Full reptend prime and cyclic numbers.

1. Some definitions

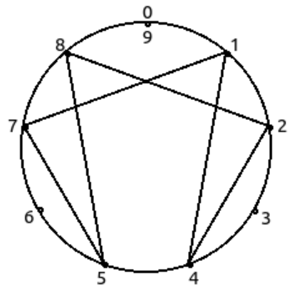
a. Digital Spectrum of number in decimal scale of notation is vector of 10 integers, each represents amount of each digit from 0 to 9 met in number.

b. Regularity of number in decimal scale of notation is vector of N-1 integers, where N is amount of digits in number, so N is integer part of lg(number), and each integer in vector represents difference between 2 digits in number.

c. Number reduction in decimal scale of notation is a single digit, that made by summation of all digits from number, and if sum is more then one digits long - then continue the summation.

Many digit sequences in this article visualized using numbered circle.

For example repeating decimal 1/7=0.(142857) could be represented as:



2. Theme definitions

P is prime number, n are integers from 1 to P-1.

If all the rational numbers n/P give repeating decimal that have equal digital spectrum and/or their regularities is the only one, but starting from different n positions – then number P is full reptend prime.

For example for P=7:

1 / 7 = 0.(1 4 2 8 5 7)
2 / 7 = 0.(2 8 5 7 1 4)
3 / 7 = 0.(4 2 8 5 7 1)
4 / 7 = 0.(5 7 1 4 2 8)
5 / 7 = 0.(7 1 4 2 8 5)
6 / 7 = 0.(8 5 7 1 4 2)

Digital spectrum of repeat decimal period: [0, 1, 1, 0, 1, 1, 0 ,1 ,1, 0]
Regularity: [+3, -2,+6, -3, +2, -6]

Cyclic number is integer number that contains all digits of single period from repeating decimal, and could be written as $10^{period}/P$, where period comes from decimal fractions of full reptend prime.

For example for P=7 cyclic number is 142857.

3. Properties of full reptend numbers

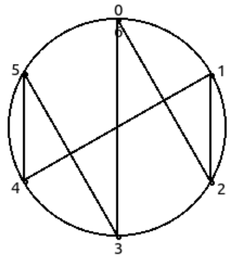
3.1 Multiplication offsets

When we multiply 1/P on n we shift repeating decimals.

For example 1/7 = 0.(1 4 2 8 5 7) * 2 = 0.(2 8 5 7 1 4) : multiply on 2 shifts 2 decimal digits.

And 1/7 * 3 = 0.(4 2 8 5 7 1) : multiply on 3 shifts 1 decimal digit.

So if we collect all that amount of shifts into vector, for P=7 it would be: [0, 2, 1, 4, 5, 3].

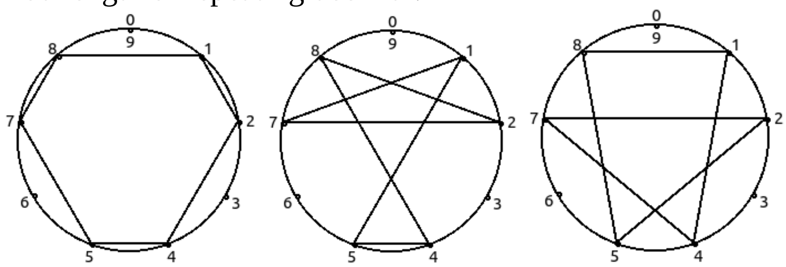
This sequence could be visualized: 

3.2 Tables

When we write all the n/P numbers – each one as a new column - we can look at them together as rows.

For P=7 there would be 6 vectors: [1,2,4,5,7,8], [4, 8, 2, 7, 1, 5], [2, 5, 8, 1, 4, 7] and 3 exactly inverted [8,7,5,4,2,1], [5,1,7,2,8,4], [7,4,1,8,5,2].

Amount of vectors comes from the period length of repeating decimal.

Those sequences could be visualized: 

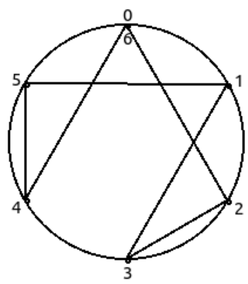
3.3 Remains of division

While we calculate repeating decimal we also get repeating remains of division.

