} - 1 = x + \frac{x^2}{2}$ is accurate to 16 digits.

Question 4


Parametric Curve for R=1.2, δr=0.1, f=15, p=0
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

(b)


Wavy Circles with Varying Parameters
10
5
0
−5
−10
R=1, δr=0.05, f=3, p=1.35
R=2, δr=0.05, f=4, p=0.42
R=3, δr=0.05, f=5, p=1.27
R=4, δr=0.05, f=6, p=0.90
R=5, δr=0.05, f=7, p=1.70
R=6, δr=0.05, f=8, p=1.33
R=7, δr=0.05, f=9, p=0.62
R=8, δr=0.05, f=10, p=0.69
R=9, δr=0.05, f=11, p=1.35