Derivation of the Imputation-Generating (Posterior Predictive Distribution) of a Latent Variable

Factored Structural Equation Modeling in Blimp

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Analysis Model:

This document describes the derivation of a latent variable's imputation-generating (posterior predictive) distribution for a simple structural model with four normal indicators and a normal distal outcome. The following model is from Figure 1a in the paper.

$$\eta_i = \mu_\eta + \varepsilon_{\eta i} = \hat{\eta}_i + \varepsilon_{\eta i}$$

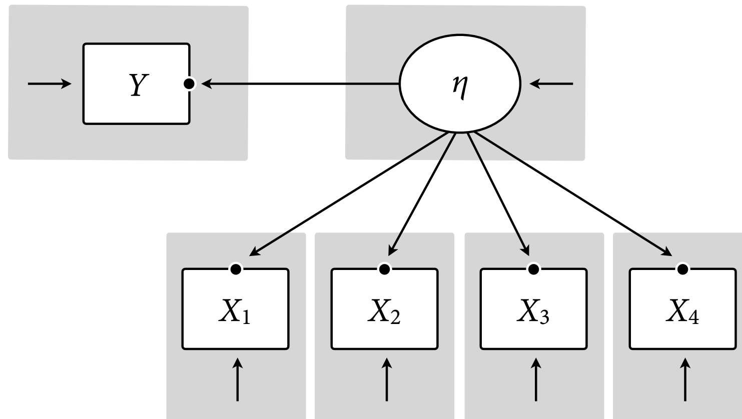
$$f(\eta) = N_1\left(\hat{\eta}_i, \sigma^2_{\varepsilon_\eta}\right)$$

$$X_{pi} = \upsilon_p + \lambda_p(\eta_i) + \varepsilon_{Xpi} = \hat{X}_{pi} + \varepsilon_{Xpi}$$

$$f\left(X_p|\eta\right) = N_1\left(\hat{X}_{pi}, \sigma^2_{\varepsilon_{Xp}}\right)$$

$$Y_i = \beta_0 + \beta_1(\eta_i) + \varepsilon_{Yi} = \hat{Y}_i + \varepsilon_{Yi}$$

$$f(Y|\eta) = N_1\left(\hat{Y}_i, \sigma^2_{\varepsilon_Y}\right)$$



Posterior Predictive Distribution of η_i Derivation:

1. Define all Terms not Involving η_i as Constants

$$f(\eta_i | \ldots) \propto \exp\left(-\frac{(Y_i - \beta_0 - \beta_1\eta_i)^2}{2\sigma^2_{\varepsilon_Y}}\right) \times \prod_{p=1}^{4} \exp\left(-\frac{(X_{pi} - \upsilon_p - \lambda_p\eta_i)^2}{2\sigma^2_{\varepsilon_{Xp}}}\right) \times \exp\left(-\frac{(\eta_i - \mu_\eta)^2}{2\sigma^2_{\varepsilon_\eta}}\right)$$

$$\exp\left(-\frac{(C_Y - \beta_1\eta_i)^2}{2\sigma^2_{\varepsilon_Y}}\right) \times \prod_{p=1}^{4} \exp\left(-\frac{(C_p - \lambda_p\eta_i)^2}{2\sigma^2_{\varepsilon_{Xp}}}\right) \times \exp\left(-\frac{(\eta_i - C_\eta)^2}{2\sigma^2_{\varepsilon_\eta}}\right)$$

$$C_Y = Y_i - \beta_0 \qquad C_p = X_{pi} - \upsilon_p \qquad C_\eta = \mu_\eta$$

