Error Analysis

1 Introduction and some useful terminology

Every measurement in experimental science has an error or uncertainty associated with it. This uncertainty is the difference between the true value and a calculated value. Sometimes we do not know what the true value is and therefore quantitative results must include the uncertainty associated with the result. Other times, we know what the true value is, but repeated measurements by different observers and techniques can give varying results leading to a spread or discrepancy in the result. Therefore, the error must be reported to determine the reliability of the experimental measurement.

- There are usually two types of uncertainties:

1. Non-random or systematic error: this type of error is reproducible and results from faulty equipment, incorrect calibration, bias from the observer, etc.
2. Random error: the error arising due to random fluctuations in the apparatus, observations or both give rise to irreproducible errors.

- Reporting uncertainty in measurement:

1. Absolute precision (or uncertainty): when the result of N measurements of quantity R is reported as <R> $\pm$ δR, where <R> is the average value and δR is the uncertainty, then one claims absolute uncertainty.
2. Relative precision (or uncertainty) If the above result were expressed as <R> $\pm \frac{\delta R}{<R>} \times 100$ then one claims relative precision on the measurement.

- What uncertainty δR should one use?

1. *Case 1*: Consider *one* set of N measurements of quantity R. If we assume that the only source of error is random error then the deviation of the independent measurements from the mean value is a good indicator of the uncertainty. The various quantities are:
   (a) Mean, <R> = $\frac{1}{N} \times \Sigma_i \, R_i$
   (b) Variance, σ^2 = $\frac{1}{N} \times \Sigma_i \, (R_i - < R >)^2$

(c) Sample standard deviation,

$$\sigma = \sqrt{\frac{1}{(N-1)} \times \Sigma_i \left(R_i - < R >\right)^2} \tag{1}$$

In this situation, a good estimate of the error δR is σ, i.e. R = <R> $\pm\sigma$.

2. *Case 2*: Consider *m* sets of *N* measurements of quantity R. Now we have *m* different values of R and σ. How do we report the final result and error. For a situation where each measurement has the same uncertainty σ, a new term called the Standard Deviation of the Mean (*SDM*), σ_N, for each set of N measurements, can be used to estimate the true error in the mean of the total N×m measurements, σ_{Nm}. (Bevington, Chap 4) These are given by:

(a) SDM for each set of *N* measurements, $\sigma_N \sim \sigma/\sqrt{N}$. The error for each set of N measurements is reported as R = $\langle R\rangle_N \pm \sigma_N$.

(b) SDM for *m* sets of *N* measurements, $\sigma_{Nm} \sim \sigma_N/\sqrt{m} = \sigma/\sqrt{Nm}$. This is the same result as part (a) for a total of *Nm* measurements.

(c) Most probable value, $\langle R\rangle_{Nm} = \sum_m \frac{< R_m >}{m}$, where $\langle R\rangle_{Nm}$ is now the average from the m sets of $\langle R_m\rangle$ values.

In this case, the error is reported as R = $\langle R\rangle_{Nm} \pm \sigma_{Nm}$.

3. *Case 3:* In the above example, if the SDM σ_N in each set is not equal, then the corresponding estimation of σ_{Nm} must be modified accordingly. For the case of random error, the new SDM can be written as:

(a) SDM, $\sigma_{Nm} = \sqrt{\sum_i \frac{1/\sigma_i^2}{[\sum(1/\sigma_i^2)]^2}}$.

• How many significant digits should be used in reporting data?

The # of significant digits is calculated as

1. Leftmost non-zero digit is most significant digit.
2. If there is no decimal point, the rightmost nonzero digit is the least significant. For example, 1,234 and 1.234×10^3 both have 4 significant digits.
3. If there is a decimal point, the right most digit is least significant, even if it is 0 (e.g 10.10 has 4 significant digits)

The # of significant figures should be at least one more than the uncertainty. For example, if <R> = 10.88 cm and σ = 3.4 cm, then the answer should be reported as 10.9 $\pm$ 3.4 cm.