For example for P=7, lets calculate 1/7 by hand:

10 / 7 = 1	10 % 7 = 3
30 / 7 = 4	30 % 7 = 2
20 / 7 = 2	20 % 7 = 6
60 / 7 = 8	60 % 7 = 4
40 / 7 = 5	40 % 7 = 5
50 / 7 = 7	50 % 7 = 1

So here we got period : [3, 2, 6, 4, 5, 1]. It could be visualized:



3.4 Geometric progressions

Rational number 1/P could be represented as summation of geometrical progression.

For example for P=7 there are infinite amount of geometric progressions, first of them is:

$$1/7 = \Sigma(3^n\, 10^{n+1}) = 1/10 + 3/100 + 9/1000 + 27/10000 + \ldots$$

The first element of each of progressions is rational number, that contains N digits, from repeating decimal 0.(142857).

Each progression maybe set by first element, multiplier and divisor.

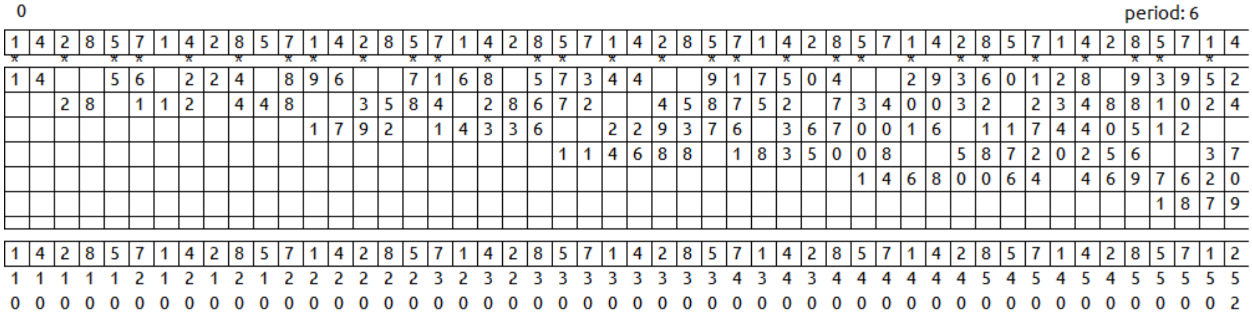
Examples of first element = 1, 14, 142, 1428, 14285, 142857, 1428571, 14285714 and so on.

Multiplier comes from remains of division, and depends on amount of digits in first element. Index = (amount of digits – 1) % 6.

It means for first element 1 it’s 3, for 14 it’s 2, for 142 it’s 6, for 1428 it’s 4, for 14285 it’s 5. Next one is 1, and then it cycles, repeating again: 3 again for 1428571.

Divisor comes from amount of digits of first element, and equal to $10^{\text{amount of digits}}$.

There is visualization of geometric progression summation:

