

Regularization

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Introduction to Machine Learning

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When Linear Models Are Too Flexible

In the old days

Typically $n > p$ (much more data than predictors)

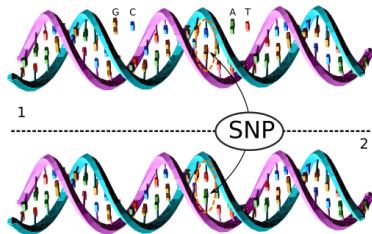
For example: predict blood pressure based on age, gender and body mass index (BMI)
(e.g. $n = 200$ patients, $p = 3$).

Nowadays: Big Data

Often $n \approx p$ or $n < p$

For example: predict blood pressure based on
500 000 single nucleotide polymorphisms (SNP)
($n = 200$, $p = 500\,000$).

⇒ **Linear Model perfectly fits the training data.**



Making Linear Models Less Flexible

Idea 1: Fix some parameters at zero

$$\hat{y} = f(x) = f(x_1, x_2, \dots, x_p) = \beta_0 + \underbrace{\beta_1}_{\neq 0} x_1 + \underbrace{\beta_2}_{\neq 0} x_2 + \beta_3 x_3 + \dots + \underbrace{\beta_{p-1}}_{\neq 0} x_{p-1} + \beta_p x_p$$

Problem: Many different models to fit; $\binom{p+1}{m}$ combinations of m non-fixed parameters.

Idea 2: constrain the parameters

Minimize the original loss $L(\beta)$ under the constraint $\|\beta\|_2^2 = \sum_{i=1}^p \beta_i^2 \leq S$.

This is equivalent to replacing the original loss $L(\beta)$ by

$$L_{L2}(\beta) = L(\beta) + \lambda \|\beta\|_2^2$$

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Ridge Regression (L2 Regularization)

$$L_{L2}(\theta) = L(\theta) + \lambda \|\theta\|_2^2$$

with **regularization constant** λ and (squared) **L2 norm** $\|\theta\|_2^2 = \sum_{i=1}^p \theta_i^2$.

1. The regularization constant λ is a hyper-parameter.
2. Often the intercept θ_0 is not regularized.
3. If $\lambda = 0$: original loss (no penalty)
4. The larger λ , the stronger the impact of the penalty on the result.
5. With increasing λ the model becomes less flexible.
6. With increasing λ all parameters tend to zero; it happens rarely that one is exactly zero.

Lasso (L1 Regularization)

$$L_{L1}(\theta) = L(\theta) + \lambda \|\theta\|_1$$

with **regularization constant** λ and **L1 norm** $\|\theta\|_1 = \sum_{i=1}^p |\theta_i|$.

Points 1-5 from ridge regression are also valid for the Lasso. However:

6. With large λ some parameters are exactly zero (in contrast to ridge regression).

An Alternative Formulation of Regularization

Thanks to a result from constraint optimization (see Karush-Kuhn-Tucker conditions, a generalization of Lagrange multipliers) the above formulations of Ridge Regression and the Lasso are equivalent to a constraint optimization problem:

Ridge Regression

minimize $L(\theta)$ under the constraint that $\|\theta\|_2^2 \leq S$.

The parameters are confined to a p -ball of radius S with center at the origin.

Lasso

minimize $L(\theta)$ under the constraint that $\|\theta\|_1 \leq S$.

The parameters are confined to a hypercube with edge length S , center at the origin and corners on the axes.

S is a (complicated) function of λ and the original loss $L(\theta)$.

With increasing S the model becomes more flexible.

Analytical Solutions for Simple Linear Regression

Notation: $\langle x \rangle = \frac{1}{n} \sum_{i=1}^n x_i$

Ridge Regression

$$L(\theta, \lambda) = \langle (y - \theta_0 - \theta_1 x)^2 \rangle + \lambda \theta_1^2$$

$$\theta_1 = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle}{\langle x^2 \rangle - \langle x \rangle^2 + \lambda}, \quad \theta_0 = \langle y \rangle - \theta_1 \langle x \rangle$$

Lasso

$$L(\theta, \lambda) = \frac{1}{2} \langle (y - \theta_0 - \theta_1 x)^2 \rangle + \lambda |\theta_1|$$

$$\theta_1 = \frac{\langle xy \rangle - \langle x \rangle \langle y \rangle - \text{sign}(\theta_1) \lambda}{\langle x^2 \rangle - \langle x \rangle^2} \text{ or } 0 \text{ if } |\langle xy \rangle - \langle x \rangle \langle y \rangle| < \lambda$$

Standardized Inputs for Regularization

Problem

Assume we find in multiple linear regression on the weather data the following parameters

$$\begin{array}{lll} X_1 & \text{LUZ_pressure} & [\text{hPa}] \\ X_2 & \text{LUZ_temperature} & [^\circ\text{C}] \end{array} \left| \begin{array}{ll} \theta_1 = -1 & [\text{km/h/hPa}] \\ \theta_2 = 0.5 & [\text{km/h/}^\circ\text{C}] \end{array} \right.$$

We could have measured the pressure in Pa and get the equivalent result

$$\begin{array}{lll} X_1 & \text{LUZ_pressure} & [\text{Pa}] \\ X_2 & \text{LUZ_temperature} & [^\circ\text{C}] \end{array} \left| \begin{array}{ll} \theta_1 = -1/100 & [\text{km/h/Pa}] \\ \theta_2 = 0.5 & [\text{km/h/}^\circ\text{C}] \end{array} \right.$$

With regularization $\lambda(\theta_1^2 + \theta_2^2)$ we would get different results for measurements in hPa and in Pa, because θ_1 contributes less to the penalty in the latter case.