2 Error Propagation

In the above example, the quantity R was measured N times. Consider now the situation where the quantity R depends on two independent variables x and y, i.e. R = R(x,y). Now let us make N measurements of (x,y). If the errors in x and y are independent of each other and random, then how do we determine the final error in R. To estimate this we make use of error propagation, whose mathematical formulation is given below.

Consider a box having length *L*, width *W* and height *H*. How is the change of volume (*V*) of the box related to the changes in length, height and width? Since *V* = *LHW*, if only the length varies the change in volume is $\Delta V = \Delta LHW$, and $L\Delta HW$ if only the height changes and $LH\Delta W$ if only the width changes. If all three dimensions change, then the change in volume is simply the sum of these individual changes:

$$\Delta V = \Delta LHW + L\Delta HW + LH\Delta W \tag{2}$$

The variation of a function of several variables due to the change in only one of the variables is how we define partial derivatives. The first term in the equation above is simply the partial derivative of *V* with respect to *L* times the change in length *L*. We can write the total differential change in volume in terms of partial derivatives as:

$$dV = \frac{\delta V}{\delta L}dL + \frac{\delta V}{\delta H}dH + \frac{\delta V}{\delta W}dV \tag{3}$$

Now suppose g(x,y,z) is a quantity that is a function of the measured variables *x,y,z*. Then *g* itself is uncertain due to the uncertainties of each of these measured values. A measure of the scatter of the individual measured values of each variable about their mean is provided by the average square deviation of *g* and is given by the sum of the square deviations divided by *N*. In fact, this becomes the commonly accepted definition of the square of the standard deviation,, when *N* is replaced by *N-1*.

$$\sigma_g^2 = \frac{1}{N-1}\sum_{-i}(g_i - \overline{g})^2 \tag{4}$$

(where g_i is an individual value and $\overline{g}$ is the average value)

This definition keeps σ meaningful for small populations (when *N* = 1, or a small number). Statistically, 68% of the measurements will fall within 1σ and 95% within 2σ. The variation (deviation from the mean) of the i^{th} value of *g* is related to the variations of the measured variables by:

$$(g_i - \overline{g}) = (x_i - \overline{x})\frac{\delta g}{\delta x} + (y_i - \overline{y})\frac{\delta g}{\delta y} + (z_i - \overline{z})\frac{\delta g}{\delta z} \tag{5}$$

Therefore, the square of the standard deviation is given by:

$$\sigma_g^2 = (\frac{1}{N-1})\sum_i[(x_i - \overline{x})\frac{\delta g}{\delta x} + (y_i - \overline{y})\frac{\delta g}{\delta y} + (z_i - \overline{z})\frac{\delta g}{\delta z}]^2 \tag{6}$$

or after expansion by:

$$\sigma_g^2 = (\frac{1}{N-1})[\sum_i(x_i - \overline{x})^2(\frac{\delta g}{\delta x})^2 + \sum_i(y_i - \overline{y})^2(\frac{\delta g}{\delta y})^2 + \sum_i(z_i - \overline{z})^2(\frac{\delta g}{\delta z})^2] \tag{7}$$

The cross terms (second line in Eq. 7) vanish in the summation if the variables are *linearly independent*. That is, if the variation in *x*, for instance, is independent of the variations in *y* and *z*. Note also that:

$$\sigma_x^2 = \frac{1}{N-1}\sum_i (x_i - \overline{x})^2 \; ; \sigma_y^2 = \frac{1}{N-1}\sum_i (y_i - \overline{y})^2 \; ; \sigma_z^2 = \frac{1}{N-1}\sum_i (z_i - \overline{z})^2 \tag{8}$$

Substituting the expressions from Eq. 8 into Eq. 7 yields:

$$\sigma_g^2 = \sigma_x^2(\frac{\delta g}{\delta x})^2 + \sigma_y^2(\frac{\delta g}{\delta y})^2 + \sigma_z^2(\frac{\delta g}{\delta z})^2 \tag{9}$$

which relates the standard deviation of the computed function σ_g to the standard deviations of the measured quantities. Since the standard deviation σ is related to the *standard deviation of the mean* (or standard error) σ_m by the relationship $\sigma_m = \sigma/\sqrt{N}$, we can find the expected uncertainty of the computed quantity *g* from the uncertainties of the measured quantities *x,y,z*. This quantity is also known as the *standard deviation of the mean* or the *standard error of the mean*. Now lets see how this works for the different functional forms.