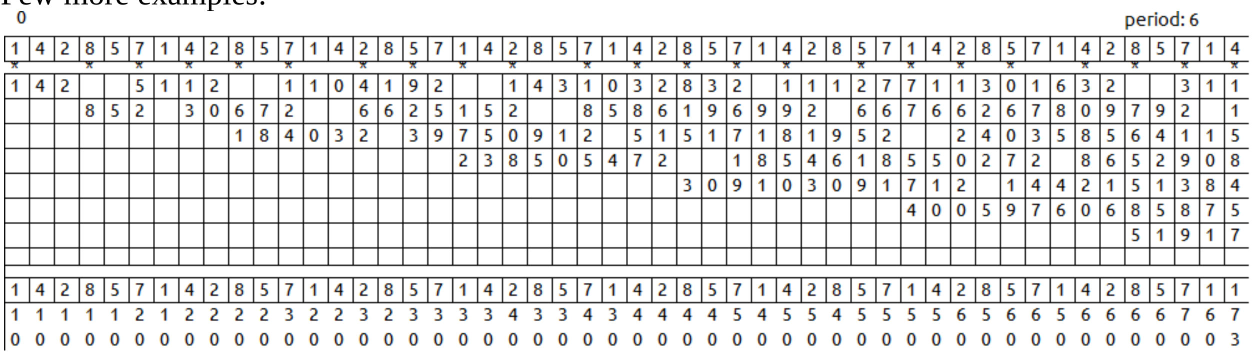


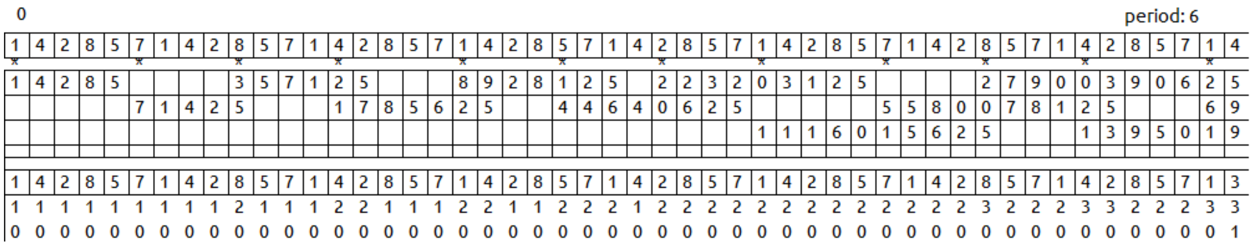
First string is the 1/7 itself, asterisk below it shows the points when new elements of progression join the summation;

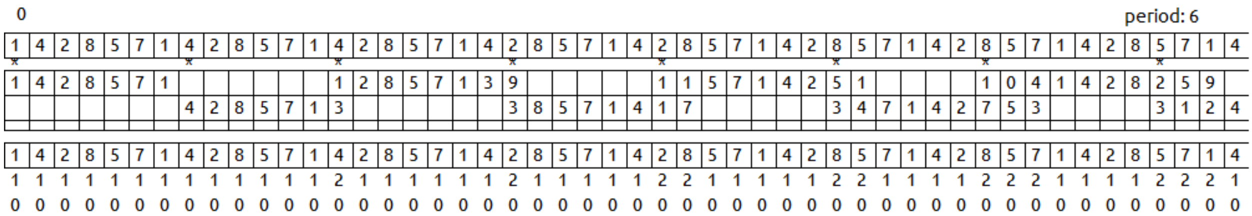
In the middle we can see members of progression, below them there is result of summation;

String below the result is amount of progression members that took part in forming each digit, and last string is difference between 1/7 and the summation of visible progression members.

Few more examples:







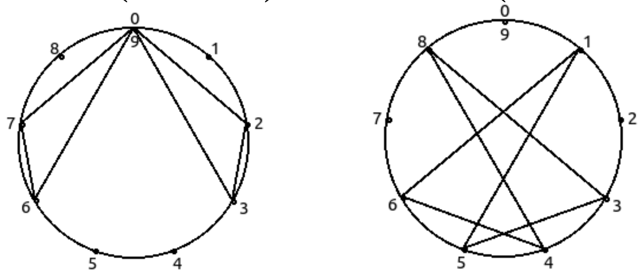
Also there is something special about geometrical progressions, some of them has musical interpretation, there would be written about it down that paper.

3.5 Each full cyclic number has its own pair of 2-cyclic number

For example P=7, the pair is 13.

It is not full reptend prime. But it is reptend prime, it have 2 different cycles:

1/13 = 0. (0 7 6 9 2 3) 2/13 = 0. (1 5 3 8 4 6)



There are 3 connections between 7 and 13:

a) Repunit 111111 has 13 and 7 in its integer factorization, the length of repunit is 6,

and period of n/7 and m/13 is also 6, where m = integers from 1 till 13-1.

b) Reverse repeating decimals from n/7 and m/13

If original 1/7 = 0.(142857) then reverse is 0.(758241).

All the reverses from n/7 and m/13 are rational numbers,

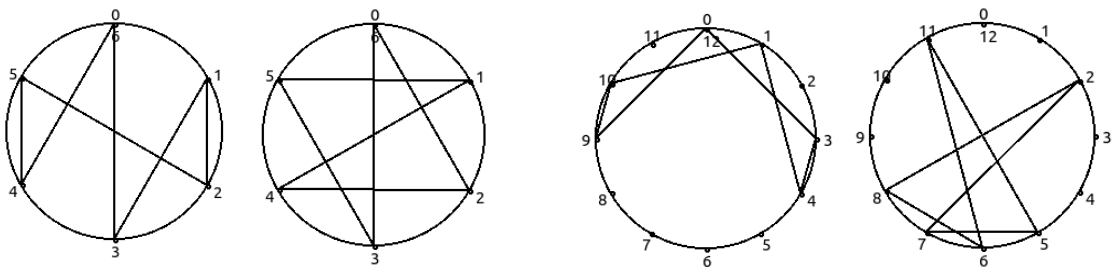
They can be found between 1/91 and 90/91, where 91 is 7*13.

