Lecture #2 Entanglement, Teleportation & Noise

I. entanglement

In order to "understand" entanglement we need to review how we represent composite systems.

The state space of a composite system is the tensor product of state spaces of the component physical systems.

In the specific case of n systems with each system i prepared in $|\psi_i\rangle$, the joint state of the total system is

$$|\psi_1\rangle \otimes |\psi_2\rangle \otimes \cdots \otimes |\psi_n\rangle$$

If you can write a state like this, i.e. as a tensor product, then the system is seperable.

If you cannot, then the system is entangled.

→ there are different amounts of entanglement & different measure of entanglement. This is an active area of research. It is still not quantitatively clear how entanglement should be quantified as a resource.

The seperable state is called a product state.

For concreteness, let's take a two qubit state.

$$|a\rangle = a_0|0\rangle + a_1|1\rangle \qquad |b\rangle = b_0|0\rangle + b_1|1\rangle \qquad |c\rangle = c_0|0\rangle + c_1|1\rangle$$

$$= \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \qquad = \begin{pmatrix} b_0 \\ b_1 \end{pmatrix} \qquad = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix}$$

$$|cba\rangle = |c\rangle \otimes |b\rangle \otimes |a\rangle = \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \otimes \begin{pmatrix} b_0 \times \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \\ b_1 \times \begin{pmatrix} a_0 \\ a_1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} \times \begin{pmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{pmatrix}$$

$$= \begin{pmatrix} c_0 \times \begin{pmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{pmatrix} \\ c_1 \times \begin{pmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} c_0 b_0 a_0 \\ c_0 b_0 a_1 \\ c_0 b_1 a_0 \\ c_0 b_1 a_1 \\ c_1 b_0 a_0 \\ c_1 b_0 a_1 \\ c_1 b_1 a_0 \\ c_1 b_1 a_1 \end{pmatrix}$$

What is the vector representation of $\frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$?

Since we are dealing with <u>real</u> quantum systems, we need to be comfortable with the density matrix representation of a composite system

$$\rho \equiv \sum_i p_i \underset{\uparrow}{|\psi_i\rangle}\langle\psi_i|$$

now this are the 2^n length vectors.

for a two qubit system

$$|ba\rangle\langle ba| = \begin{pmatrix} b_0 a_0 \\ b_0 a_1 \\ b_1 a_0 \\ b_1 a_1 \end{pmatrix} \left((b_0 a_0)^* \ (b_0 a_1)^* \ (b_1 a_0)^* \ (b_1 a_1)^* \right)$$

$$\begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array}$$

$$= \begin{pmatrix} |b_0|^2|a_0|^2 & |b_0|^2 a_0 a_1^* & b_0 b_1^* |a_0|^2 & b_0 b_1^* a_0 a_1^* \\ |b_0|^2 a_0^* a_1 & |b_0|^2 |a_1|^2 & b_0 b_1^* a_0^* a_1 & b_0 b_1^* |a_1|^2 \\ & & \text{etc,} & \\ & & & \end{pmatrix}$$

two systems in ρ are seperable if $\rho = \sum_k p_k \rho_1^k \otimes \rho_2^k$

As part of your homework you will create & measure the fidelity of an entangled state.

If the target state is a pure state $F(|\psi\rangle, \rho) = \sqrt{\langle\psi|\rho|\psi\rangle}$

3. Teleportation

→ extremely valuable concept

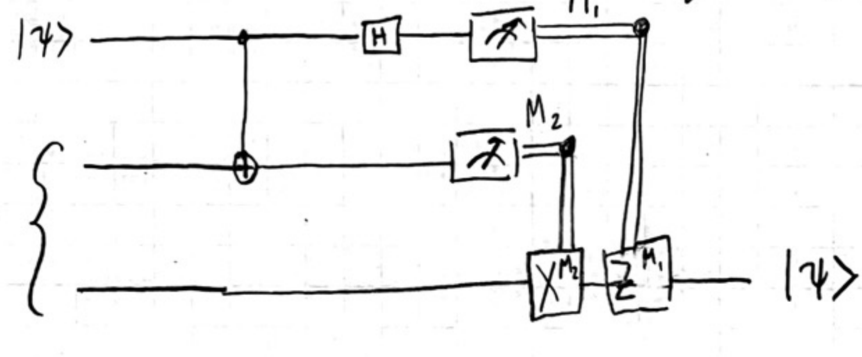
→ gate teleportation is used in Linear Optics QC