2.1 Addition and Subtraction

First we consider a quantity g(x,y,z) that is a function of three independent parameters consisting of sums and differences:

$$g(x,y,z) = ax + by + cz \;\; or \;\; g(x,y,z) = ax - by - cz \tag{10}$$

The partial derivatives are

$$\frac{\delta g}{\delta x} = a, \; \frac{\delta g}{\delta y} = \pm b \; \frac{\delta g}{\delta z} = \pm c \tag{11}$$

The sign depends upon whether the terms are added or subtracted. We can use the expression derived above (Eq. 9) to find σ_g as a function of σ_x, σ_y and σ_z

$$\sigma_g^2 = \sigma_x^2(\frac{\delta g}{\delta x})^2 + \sigma_y^2(\frac{\delta g}{\delta y})^2 + \sigma_z^2(\frac{\delta g}{\delta z})^2 = a^2\sigma_x^2 + b^2\sigma_y^2 + c^2\sigma_z^2 \tag{12}$$

Note here that the minus signs vanish because of the squares. The square of the standard error is

$$\sigma_m^2 = \frac{\sigma_g^2}{N} = \frac{a^2\sigma_x^2 + b^2\sigma_y^2 + c^2\sigma_z^2}{N} = a^2\sigma_{mx}^2 + b^2\sigma_{my}^2 + c^2\sigma_{mz}^2 \tag{13}$$

Thus, the uncertainty for measurements that have a functional dependence that involves sums and differences is

$$\sigma_m = \sqrt{a^2\sigma_{mx}^2 + b^2\sigma_{my}^2 + c^2\sigma_{mz}^2} = \sqrt{\frac{1}{N}(a^2\sigma_x^2 + b^2\sigma_y^2 + c^2\sigma_z^2)} \tag{14}$$

Rule for Addition and Subtraction

The overall uncertainty is equal to the square root of the sum of the squares of the uncertainties of each of the individual terms.

2.2 Multiplication and Division

Next we consider a quantity g(x,y) that is a function of two independent parameters consisting of a single multiplication or division

$$g(x,y) = \pm axy \;\; or \;\; g(x,y) = \pm ax/y \tag{15}$$

For the case of multiplication we have

$$\frac{\delta g}{\delta x} = \pm ay \; and \; \frac{\delta g}{\delta y} = \pm ax$$

and

$$\sigma_g^2 = \sigma_x^2 (\frac{\delta g}{\delta x})^2 + \sigma_y^2 (\frac{\delta g}{\delta y})^2 = a^2 y^2 \sigma_x^2 + a^2 x^2 \sigma_y^2$$

and for division

$$\frac{\delta g}{\delta x} = \pm \frac{a}{y} \; and \; \frac{\delta g}{\delta y} = \pm \frac{ax}{y^2}$$

and

$$\sigma_g^2 = \sigma_x^2 (\frac{\delta g}{\delta x})^2 + \sigma_y^2 (\frac{\delta g}{\delta y})^2 = \frac{a^2}{y^2} \sigma_x^2 + \frac{a^2 x^2}{y^4} \sigma_y^2$$

Dividing by g^2 (in each case) results in the following expression

$$\frac{\sigma_g^2}{g^2} = \frac{\sigma_x^2}{x^2} + \frac{\sigma_y^2}{y^2}$$

for both multiplication and division. Now, recalling that the standard error is $\sigma_m = \sigma/\sqrt{N}$, and that σ_m/g is the fractional error of *g*, we have

$$\frac{\sigma_m}{g} = \sqrt{(\frac{\sigma_{mx}}{x})^2 + (\frac{\sigma_{my}}{y})^2}$$

for *both* multiplication and division.

Rule for Multiplication and Division

The Fractional Error of the quantity (fractional overall uncertainty) is equal to the square root of the sum of the squares of the individual fractional errors (note that σ_{mx}/x is the fractional error of *x*, etc.).