For example 0.(758241) = 69/91.

c) Musical interpretation also works for one of geometrical progressions.

Because of 2 cycles some of properties of number 13 appear twice.

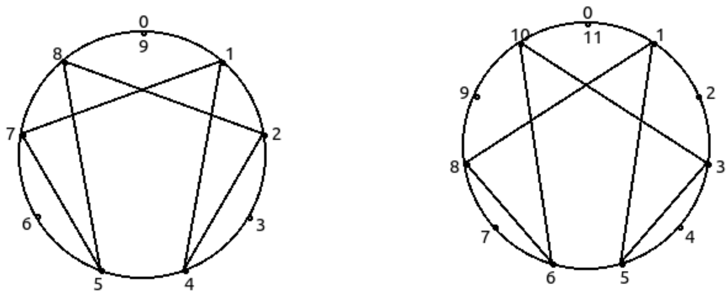
Regularities are: [+7, -1, +3, -7, +1, -3] and [+4, -2, +5, -4, +2, -5].
Multiplication offsets are: [0, 4, 5, 2, 1, 3] and [0, 2, 4, 1, 5, 3].
Remains of division are: [10, 9, 12, 3, 4, 1] and [7, 5, 11, 6, 8, 2].



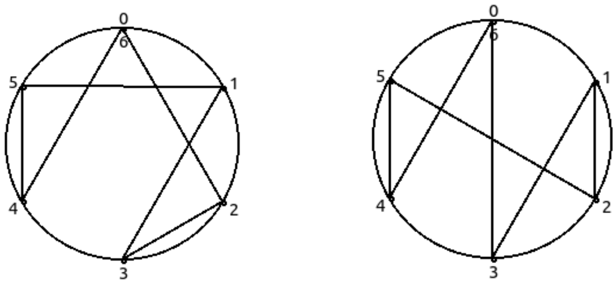
3.6 Decimal full reptend prime has a pair in other scale of notation

For example for 1/7 = 0.(142857) in decimal, in 12 scale of notation would be = 0.(186A35).

If we visualize that pattern on circle we can see it looks similar, but got opposite direction.



Remains of divisions [5, 4, 6, 2, 3, 1] and multiply offsets [0, 4, 5, 2, 1, 3] in 12 scale of notation. We can see multiplication offset is same as with 1/13, and remains of division are reversed from 1/7 in 10 scale of notation.



3.7 Full reptend prime and its 2-cyclic pair on multiplication

If we take 1/7 and multiply it on 2 in cycle, then in fraction we will only have [1/7, 2/7, 4/7]. And if 1/7 would be multiplied on 4 in cycle, in fraction we will have only [1/7, 4/7, 2/7]. If 1/7 multiplied on 6 in cycle in fraction we got [1/7, 6/7].

But if we multiply on 3, we would have all the possible fractions, in special sequence:

[1/7, 3/7, 2/7, 6/7, 4/7, 5/7]

We can see that numerators come from division remains.

Reversed sequence appear if we multiply on 5: [1/7, 5/7, 4/7, 6/7, 2/7, 3/7].

There is also an exploration why 3 and 5 work that way, that can be obtained with help of number reduction, but it’s rather long, so you can find it down this paper.

In case of 2-cyclic pair multiply by 2 goes though all possible the rationals, each next got different cycle, written only numerators: [1, 2, 4, 8, 3, 6, 12, 11, 9, 5, 10, 7].

Multiply by 7 gives the reversed sequence: [1, 7, 10, 5, 9, 11, 12 ,6 ,3, 8, 4, 2].

Multiply by 3 gives: [1, 3, 9]. And multiply by 9 gives reversed one: [1,9,3].

Multiply by 4 gives: [1, 4, 3, 12, 9, 10]. And reversed is multiply by 10: [1, 10, 9, 12, 3, 4]. It contains only one cycle, so if 2/13 would be multiplied on 4, it would give another sequence, containing only another cycle: [2, 8, 6, 11, 5, 7], same with multiply by 10.