→ universal set of "gates" are single qubit, Bell meas. & GHZ state

→ a way to connect quantum registers in a heralded manner

The standard teleportation circuit is given by

$\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$



① initial state : $|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$

note: initial state is $|000\rangle$ for IonQ circuit so need to prepare initial state!

$\frac{1}{\sqrt{2}}(\alpha|0\rangle+\beta|1\rangle)\otimes(|00\rangle+|11\rangle$

$\frac{1}{\sqrt{2}}\Big(\alpha|0\rangle(|00\rangle+|11\rangle+\ \beta|1\rangle(|00\rangle+|11\rangle\Big)$

CNOT : $\frac{1}{\sqrt{2}}\Big(\alpha|0\rangle(|00\rangle+|11\rangle)+\beta|1\rangle(|10\rangle+|01\rangle\Big)$

$=\frac{1}{\sqrt{2}}\Big(\alpha\big(\frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)\big)(|00\rangle+|11\rangle)+\beta\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle)(|10\rangle+|01\rangle)\Big)$

$$\begin{aligned}=\frac{1}{2}\Big[&|00\rangle(\alpha|0\rangle+\beta|1\rangle)\\ &+|01\rangle(\alpha|1\rangle+\beta|0\rangle)\\ &+|10\rangle(\alpha|0\rangle-\beta|1\rangle)\\ &+|11\rangle(\alpha|1\rangle-\beta|0\rangle)\Big]\end{aligned}$$

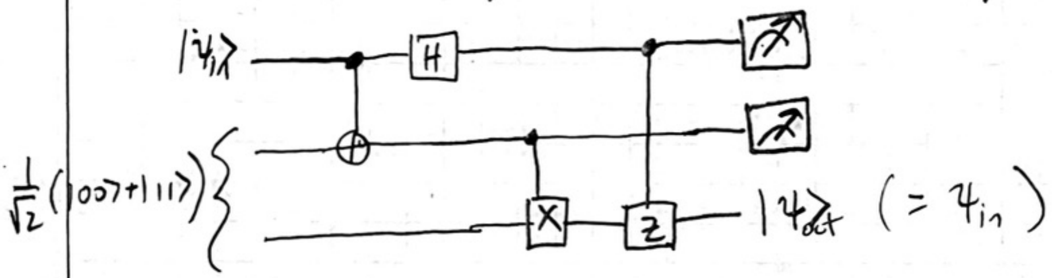
↑

depending on outcome perform unitaries!

IonQ cannot seem to handle conditional gates based on measurement. Luckily for us there is the Principle of Deferred Measurement:

Measurements can always be moved from an intermediate stage of a quantum circuit to the end of the circuit; if the measurement results are used at any stage of the circuit then the classically controlled operations can be replaced by conditional quantum operations.

In the Homework, please use the measurement deferred circuit

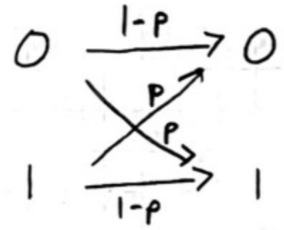


remember → set up a circuit with three classical meas. channels!

you will need to perform tomography on the teleported qubit. you need to do this "by hand" setting up the the three circuits to measure $\langle Z\rangle$, $\langle X\rangle$, & $\langle Y\rangle$ of the teleported qubit.

Noise! The 2^{nd} problem in the HW asks you to build a "realistic" noise model in qiskit for the IonQ computer.

