Final review

$$Pr(\text{something happens}) = \frac{\#\text{ outcome if happens}}{\#\text{ possible outcomes.}}$$

→ Equally likely.

What if not equally likely?

{HH, HT, TT}

⇒ Find ways to rewrite problem.

(*) add ordering / labeling (balls in boxes)

(*) equivalent experiment. (take out 2 cards A ♡)

P(E) as a set function

(*) Axioms

(*) Using set functions to define complicated events

(unions, intersections, exclusion-inclusion, etc.

e.g. $P(\text{at least one} \dots) = P(E_1 \cup E_2 \cup \cdots)$

$$P(E|F) = \frac{P(EF)}{P(F)}$$

(*) $P(ABC) = P(A)\,P(B|A)\,P(C|AB)$

(*) Baye's formula

$$P(E) = P(E|F)P(F) + P(E|F^c)P(F^c)$$

(recall equally likely condition).

(*). prob $\longleftrightarrow$ conditional prob

(*) depending on which question asked.

(*) $\underbrace{P(E|F)}P(F) = \underbrace{P(F|E)}P(E)$

(*) Independence

(*) $\underbrace{\text{2 events}}_{\text{pairwise}}$ / $\underbrace{\text{multiple events}}_{\text{mutually}}$.

(*) $P(\cdot|F)$ is a probability

(*) conditional on more than 1 event.

e.g. Simpson's paradox

$$P(E|A,F) > P(E|B,F)$$
$$P(E|A,F^c) > P(E|B,F^c)$$

$$\Rightarrow P(E|A) \not> P(E|B)$$

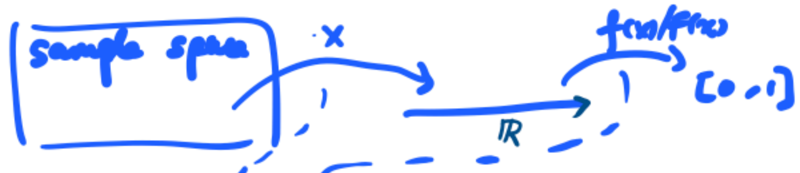
$$P(E|A) = P(E|F\cap A)P(F|A) + P(E|F^c\cap A)P(F^c|A)$$

Random variable

1. r.v. & distribution.

r.v. X(event) $\rightarrow$ real number

distribution $\begin{cases} \text{p.m.f} \\ \text{c.d.f.} \end{cases}$ is a function from all possible X values to $[0, 1]$



events	X	$p(x)$	X^2
A	-2	0.1	4
B	-1	0.2	1
C	1	0.3	1
D	2	0.2	4
E	3	0.2	9

$Y=X^2$	event	Prob
1	B, C	0.5
4	A, D	0.3
9	E	0.2

2. r.v & transformed r.v

$g(X)$ is also a r.v.

$\rightarrow P(X=k) \not= P(g(X)=g(k))$

e.g. $P(X=-2)=0.1$

$P(X^2=4) = P(X=-2)+P(X=2) = 0.3$

two events are equivalent

(*) $P(X+1=k+1) = P(X=k)$

$E(g(x)) = \sum_x g(x)\,p(x)$

or. $E(g(x)) = \sum_y y\,p(y)$ $\quad p(y) \rightarrow Pr(g(X)=y)$

y : all unique values of $g(X)$

$p(y) = \sum_{x:\, g(x)=y} p(x)$

3. r.v. & the values of r.v.

$P(X = x) = f(x)$

(*) Expectation & Variance.

$E(X) = \sum_x x\, p(x)$, $Var(X) = E(X^2) - (E(X))^2$

- $E(aX+b) = aE(X) + b$
- $Var(aX+b) = a^2 Var(X)$
- $E(X_1 + X_2) = E(X_1) + E(X_2)$, but this does not hold for variance.

Discrete r.v.