2.3 Powers

At first glance one may think that powers are just products and we proceed as described above for multiplication. For instance, the function, g(x,y) = Cxy^2 = Cxyy is a constant *C* times the product of three variables *x*, *y* and *y*, but the last two are obviously *not independent variables.* Therefore, the treatment above is no longer valid and we must develop the proper expression for variables raised to some power.

Consider a function of a single variable given by

$$g(x) = ax^{\pm b}$$

which has the following partial derivative

$$\frac{\delta g}{\delta x} = \pm abx^{\pm(b-1)}$$

Since

$$\sigma_g^2 = \sigma_x^2(\frac{\delta g}{\delta x})^2 \;\; or \;\; \sigma_g = \sigma_x \frac{\delta g}{\delta x} = \sigma_x abx^{\pm(b-1)} \tag{16}$$

we can obtain the fractional uncertainty by dividing Eq. 16 by $g = ax^{\pm b}$

$$\frac{\sigma_g}{g} = b\frac{\sigma_x}{x}$$

Recalling that the standard error is defined as

$$\sigma_m = \frac{\sigma}{N}$$

we obtain the fractional uncertainty in terms of the standard error

$$\frac{\sigma_m}{g} = b\frac{\sigma_{mx}}{x}$$

Rule for Powers

For measurements that have the functional form , the fractional error on *g(x,y,z)* = $Cx^p y^q z^r$, the fractional error on g is given by

$$\frac{\sigma_m}{g} = \sqrt{(p\frac{\sigma_{mx}}{x})^2 + (q\frac{\sigma_{my}}{y})^2 + (r\frac{\sigma_{mz}}{z})^2}$$

3 Least Squares Fitting

In many cases the measurement of the function f(x), f(x,y), etc. is performed by varying one of the independent variables over a large range. For example, consider the function y = A + Bx. Instead of making numerous measurements of y for one particular value of x, we can make a series of N measurements of y_i , one for each of several values of the quantity x = x_i, where i runs from 1 to N. Therefore we have N sets of data points represented as (x_i, y_i). Given this situation, what will be the best estimate for the values A and B and the error in the various quantities y, A and B? One powerful approach to determine these quantities is that of least-squares fitting and is discussed below.

3.1 The true or most probable value for random or Gaussian error

It can be shown that when a quantity y_i consists of random uncertainty such that the probability $P(y_i)$ of obtaining the value y_i belongs to a Gaussian distribution, then

$$P(y_i) = \frac{1}{\sqrt{2}\sigma_i} exp^{-\frac{(y_i-\overline{y})^2}{2\sigma^2}} \tag{17}$$

where σ_i and $\overline{y}$ are the standard deviation and mean of the the quantity y.

Under these circumstances, the most probably value of y is when the probability $P(y_i)$is at a maximum. This is called the *method maximum likelihood*. Now, $P(y_i)$ is at its maximum when the term in the exponential, i.e. $(y_i- \overline{y})^2$is a minimum, or in other words, when $y_i= \overline{y}$. There it is clear that for a random Gaussian situation, the most probably value of y is its mean value. This is the approach which is utilized in the method of least squares, i.e. obtaining the minima the the squared term of the exponential.

3.2 Applying the least-squares

Returning back to the situation of y = A +Bx, assuming that the experimental measurements suffer from a random or Gaussian error, the probability of measuring a value y_i can now be expressed as:

$$P(y_i) = \frac{1}{\sqrt{2}\sigma_i} exp^{-\frac{[y_i-(A+Bx_i)]^2}{2\sigma^2}} \tag{18}$$

where the true value of y is given as $\overline{y} = A + Bx_i$.