Noise description in a classical system.



(symmetric case)

p is determined by understanding bit-environment interaction or by measuring.

Let p_0 & p_1 be the initial probabilities to be in state 0 & 1, respectively.

$X \equiv$ initial state $\quad Y \equiv$ final state

$x = 0, 1 \qquad y = 0, 1$

$$p(Y=y) = \sum_x p(Y=y \mid X=x) \underbrace{p(X=x)}^{p_0 \text{ or } p_1}$$

$$\begin{bmatrix} q_0 \\ q_1 \end{bmatrix} = \begin{bmatrix} 1-p & p \\ p & 1-p \end{bmatrix} \begin{bmatrix} p_0 \\ p_1 \end{bmatrix}$$

This discussion is for $X \to Y$.

What if we have a process $X \to Y \to Z$

Assume Markov process $\to$ consecutive noise processes are independent $\}$ time

$\to$ parallel noise processes are independent $\}$ space

for a single stage process

$$\vec{q} = E \vec{p}$$

vector, evolution matrix, vector

properties of E

$E\vec{p}$ must be a valid <u>dist</u> function.

$\Rightarrow$ all entities in E must be ≥ 0 $\Rightarrow$ positivity req.

$\Rightarrow$ all columns must sum to 1 $\Rightarrow$ completeness

Quantum Operators

$$\rho' = \mathcal{E}(\rho)$$

Examples : Unitary operator : $\mathcal{E}(\rho) = U\rho U^\dagger$

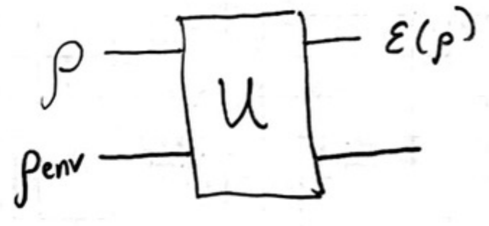
measurement : $\mathcal{E}_m(\rho) = M_m \rho M_m^\dagger$

(example $M_1 = |0\rangle\langle 0|$, $M_{-1} = |1\rangle\langle 1|$)

So far we have discussed closed systems

$$\rho - \boxed{U} - U\rho U^\dagger$$

now systems are open



$$\mathcal{E}(\rho) = tr_{env}\left[U(\rho \otimes \rho_{env}) U^\dagger \right]$$

But typically we don't know ρ_{env} or how U acts on it!

~~Before going on to removing explicit environment, first let's look at simple example on the role of the environment~~

In order to get this into the more useful operator-sum formalism lets consider a bipartite system

$$\rho_{12} = \rho_1 \otimes \underbrace{|0\rangle_2 {}_2\langle 0|}_{\text{environment}}$$

→ we can always assume a pure state. why?

why can we assume seperable?

$$\rho_{12}' = U \rho_{12} U^{\dagger} = U \left(\rho_1 \otimes |0\rangle_2 {}_2\langle 0|\right) U^{\dagger}$$

$$\rho_1' = \mathrm{Tr}_2\, \rho_{12}' = \mathrm{Tr}\left[U \left(\rho_1 \otimes |0\rangle_2 {}_2\langle 0|\right) U^{\dagger}\right]$$
$$= \sum_k {}_2\langle k|U|0\rangle_2\, \rho_1\, {}_2\langle 0|U^{\dagger}|k\rangle_2$$

all environment states $\{|k\rangle_2\}$ is a basis for Hibert space H_2 of subsystem 2