2. Expand Parentheses, Drop Terms not Involving η_i, Combine exp Functions

$$\exp\left(-\frac{C_Y^2 - 2C_Y\eta_i\beta_1 + \eta_i^2\beta_1^2}{2\sigma_{\varepsilon_Y}^2}\right) \times \prod_{p=1}^{4} \exp\left(-\frac{C_p^2 - 2C_p\eta_i\lambda_p + \eta_i^2\lambda_p^2}{2\sigma_{\varepsilon_{Xp}}^2}\right) \times \exp\left(-\frac{\eta_i^2 - 2C_\eta\eta_i + C_\eta^2}{2\sigma_{\varepsilon_\eta}^2}\right)$$

$$\exp\left(-\frac{-2C_Y\eta_i\beta_1 + \eta_i^2\beta_1^2}{2\sigma_{\varepsilon_Y}^2}\right) \times \prod_{p=1}^{4} \exp\left(-\frac{-2C_p\eta_i\lambda_p + \eta_i^2\lambda_p^2}{2\sigma_{\varepsilon_{Xp}}^2}\right) \times \exp\left(-\frac{\eta_i^2 - 2C_\eta\eta_i}{2\sigma_{\varepsilon_\eta}^2}\right)$$

$$\exp\left(-\frac{-2C_Y\eta_i\beta_1 + \eta_i^2\beta_1^2}{2\sigma_{\varepsilon_Y}^2} - \sum_{p-1}^{4} \frac{-2C_p\eta_i\lambda_p + \eta_i^2\lambda_p^2}{2\sigma_{\varepsilon_{Xp}}^2} - \frac{\eta_i^2 - 2C_\eta\eta_i}{2\sigma_{\varepsilon_\eta}^2}\right)$$

3. **Common Denominator**

$$\exp\left(-\frac{-2C_Y\eta_i\beta_1 + \eta_i^2\beta_1^2}{2\sigma_{\varepsilon_Y}^2} - \frac{-2C_1\eta_i\lambda_1 + \eta_i^2\lambda_1^2}{2\sigma_{\varepsilon_{X1}}^2} - \frac{-2C_2\eta_i\lambda_2 + \eta_i^2\lambda_2^2}{2\sigma_{\varepsilon_{X2}}^2} - \frac{-2C_3\eta_i\lambda_3 + \eta_i^2\lambda_3^2}{2\sigma_{\varepsilon_{X3}}^2} - \frac{-2C_4\eta_i\lambda_4 + \eta_i^2\lambda_4^2}{2\sigma_{\varepsilon_{X4}}^2} - \frac{\eta_i^2 - 2C_\eta\eta_i}{2\sigma_{\varepsilon_\eta}^2}\right)$$

$$V_Y = \sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2$$

$$V_1 = \sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2$$

$$V_2 = \sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2$$

$$V_3 = \sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2$$

$$V_4 = \sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_\eta}^2$$

$$V_\eta = \sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2$$

$$\exp\left(-\frac{-2C_Y\eta_i\beta_1V_Y+\eta_i^2\beta_1^2V_Y}{2\sigma_{\varepsilon_Y}^2V_Y}-\frac{-2C_1\eta_i\lambda_1V_1+\eta_i^2\lambda_1^2V_1}{2\sigma_{\varepsilon_{X1}}^2V_1}-\frac{-2C_2\eta_i\lambda_2V_2+\eta_i^2\lambda_2^2V_2}{2\sigma_{\varepsilon_{X2}}^2V_2}-\frac{-2C_3\eta_i\lambda_3V_3+\eta_i^2\lambda_3^2V_3}{2\sigma_{\varepsilon_{X3}}^2V_3}-\frac{-2C_4\eta_i\lambda_4V_4+\eta_i^2\lambda_4^2V_4}{2\sigma_{\varepsilon_{X4}}^2V_4}-\frac{\eta_i^2V_\eta-2C_\eta\eta_iV_\eta}{2\sigma_{\varepsilon_\eta}^2V_\eta}\right)$$