(*) Bernoulli / binomial → # success in n trials

↳ 1 trial with success prob = p

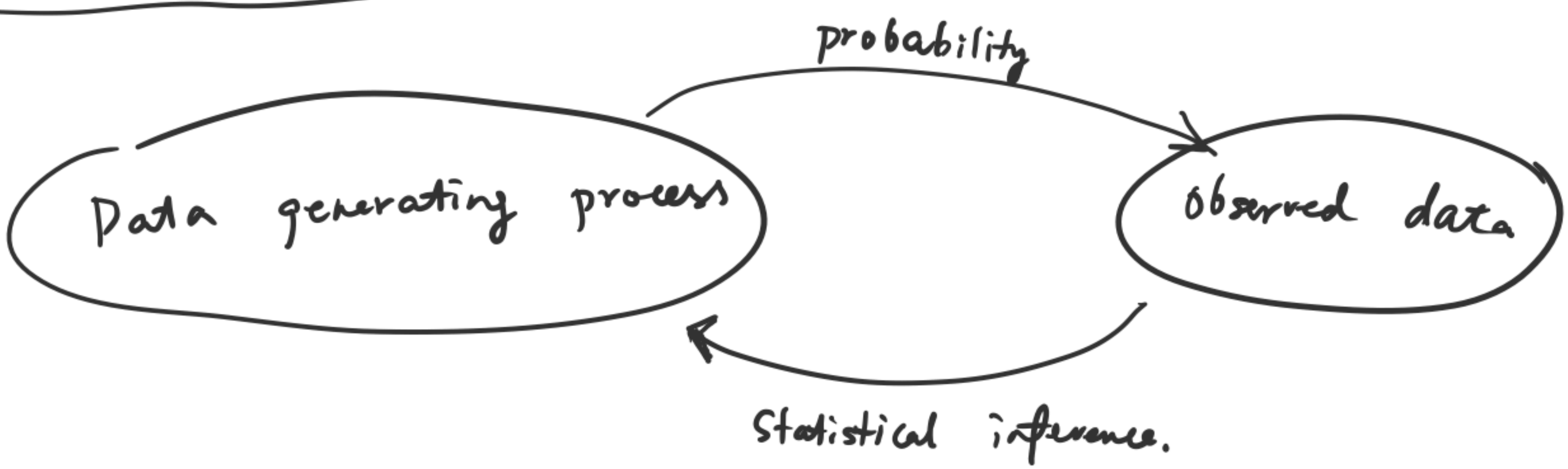
(*) Poisson
- (*) approx binomial with $np = \lambda$, $n \to \infty$ — relaxing equal p.
- (*) (poisson paradigm) # of events with probability $p_1, p_2 \cdots p_n$ happens with $\lambda = \sum_{i=1}^{n} p_i$
- (*) events happening in continuous time with constant rate → $P(N(t)) \sim Poi(\lambda t)$

(*) geometric → # of trials until the 1st success

(*) negative binomial → # of trials until the r-th success.

(*) hypergeometric → . . . . .

How probability relates to statistics



Problem solving.

1. write down what we know

   - $P(E)$, $P(E|F)$, $P(EF)$ . . . . .
   - X : ? , $X \sim$?

2. —— // ———— we want to know

   - P(some event)
     $\longrightarrow$ unions / intersects / . . . .
   - P ((some functions) of X | (some conditions))

3. connect these two

Two envelope problem

- Suppose I have two envelopes with money in them.
- One contains twice the money than the other.
- I give you one; you open it and see 100 dollars.
- Now I say
  I can give you a chance to swap the envelope. You have 50% chance of gaining another 100 dollars, and 50% chance of losing 50 dollars. So it is to your advantage if you swap, as your expectation of the money in the other envelope is 200*0.5 + 50 * 0.5 > 100.

X_1 : \$ in envelope 1 , X_2 : \$ 2

$$E(X_2) = 0.5\, E(X_2 \mid X_2 > X_1) + 0.5\, E(X_2 \mid X_2 < X_1)$$

$$= 0.5\, E(2X_1 \mid X_2 > X_1) + 0.5\, E(\tfrac{1}{2}X_1 \mid X_2 < X_1)$$

$$= 0.5 \times 2\, E(X_1 \mid X_2 > X_1) + 0.5 \times 0.5\, E(X_1 \mid X_2 < X_1)$$

$$= E(X_1 \mid X_2 > X_1) + \tfrac{1}{4} E(X_1 \mid X_2 < X_1)$$

Suppose a , $2a$

$$E(X_1 \mid X_2 > X_1) = a$$

$$E(X_1 \mid X_2 < X_1) = 2a$$

$$E(X_2) = a + \tfrac{1}{4} \cdot 2a$$

$$= \tfrac{3}{2} a$$

$$E(X_1) = \cdots = \tfrac{3}{2} a$$

St Petersburg paradox

- Consider the following game of flipping a coin until the first heads appear
  - You will win \$2 if the game stops at the first throw,
  - You will win \$4 if the game stops at the second throw,
  - You will win \$8 if the game stops at the third throw,
  - ...
- How much are you willing to pay to play this game?