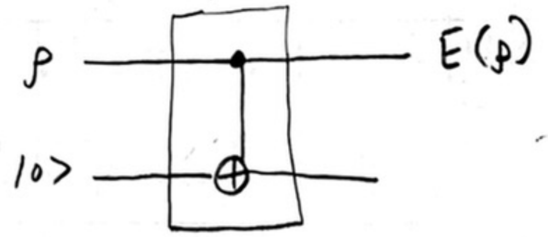
we see ${}_2\langle k|U|0\rangle_2$ is an operator acting on subsystem 1 in Hilbert space H_1

define the Kraus operators

$$E_k = {}_2\langle k|U|0\rangle_2$$

$$\rho_1' = \boxed{\sum_k E_k \rho_1 E_k^{\dagger} = \mathcal{E}(\rho_1)}$$

Let's look at a specific example



We can show directly $\mathcal{E}(\rho) = \mathrm{tr}_{env}\left[U(\rho \otimes env)U^{+}\right]$

using $U = P_0 \otimes I + P_1 \otimes X$

Or calculate

$$E_k = \langle k|U|0\rangle$$

$$E_0 = {}_e\langle 0|P_0 \otimes I + P_1 \otimes X|0\rangle_e \qquad E_1 = {}_e\langle 1|P_0 \otimes I + P_1 \otimes X|0\rangle_e$$

$$= \langle 0|0\rangle P_0 \qquad\qquad = \langle 1|X|0\rangle P_1$$

$$= P_0 \qquad\qquad = P_1$$

$$\mathcal{E}(\rho) = P_0 \rho P_0 + P_1 \rho P_1$$

Important examples. Geometric picture for single qubit

First: recall that
$$\rho = \frac{I + \vec{r}\cdot\vec{\sigma}}{2}$$
$$= \frac{1}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}$$

$$\vec{r} \xrightarrow{\mathcal{E}} \vec{r}\,' = M\vec{r} + \vec{c}$$

$\Rightarrow$ all single qubit operators map Bloch sphere to itself

(we do not prove this!)

Important examples.

Bit flip & phase flip channels

$$E_0 = \sqrt{p}\, I = \sqrt{p}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad E_1 = \sqrt{1-p}\, X = \sqrt{1-p}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

What kind of interaction is this??

$\Rightarrow$ Homework $\rightarrow$ visualize the Bloch sphere transformation

Let $|\psi\rangle = \cos\theta\,|0\rangle + e^{i\phi}\sin\theta\,|1\rangle$

$$\rho = \frac{1}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}$$

with
$r_x = \cos\varphi\sin\theta$
$r_y = \sin\varphi\sin\theta$
$r_z = \cos\theta$

calculate $\mathcal{E}(\rho) = \sum_i E_i \rho E_i^{\dagger}$

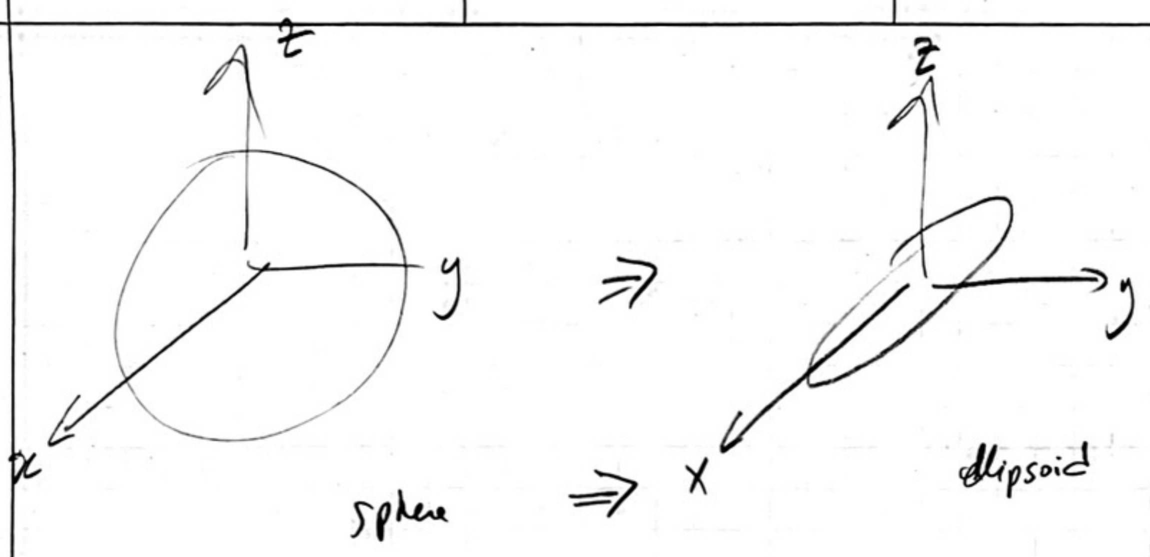
$$= p\rho + (1-p)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\rho\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= p\rho + \frac{(1-p)}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$+ \frac{1-p}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} r_x - ir_y & 1+r_z \\ 1-r_z & r_x + ir_y \end{bmatrix}$$

