STAT 394
Probability I
Summer 2017

Problem Session # 4

Instructions: Find the group with the circled number. Start working together on the circled question. Continue with question 1 when you finish. Do as many as you have time for.

1. Assume that two equally matched teams, A and B, play a series of games and that the first team that wins four games is the overall winner of the series. As it happens, team A lost the first game. What is the probability it will win the series? Assume that the games are Bernoulli trials with success probability 0.5. (Exercise: if after the first game, team A can request to change the rule to best of 5 instead of best of 7, should team A make the request? What if A and B have different skill levels and the winning probability for team A is 0.2 and for team B is 0.8?)

   For this problem we can consider the scenario that all 7 games have to be played (it is equivalent to end the series whenever a team has 4 wins, why?). Then what we need to find is the probability that team A wins at least 4 games. Since the first game is a loss for team A, let X denote the number of wins for team A after playing out all 7 games, then $X \sim Bin(6, 0.5)$.

$$P(X = k) = \binom{6}{k} p^k (1-p)^{6-k} = \binom{6}{k} 0.5^6$$

   So the probability of winning the series for team A is

$$P(win) = \sum_{k=4}^{6} P(X = k) = \sum_{k=4}^{6} \binom{6}{k} 0.5^6 \approx 0.3438$$

   Of course, we could also look at the problem directly, and calculate probability of winning in 5, 6, and 7 games respectively. In this case,

$$P(win) = \sum_{k=5}^{7} P(\text{win in k games}) = 0.5^4 + \binom{4}{1} 0.5^5 + \binom{5}{2} 0.5^6 \approx 0.3438$$

   The reason we need to use $\binom{4}{1}$ and $\binom{5}{2}$ is because in order for the series to finish at game 6 and 7 respectively, the last game need to be a win for team A (otherwise the series will end prior to the last game). Exercise problem is similar and omitted.

2. Suppose the number of typos a writer makes in a given chapter of his book is a Poisson random variable with parameter $\lambda = 5$. His editor suggests him to take a writing class, which has been marketed that it reduces the Poisson parameter to $\lambda = 1$ for 80 percent of the people. For the other 20 percent, it does not have any appreciable effect on reducing typos. So this writer tries the class and writes a new chapter in this book. It has k typos in the chapter. How likely is it that the class is beneficial for him?

   Let X denote the number of typos in a chapter, and let E denote that the class is beneficial for him. Then

$$P(E|X = k) = \frac{P(X = k|E)P(E)}{P(X = k|E)P(E) + P(X = k|E^c)P(E^c)}$$

   As $P(E) = 0.8$, $P(E^c) = 0.2$, $P(X = k|E) = \frac{e^{-1}}{k!}$, and $P(X = k|E^c) = \frac{5^k e^{-5}}{k!}$, we can plug in every term and obtain the result

$$P(E|X = k) = \frac{4}{4 + 5^k e^{-4}}$$

3. On rainy days, Allen is late to work with probability 0.3; on non-rainy days, he is late with probability 0.1. With probability 0.7, it will rain tomorrow. (a) Find the probability that Allen is late tomorrow. (b) If we know Allen is late today, what is the conditional probability that it rained?

   Let R be the event that it rains tomorrow and let L the event that Allen is late tomorrow. Then,

$$P(L) = P(L|R)P(R) + P(L|R^c)P(R^c) = 0.3 \times 0.7 + 0.1 * (1 - 0.7) = 0.24$$

4. There are 3 coins in a box: a fair coin, a two-headed coin, a biased coin that come up heads 70 percent of the time. If one of the coin is selected at random and flipped twice. It shows both heads. What is the probability that it is the two-headed coin?

   Let E denote the event that a random coin shows two heads in two flips, and let F_i denote the event that i-th coin was selected.

$$P(F_2|E) = \frac{P(E|F_2)P(F_2)}{\sum_{i=1}^{3} P(E|F_i)P(F_i)} = \frac{1 * \frac{1}{3}}{0.5^2 * \frac{1}{3} + 1 * \frac{1}{3} + 0.7^2 * \frac{1}{3}} = \frac{1}{1.74} \approx 0.5747$$

5. Three buses carrying 100 students from the same school arrive at a football stadium. The buses carry, respectively, 40, 25, and 35 students. One of the students is randomly selected. Let X denote the number of students that were on the bus carrying the randomly selected student. One of the 3 bus drivers

is also randomly selected. Let Y denote the number of students on her bus. Compute $E(X)$, $E(Y)$, $Var(X)$, and $Var(Y)$.

The p.m.f of the two random variables are

$$P(X = 40) = 0.4, \quad P(X = 25) = 0.25, \quad P(X = 35) = 0.35,$$

and

$$P(Y = 40) = P(Y = 25) = P(Y = 35) = 1/3,$$

so the expectations are

$$E(X) = 0.4 \times 40 + 0.25 \times 25 + 0.35 \times 35 = 34.5$$

and

$$E(Y) = 1/3 \times (40 + 25 + 35) = \frac{100}{3}$$

Similarly, since

$$E(X^2) = 0.4 \times 40^2 + 0.25 \times 25^2 + 0.35 \times 35^2 = 1225$$

and

$$E(Y^2) = 1/3 \times (40^2 + 25^2 + 35^2) = 1150$$

we have

$$Var(X) = E(X^2) - E(X)^2 = 34.75$$

$$Var(Y) = E(Y^2) - E(Y)^2 = \frac{350}{9} \approx 38.89$$