Remember, our goal is to determine A and B and the associated errors. Now let us make N measurements of y_i. The probability of all these measurements can be expressed as:

$$P(y_1, y_2...y_N) \propto \frac{1}{\sigma^N} e^{-\sum_i^N \frac{(y_i-A-Bx_i)^2}{2\sigma^2}} \propto \frac{1}{\sigma^N} e^{-\frac{\chi^2}{2}} \tag{19}$$

Now the best set of measurement of y will occur when $P(y_1, y_2...y_N)$ is a maximum or:

$$\frac{\chi^2}{2} = minimum \tag{20}$$

This is the concept of Least-squares, i.e. the sum of the squares should be "least" and is done as follows:

$$\frac{\delta\chi^2}{\delta A} = \frac{-2}{\sigma_y^2}\sum_i^N (y_i - A - Bx_i) = 0; \;\; and \;\; \frac{\delta\chi^2}{\delta B} = \frac{-2}{\sigma_y^2}\sum_i^N (y_i - A - Bx_i)x_i = 0 \tag{21}$$

These two can be simplified to:

$$AN + B\sum x_i = \sum y_i \;\; and \;\; A\sum x_i + B\sum x_i^2 = \sum x_i y_i$$

These two equation can be solved for A and B, giving:

$$A = \frac{(\sum x_i^2)(\sum y_i) - (\sum x_i)(\sum x_i y_i)}{[N(\sum x_i^2) - (\sum x_i)^2] = \Delta}; \quad and \quad B = \frac{N(\sum x_i y_i) - (\sum x_i)(\sum y_i)}{\Delta} \tag{22}$$

Equation 22 can be solved by tabulating the various quantities as shown in table 1:

x_i	y_i	x_i^2	x_iy_i	$y_i - A - Bx_i$
$\sum x_i$	$\sum y_i$	$\sum x_i^2$	$\sum x_iy_i$	$\sum(y_i - A - Bx_i)^2$

Table 1: Least-squares analysis for straight line fitting

3.3 Uncertainties in A, B and y

The standard deviation for the various quantities can be evaluated as under:

$$\sigma_y^2 = \frac{1}{N-2}\sum_{i=1}^{N}(y_i - A - Bx_i)^2 \tag{23}$$

The denominator of N-2 comes from the fact that there are only N -2 independent variable in the N measurements, as we have determined A and B.

$$\sigma_A^2 = \sigma_y^2\frac{\sum x_i^2}{\Delta}; \quad and \quad \sigma_B^2 = \frac{N\sigma_y^2}{\Delta} \tag{24}$$

where $\Delta$is the denominator of equation 22.

4 Data Rejection

Sometimes when we make a series of measurements of a specific quantity one of the measured values disagrees strikingly with all the other measured values. When this happens the experimenter is presented with the situation where he/she must decide whether the anomalous measurement resulted from some mistake (glitch in the measurement system) and should be rejected or was a bona fide measurement that should be kept. If careful records were kept sometimes we can establish a definite cause for the anomalous measurement and therefore justifiably reject the measurement.

If an external cause can not be found for the anomalous result, then the truly honest course of action is to repeat the measurement many times. If the anomaly shows up again then hopefully the cause may be found. Either as a glitch in the measurement system or as a real physical effect. If the anomaly does not recur, then due to the increased number of measurements made there will be no significant difference in our final answer whether we include the anomaly or not.

If it is impossible to re take the measurements then the experimenter must decide whether or not to reject the anomaly by examining the measured data and the properties of a Gaussian distribution. The rejection of data is a subjective controversial question, on which experts disagree. The experimenter who rejects data may reasonably be accused of *fixing* his/her data. The situation is made worse by the possibility that the anomalous result may reflect some important physical effect. One criterion for rejecting suspect data is Chauvenet's criterion.

4.1 Chauvenet's Criterion for Data Rejection

Suppose we make *N* measurements x_1, x_2, ... x_N of the same quantity *x*.

1. Using ***all*** the values of the *N* measurements made calculate the mean ($\overline{x}$) and standard deviation (σ_x). If one of the measurements (call it $x_{suspect}$) greatly differs from $\overline{x}$ and looks suspicious, then calculate

   $$t_{suspect} = \frac{x_{suspect} - \overline{x}}{\sigma_x}$$

   the number of standard deviations by which $x_{suspect}$ differs from $\overline{x}$.

2. 2. We next find the probability P(outside $t_{suspect}\,\sigma_x$) that a legitimate measurement will differ from $\overline{x}$ by or $t_{suspect}$ or more standard deviations.

   $$P(outside\ t_{suspect}\sigma_x) = 1 - P(within\ t_{suspect}\sigma_x)$$

3. Finally, we multiply by *N*, the total number of measurements, to arrive at

   $$n(worse\ than\ x_{suspect}) = NP(outside\ t_{suspect}\sigma_x)$$

This *n* is the number of measurements expected to be at least as bad as $x_{suspect}$.