Multiply by 6 gives: [1, 6, 10, 8, 9, 2, 12, 7, 3, 5, 4, 11]. Reversed is from multiply by 11. And finally multiply by 12 gives: [1,12].

4. Other special cases

Sometimes full reptend primes also connected to another theme, for example 1/89 can be represented as summation of geometric progression:

1/89 = 1/100 + 11/10000 + 121/1000000 + …

But also the same repeated decimal could be represented as sum of Fibonacci numbers, starting from 0.

1/89 = 0/10 + 1/100 + 1/1000 + 2/10000 + 3/100000 + …

4.1 Musical interpretations.

For P=7, the geometrical progression $14*2^n / 10^{2(n+1)}$ has musical interpretation.

If we would number all the digits in 1/7 from 0 till infinity like an axis, and then take a look at which points members of geometrical progression join the summation. As you may remember there where asterisks for those positions on summation visualization.

First 7 members give the sequence: [0, 2, 4, 5, 7, 9, 11] – if you know guitar and could play all the frets from the sequence you would hear major scale from open string.
We can also write it as difference between members of that sequence +1 member: [2,2,1,2,2,2,1].
If you don't know it: it's formula of major scale in semitones.

So it happens that on 12 possible positions we have 7 actual.
This proportion corresponds to diatonic scale in music.

If we continue to group 7 geometric progression members into scale we would have all the 7 modern modes. There is link about it. And there also would be one special scale, so 8 total.

Formula: $$fScales(n) = 2n - ([\lg(7 * 2^{n+1})] - 2)$$

My favorite thing about the scales sequence is what they don't come just in order 1-8.

If each time when we would find a major scale again – we separate all the scales from previous major scale into different sequence, and it happens to be 8 sequences.

Lets look now both at scales, and sequences:

1. Ionian	[2, 2, 1, 2, 2, 2, 1]
2. Mixolydian	[2, 2, 1, 2, 2, 1, 2]
3. Dorian	[2, 1, 2, 2, 2, 1, 2]
4. Aeolian	[2, 1, 2, 2, 1, 2, 2]
5. Phrygian	[1, 2, 2, 2, 1, 2, 2]
6. Special symmetric	[1, 2, 2, 1, 2, 2, 1]
7. Lydian	[2, 2, 2, 1, 2, 2, 1]
8. Locrian	[1, 2, 2, 1, 2, 2, 2]

1. [1, 2, 2, 3, 4, 4, 5, 6, 7]
2. [1, 2, 2, 3, 4, 4, 5, 8, 6, 7]
3. [1, 2, 2, 4, 4, 5, 8, 6, 7]
4. [1, 2, 3, 4, 4, 5, 8, 6, 7]
5. [1, 2, 3, 4, 4, 8, 6, 7]
6. [1, 2, 2, 3, 4, 4, 5, 8, 6, 7, 2, 2, 3, 4, 4, 5, 8, 6, 7]
7. [1, 2, 2, 3, 4, 5, 8, 6, 7]
8. [1, 2, 2, 3, 4, 4, 5, 8, 6]

There are a lot of regularities seen. But if we continue to group elements together, then 8 sequences come in 8 groups, each starting from 41 sequence. First 41 sequences is the distance are significant because it's the moment when all the 8 sequences met, it means last one comes at exactly 41th place.

If we would take this 41 sequences as starting point of groups, and each group will start with it, then… We get again 8 groups…

I haven't tried my best, but I wasn't able to find the cycle. I can't completely predict next element, because there is no obvious cycle in that way of exploring that geometric progression.

Now I'm rewriting my code from C++ to Python, and suddenly it was easy to make it work much faster, so I'm going explore deeper again.

I will publish new paper made with new version and include the link to code on GitHub.

4.2 Number reduction for multiplication table.

Above was noticed that there is a reason why 1/7 multiplied in cycle on 3 or 5 walks through all the fractional parts.

And also there where found special multipliers for 1/13, that also go through all possible rational numbers m/13, and changing cycle on

To show it there must be done just number reduction analyze in expected scale of notation.

When we investigate about 7 we need to make exp table in octal scale of notation and take number reduction from it.

For example: $1*2^n$ =1, 2, 4, 10, 20, 40, 100… numbers in octal scale of notation, cycle is seen clear.