$$\exp\left(-\frac{-2C_Y\eta_i\beta_1V_Y+\eta_i^2\beta_1^2V_Y}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}-\frac{-2C_1\eta_i\lambda_1V_1+\eta_i^2\lambda_1^2V_1}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}-\frac{-2C_2\eta_i\lambda_2V_2+\eta_i^2\lambda_2^2V_2}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}-\frac{-2C_3\eta_i\lambda_3V_3+\eta_i^2\lambda_3^2V_3}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}-\frac{-2C_4\eta_i\lambda_4V_4+\eta_i^2\lambda_4^2V_4}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}-\frac{\eta_i^2V_\eta-2C_\eta\eta_iV_\eta}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}\right)$$

$$\exp\left(\frac{2C_Y\eta_i\beta_1V_Y-\eta_i^2\beta_1^2V_Y+2C_1\eta_i\lambda_1V_1-\eta_i^2\lambda_1^2V_1+2C_2\eta_i\lambda_2V_2-\eta_i^2\lambda_2^2V_2+2C_3\eta_i\lambda_3V_3-\eta_i^2\lambda_3^2V_3+2C_4\eta_i\lambda_4V_4-\eta_i^2\lambda_4^2V_4-\eta_i^2V_\eta+2C_\eta\eta_iV_\eta}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}\right)$$

$$\exp\left(-\frac{-2C_Y\eta_i\beta_1V_Y+\eta_i^2\beta_1^2V_Y-2C_1\eta_i\lambda_1V_1+\eta_i^2\lambda_1^2V_1-2C_2\eta_i\lambda_2V_2+\eta_i^2\lambda_2^2V_2-2C_3\eta_i\lambda_3V_3+\eta_i^2\lambda_3^2V_3-2C_4\eta_i\lambda_4V_4+\eta_i^2\lambda_4^2V_4+\eta_i^2V_\eta-2C_\eta\eta_iV_\eta}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}\right)$$

4. Combine and Group Terms Involving η_i^2 and $2\eta_i$

$$\exp\left(-\frac{\eta_i^2\beta_1^2V_Y+\eta_i^2\lambda_1^2V_1+\eta_i^2\lambda_2^2V_2+\eta_i^2\lambda_3^2V_3+\eta_i^2\lambda_4^2V_4+\eta_i^2V_\eta-2\eta_iC_Y\beta_1V_Y-2\eta_iC_1\lambda_1V_1-2\eta_iC_2\lambda_2V_2-2\eta_iC_3\lambda_3V_3-2\eta_iC_4\lambda_4V_4-2\eta_iC_\eta V_\eta}{2\sigma_{\varepsilon_Y}^2\sigma_{\varepsilon_{X1}}^2\sigma_{\varepsilon_{X2}}^2\sigma_{\varepsilon_{X3}}^2\sigma_{\varepsilon_{X4}}^2\sigma_{\varepsilon_\eta}^2}\right)$$

5. Isolate η_i^2 and Arrange Kernel to Match a Quadratic Form

$$= \exp\left(-\frac{\eta_i^2 - 2\eta_i A}{2B}\right) = \exp\left(-\frac{\eta_i^2 - 2\eta_i E(\eta_i|\ldots)}{2VAR(\eta_i|\ldots)}\right)$$

$$\exp\left(-\frac{\eta_i^2\left(\beta_1^2 V_Y + \lambda_1^2 V_1 + \lambda_2^2 V_2 + \lambda_3^2 V_3 + \lambda_4^2 V_4 + V_\eta\right) - 2\eta_i\left(C_Y\beta_1 V_Y + C_1\lambda_1 V_1 + C_2\lambda_2 V_2 + C_3\lambda_3 V_3 + C_4\lambda_4 V_4 + C_\eta V_\eta\right)}{2\sigma_{\varepsilon Y}^2\sigma_{\varepsilon X1}^2\sigma_{\varepsilon X2}^2\sigma_{\varepsilon X3}^2\sigma_{\varepsilon X4}^2\sigma_{\varepsilon\eta}^2}\right)$$