X: amount you win

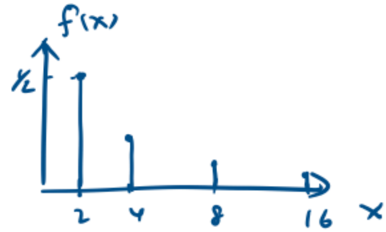
$$\text{p.m.f } f(X) \begin{cases} 2 & \text{w.p. } 1/2 \\ 4 & \text{w.p. } 1/4 \\ 8 & \vdots \\ 16 & \\ \vdots & \end{cases}$$

$$E(X) = \sum_{x=1}^{\infty} x\,p(x) = 2 \times \frac{1}{2} + 4 \times \frac{1}{4} + 8 \times \frac{1}{8} + \cdots$$

$$= 1 + 1 + 1 + \cdots$$

$$= \infty$$

$\Rightarrow$ at any price - you should play.



median

$P(X \le 3) = 0.5$

$P(X \ge 3) = 0.5$

- A modified version:
  - If you bet on the outcome of a flip of coin is heads
  - You bet \$1 on the first flip
  - If you lose, you then bet \$2 on the next flip
  - If you lose again, you the bet \$4 on the next flip, and then \$8, \$16, ...

First H	Winning
1	\$1
2	$-1+2=\$1$
3	$-1-2+4=\$1$
$\vdots$	$\vdots$

- Another modified version
  - Instead of betting on flip of coin, you bet on an outcome with probability 0.01 of occurring

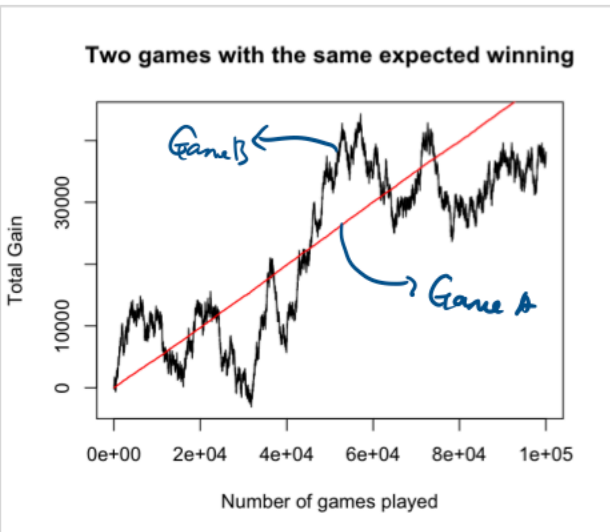
same

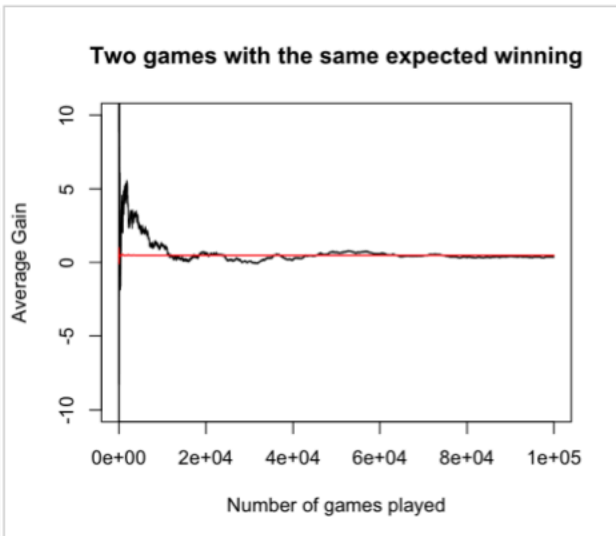
Gambler's fallacy (Laplace, 1820)

Imagine repeatedly flipping a (fair) coin and guessing the outcome. If you observe k heads in a roll, will you believe a tail is "due," that is, more likely to appear on the next flip than a head?

At the roulette wheel at Le Grande Casino in Monte Carlo, Monaco, the color black came up 26 times in a row. The probability of the occurrence was 1 in 136,823,184[44] The incident is cited as an illustration of the gambler's fallacy, because after the wheel stopped at black ten straight times, casino patrons began betting large sums of money on red, on the logic that black could not possibly come up again. The odds of red or black coming up on any individual spin were the same each time—18 out of 37; to no surprise of statisticians, "the casino made several million francs that night".

		Game A	Game B
a game	50%	win \$2	win \$101
	50%	lose \$1	lose \$100





Gambler's ruin

You are playing a game with your friend, where you have probability p of winning. Suppose you have n_1 dollars and your friend has n_2 dollars, and after each game, the loser gives 1 dollar to the winner. What is the probability that you will lose all the money to your friend?

we didn't talk about this, but point is (and you can verify) that if $p < 0.5$ and $n_2 \to \infty$, the probability you lose all the money is going to 1!

So dont play with casinos unless you have larger chance of winning! (or if you don't care about losing n_1 dollars)