$$\frac{p}{2}\begin{bmatrix} 1+r_z & r_x - ir_y \\ r_x + ir_y & 1-r_z \end{bmatrix} + \frac{1-p}{2}\begin{bmatrix} 1-r_z & r_x + ir_y \\ r_x - ir_y & 1+r_z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1-(1-2p)r_z & r_x + i(1-2p)r_y \\ r_x - i(1-2p)r_y & 1+(1-2p)r_z \end{bmatrix}$$

$r_z \rightarrow -\left(\frac{1}{2}-p\right)r_z \qquad r_y \rightarrow -(1-2p)r_y \Rightarrow$ flip & shrink along r_z & r_y



phase flip : $E_0 = \sqrt{p}\, I \qquad E_1 = \sqrt{1-p}\, Z$

$\Rightarrow$ also depolarizing channel

$\hookrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

what kind of interaction is this??

bit - phase flip

$$E_0 = \sqrt{p}\, I \qquad E_1 = \sqrt{1-p}\, Y$$

$\rightarrow$ ellipsoid w/ major axis along Y

Somewhat unphysical but often used :

<u>depolarizing channel</u>

$$\mathcal{E}(\rho) = p\frac{I}{2} + (1-p)\rho$$

$$= (1-p)\rho + \frac{p}{3}\left(X\rho X + Y\rho Y + Z\rho Z \right)$$

<u>amplitude damping</u>

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-\gamma} \end{bmatrix} \qquad E_1 = \begin{bmatrix} 0 & \sqrt{\gamma} \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow$ damps to $|0\rangle$ state generalized amplitude damping

$$E_0 = \sqrt{p}\begin{bmatrix}1 & 0\\ 0 & \sqrt{1-\gamma}\end{bmatrix}$$

$$E_1 = \sqrt{p}\begin{bmatrix}0 & \sqrt{\gamma}\\ 0 & 0\end{bmatrix}$$

$$E_2 = \sqrt{1-p}\begin{bmatrix}\sqrt{1-\gamma} & 0\\ 0 & 1\end{bmatrix}$$

$$E_3 = \sqrt{1-p}\begin{bmatrix}0 & 0\\ \sqrt{\gamma} & 0\end{bmatrix}$$

$$\rho_\infty = \begin{bmatrix}p & 0\\ 0 & 1-p\end{bmatrix} \Longleftarrow \text{thermal equilibrium state}$$

HW: show that

$$(r_x, r_y, r_z) \to (r_x\sqrt{1-\gamma},\ r_y\sqrt{1-\gamma},\ \gamma(2p-1) + r_z(1-\gamma))$$

Phase damping: $E_0 = \begin{bmatrix}1 & 0\\ 0 & \sqrt{1-\lambda}\end{bmatrix}$, $E_1 = \begin{bmatrix}0 & 0\\ 0 & \sqrt{\lambda}\end{bmatrix}$ } derives from random phase kicks

$\Rightarrow$ can show $\mathcal{E}(\rho)$ for E_0, E_1 is same as

$\mathcal{E}(\rho)$ for E_0', E_1'

$$E_0' = \sqrt{\alpha}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \qquad \alpha = 1+\sqrt{1-\lambda}/2$$

$$E_1' = \sqrt{1-\alpha}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$$