$$= \exp\left(-\frac{\eta_i^2 - 2\eta_i\dfrac{\left(C_Y\beta_1 V_Y + C_1\lambda_1 V_1 + C_2\lambda_2 V_2 + C_3\lambda_3 V_3 + C_4\lambda_4 V_4 + C_\eta V_\eta\right)}{\left(\beta_1^2 V_Y + \lambda_1^2 V_1 + \lambda_2^2 V_2 + \lambda_3^2 V_3 + \lambda_4^2 V_4 + V_\eta\right)}}{2\dfrac{\sigma_{\varepsilon Y}^2\sigma_{\varepsilon X1}^2\sigma_{\varepsilon X2}^2\sigma_{\varepsilon X3}^2\sigma_{\varepsilon X4}^2\sigma_{\varepsilon\eta}^2}{\left(\beta_1^2 V_Y + \lambda_1^2 V_1 + \lambda_2^2 V_2 + \lambda_3^2 V_3 + \lambda_4^2 V_4 + V_\eta\right)}}\right)$$

6. Posterior Predictive Distribution

$$\begin{aligned} f(\eta_i|\ldots) &= N\left(E(\eta_i|\ldots), VAR(\eta_i|\ldots)\right) \\ &= N\left(\frac{\left(C_Y\beta_1 V_Y + C_1\lambda_1 V_1 + C_2\lambda_2 V_2 + C_3\lambda_3 V_3 + C_4\lambda_4 V_4 + C_\eta V_\eta\right)}{\left(\beta_1^2 V_Y + \lambda_1^2 V_1 + \lambda_2^2 V_2 + \lambda_3^2 V_3 + \lambda_4^2 V_4 + V_\eta\right)}, \frac{\sigma_{\varepsilon Y}^2\sigma_{\varepsilon X1}^2\sigma_{\varepsilon X2}^2\sigma_{\varepsilon X3}^2\sigma_{\varepsilon X4}^2\sigma_{\varepsilon\eta}^2}{\left(\beta_1^2 V_Y + \lambda_1^2 V_1 + \lambda_2^2 V_2 + \lambda_3^2 V_3 + \lambda_4^2 V_4 + V_\eta\right)}\right) \end{aligned}$$

$$C_Y = Y_i - \beta_0$$

$$C_1 = X_{1i} - \upsilon_1$$

$$C_2 = X_{2i} - \upsilon_2$$

$$C_3 = X_{3i} - \upsilon_3$$

$$C_4 = X_{4i} - \upsilon_4$$

$$C_\eta = \mu_\eta$$

$$V_Y = \sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon X4}\sigma^2_{\varepsilon \eta}$$

$$V_1 = \sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon X4}\sigma^2_{\varepsilon \eta}$$

$$V_2 = \sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon X4}\sigma^2_{\varepsilon \eta}$$

$$V_3 = \sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X4}\sigma^2_{\varepsilon \eta}$$

$$V_4 = \sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon \eta}$$

$$V_\eta = \sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon X4}$$

$$E(\eta_i|\dots) = \frac{\left(C_Y\beta_1V_Y + C_1\lambda_1V_1 + C_2\lambda_2V_2 + C_3\lambda_3V_3 + C_4\lambda_4V_4 + C_\eta V_\eta\right)}{\left(\beta_1^2V_Y + \lambda_1^2V_1 + \lambda_2^2V_2 + \lambda_3^2V_3 + \lambda_4^2V_4 + V_\eta\right)} = \frac{\left(C_Y\beta_1V_Y + C_1\lambda_1V_1 + C_2\lambda_2V_2 + C_3\lambda_3V_3 + C_4\lambda_4V_4 + C_\eta V_\eta\right)}{D}$$