Solution

Standardize all predictors, such that they have mean 0 and variance 1:

$$\tilde{X}_i = (X_i - \bar{X}_i) / \sqrt{\text{Var}(X_i)}$$

Scaling of the Regularization Constant with n

With loss $L(\theta) = \sum_{i=1}^n \ell(y_i, f(x_i)) + \lambda \|\theta\|_2^2$
the effective regularization depends on the size of the data set.

One can use instead an average loss $L(\theta) = \frac{1}{n} \sum_{i=1}^n \ell(y_i, f(x_i)) + \lambda \|\theta\|_2^2$ or
(equivalently) scale the regularization term $L(\theta) = \sum_{i=1}^n \ell(y_i, f(x_i)) + n \cdot \lambda \|\theta\|_2^2$

Quiz

- ▶ The Lasso tends to have larger variance (when fitted on different training sets from the same data generator) but smaller bias (relative to the true data generator) than linear regression.
- ▶ Indicate which is correct: as we increase S from 0 to ∞ in L2 regularized linear regression the training error will be
A) inverted U shape. B) U shape.
C) steadily increasing. D) steadily decreasing. E) constant.
- ▶ Indicate which is correct: as we increase S from 0 to ∞ in L2 regularized linear regression the test error will be
A) inverted U shape. B) U shape.
C) steadily increasing. D) steadily decreasing. E) constant.

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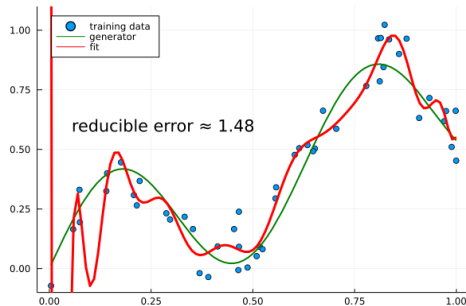
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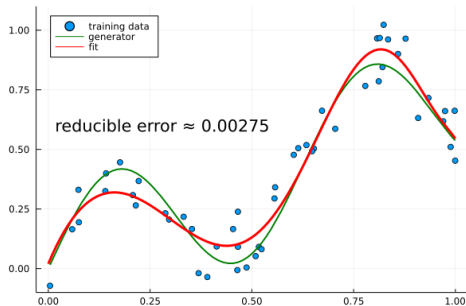
3. Regularization Examples

Polynomial Ridge Regression

$d = 20, \lambda = 0$



$d = 20, \lambda = 10^{-4}$



With a little bit of L2 regularization ($\lambda = 10^{-4}$)
one can prevent overfitting of polynomials with high degrees.

Multiple Logistic Ridge Regression on the Spam Data

$n = 2000$ emails, $p = 801$ features (size of the lexicon)

Without regularization

training misclassification rate: 0.0015

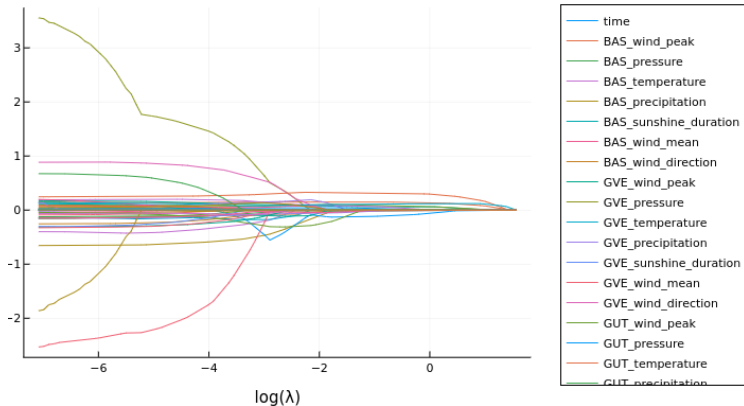
test misclassification rate: 0.048

With L2 regularization

training misclassification rate: 0.013

test misclassification rate: 0.041

The Lasso Path for the Weather Data



As we lower λ (from right to left), **BAS_wind_peak** is the first non-zero factor, **BAS_wind_mean** the second and **BAS_wind_direction** the third.

Summary

- ▶ Regularization allows to lower the flexibility of a model by restricting the parameters to certain areas of the parameter space.
- ▶ L1 regularization leads to sparse models with some parameters exactly zero
⇒ great for interpretability.

Suggested Reading

- ▶ 6.2 Shrinkage Methods
- ▶ 6.4 Considerations in High Dimensions