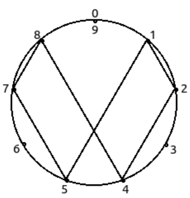
$1*2^n$ = [1, 2, 4] $1*3^n$ = [1, 3, 2, 6, 4, 5] $1*4^n$ = [1, 4, 2] $1*5^n$ = [1, 5, 4, 6, 2, 3]
$1*6^n$ = [1, 6]

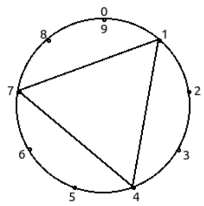
Same way we can find values for 13 in 14th scale of notation.

$1*2^n$ = 1, 2, 4, 8, 12, 24, 48, 92, 144, 288, 532, A64 = [1,2,4,8,3,6,12,11,9,5,10,7]

And if we would build the same way table for decimal scale of notation, it shows another thing related to P=7.

$1*2^n$ = [1, 2, 4, 8, 7, 5] $1*3^n$ = 1, 3, [9] $1*6^n$ = 1, 6, [9] - in these 2 cases cycles only '9'
$1*5^n$ = [1, 5, 7 ,8, 4, 2] $1*4^n$ = [1, 4, 7] $1*7^n$ = [1, 7, 4]
$1*8^n$ = [1, 8] $1*9^n$ = 1, [9]





So we can see pair: 2 and 5 have the longest cycle, and the sequences they produce are reversed to each other.

Because of this geometrical progression 1/7 = 14/100 + 28/10000 + … can also be written different.

In addition to normal geometric progression that counts from first element, to last, we just like reverse the process, and count it from last element to first. So it has to become progression made not from 14*2 but from 14/2. And here comes the number reduction.
It happens that according to number reduction is insignificant would I calculate 14/2 or 14*5, first result is 7, second one is 70, number reductions equal.

So I can make another progression, from multiplying on 5, there would be totally different members of geometric progression.

But whats interesting – now we still can make the "scale" operation. And if we number the digits in 1/7 starting from last '7' equal to 0, and going to negative numbers, just like we continue or 'number axis' that already has its own 0 in first digit '1'.

So if we would write scales in negative numbers, it would becomes something like [0, -2, -4, -5, -7, -9, -11] so we got just the same sequence, and it would also continue the same.
This pictures should be observed from right to left, as ‘digits axis’ is negative:

0 period: 6

5	7	1	4	2	8	5	7	1	4	2	8	5	7	1	4	2	8	5	7	1	4	2	8	5	7	1	4	2	8	5	7	1	4	2	8	5	7	1	4	2	8	5	7	1	4	2	8	5	7
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	0	0	0	0	0	0	0	1	1	2	3	3	4	4	5	6	6	7	7	8	9	9	10	10	11	10	10	10	9	9	8	8	8	7	7	6	6	6	5	5	4	4	3	3	3	2	2	1	1
								6	1	6	0	1	0	6	6	9	1	6	5	7	6	0	2	5	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Interesting to say, that the number pattern of repeating decimal 1/7 = 0. (1 4 2 8 5 7) has a thing for this behavior in its structure:

If we divide or multiply 14 on 2 we stay inside number pattern:

14/2 = 7 is a digit from number pattern
14*2= 28 is also from number pattern

5. Conclusion

There could be said much about full reptend primes, but this paper covers directions in the short form.
I will continue my research, and will publish next paper with code on GitHub, that would help others to repeat all the computing or play with methods I used for my research.

Most curious question, is how reptend primes may be related to life.
There are reasons why I cannot think that they aren’t related:

1) There is a huge amount of regularities, describing them in details will make a little book, and all of them can be done when you just find ratio of one day to the whole week.
2) George Gurdjieff (1866-1949) was a philosopher and mystic. He had a wide range of knowledge, for example he predicted in his writings that most of star systems are paired.

It’s kind of funny, but my love to full reptend primes began from his spiritual teaching. I have learned as much as possible about his ideas, and he talks about 1/7 and gives it very specific interpretation, which I can’t share with him.

But what attracted me much more: there done a lot of dances, that represent regularities that appear in 1/7 as in full reptend prime, you can find some of them on YouTube, and probably will even understand part of “the message” if you’ll take a look at dances named ‘multiplication’.

<u>If you know anything about reptend primes or cyclic numbers, that wasn’t noticed in that paper: please let me know!</u>

By the way, my name is Konstantin Kutsenko, you can contact me using discord: constcut#1679
Thank you for your attention