$$E(\eta_i|\dots) = \frac{\left((Y_i - \beta_0)\beta_1V_Y + (X_{1i} - \upsilon_1)\lambda_1V_1 + (X_{2i} - \upsilon_2)\lambda_2V_2 + (X_{3i} - \upsilon_3)\lambda_3V_3 + (X_{4i} - \upsilon_4)\lambda_4V_4 + \mu_\eta V_\eta\right)}{\left(\beta_1^2V_Y + \lambda_1^2V_1 + \lambda_2^2V_2 + \lambda_3^2V_3 + \lambda_4^2V_4 + V_\eta\right)}$$

$$\begin{aligned}E(\eta_i|\dots) = D^{-1}\Big(&(Y_i - \beta_0)\beta_1\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon X4}\sigma^2_{\varepsilon \eta} + (X_{1i} - \upsilon_1)\lambda_1\sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon X4}\sigma^2_{\varepsilon \eta} + (X_{2i} - \upsilon_2)\lambda_2\sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon X4}\sigma^2_{\varepsilon \eta}\\ &+ (X_{3i} - \upsilon_3)\lambda_3\sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X4}\sigma^2_{\varepsilon \eta} + (X_{4i} - \upsilon_4)\lambda_4\sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon \eta} + \mu_\eta\sigma^2_{\varepsilon Y}\sigma^2_{\varepsilon X1}\sigma^2_{\varepsilon X2}\sigma^2_{\varepsilon X3}\sigma^2_{\varepsilon X4}\Big)\end{aligned}$$

$$VAR(\eta_i|\ldots) = \frac{\sigma^2_{\varepsilon_Y}\sigma^2_{\varepsilon_{X1}}\sigma^2_{\varepsilon_{X2}}\sigma^2_{\varepsilon_{X3}}\sigma^2_{\varepsilon_{X4}}\sigma^2_{\varepsilon_\eta}}{\left(\beta_1^2 V_Y + \lambda_1^2 V_1 + \lambda_2^2 V_2 + \lambda_3^2 V_3 + \lambda_4^2 V_4 + V_\eta\right)} = \frac{\sigma^2_{\varepsilon_Y}\sigma^2_{\varepsilon_{X1}}\sigma^2_{\varepsilon_{X2}}\sigma^2_{\varepsilon_{X3}}\sigma^2_{\varepsilon_{X4}}\sigma^2_{\varepsilon_\eta}}{D}$$

$$\begin{aligned} D = {} & \beta_1^2\sigma^2_{\varepsilon_{X1}}\sigma^2_{\varepsilon_{X2}}\sigma^2_{\varepsilon_{X3}}\sigma^2_{\varepsilon_{X4}}\sigma^2_{\varepsilon_\eta} + \lambda_1^2\sigma^2_{\varepsilon_Y}\sigma^2_{\varepsilon_{X2}}\sigma^2_{\varepsilon_{X3}}\sigma^2_{\varepsilon_{X4}}\sigma^2_{\varepsilon_\eta} + \lambda_2^2\sigma^2_{\varepsilon_Y}\sigma^2_{\varepsilon_{X1}}\sigma^2_{\varepsilon_{X3}}\sigma^2_{\varepsilon_{X4}}\sigma^2_{\varepsilon_\eta} + \lambda_3^2\sigma^2_{\varepsilon_Y}\sigma^2_{\varepsilon_{X1}}\sigma^2_{\varepsilon_{X2}}\sigma^2_{\varepsilon_{X4}}\sigma^2_{\varepsilon_\eta} + \lambda_4^2\sigma^2_{\varepsilon_Y}\sigma^2_{\varepsilon_{X1}}\sigma^2_{\varepsilon_{X2}}\sigma^2_{\varepsilon_{X3}}\sigma^2_{\varepsilon_\eta} \\ & + \sigma^2_{\varepsilon_Y}\sigma^2_{\varepsilon_{X1}}\sigma^2_{\varepsilon_{X2}}\sigma^2_{\varepsilon_{X3}}\sigma^2_{\varepsilon_{X4}} \end{aligned}$$