If *n* is less than $\frac{1}{2}$, then $x_{suspect}$ fails Chauvenet's criterion and is rejected.

5 Reporting data and error

5.1 Significant figures (digits)

When reporting values that were the result of a measurement or calculated using measured values, it is important to have a way to indicated the certainty of the measurement. This is accomplished through the use of significant figures. Significant figures are the digits in a value that are known with some degree of confidence. As the number of significant figures increases, the more certain the measurement. As precision of a measurement increases, so does the number of significant figures. Consider the weight measurements made using the following three instruments. Notice that the number of significant digits increase as the measured value gets more precise and the range of uncertainty gets smaller.

Instrument	Measured Value	Precision of Measurement	Minimum Amount of Uncertainty in the Measurement	Significant Figures of Measured Value
Postage Scale	3g	1g	+/- 0.5g	1
Two-pan balance	2.53g	0.01g	+/- 0.005g	3
Analytical balance	2.531g	0.001g	+/- 0.0005g	4

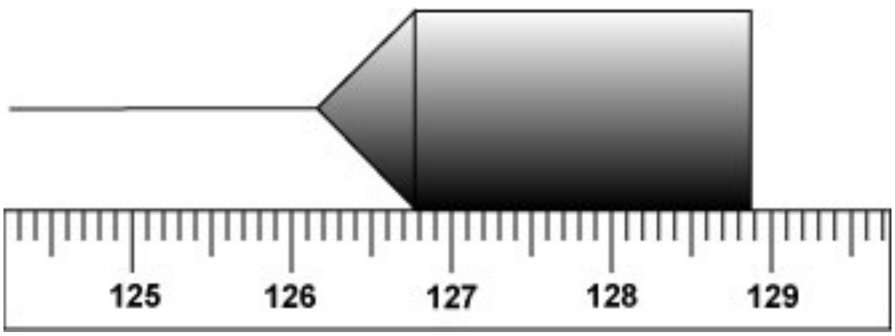
There are conventions that must be followed for expressing numbers so that their significant figures are properly indicated. These conventions are:

All non zero digits are significant.	549 has three significant figures 1.892 has four significant figures
Zeros between non zero digits are significant.	4023 has four significant figures 50014 has five significant figures
Zeros to the left of the first non zero digit are not significant.	0.000034 has only two significant figures. (This is more easily seen if it is written as 3.4×10^{-5}) 0.001111 has four significant figures.
Trailing zeros (the right most zeros) are significant when there is a decimal point in the number. For this reason it is important to give consideration to when a decimal point is used and to keep the trailing zeros to indicate the actual number of significant figures.	400. has three significant figures 2.00 has three significant figures 0.050 has two significant figures
Trailing zeros are not significant in numbers without decimal points.	470,000 has two significant figures 400 or 4×10^2 indicates only one significant figure. (To indicate that the trailing zeros are significant a decimal point must be added. 400. has three significant digits and is written as 4.00×10^2 in scientific notation.)
Exact numbers have an infinite number of significant digits but they are generally not reported. Defined numbers also have an infinite number of significant digits.	If you count 2 pencils, then the number of pencils is 2.000... The number of centimeters per inch (2.54) has an infinite number of significant digits, as does the speed of light (299792458 m/s).

5.2 How to estimate error

Errors when Reading Scales

The most common measurements in the lab are done with devices that have a marked scale. Let's look at an example. We will measure the length of the pendulum from the pivot point to the visible end of the mass. The situation is schematically shown in the figure below



(numbers are in centimeters)

Even using this idealized, zoomed-in picture, we cannot tell for sure whether the length to the end of the mass is 128.89 cm or 128.88 cm. However, it is certainly closer to 128.9 cm than to 128.8 cm or 129.0 cm. Thus we can state with absolute confidence that the length *L* is

$$L = 128.9 \text{ cm} \pm 0.1 \text{ cm}$$

We call the first term, 128.9 cm, the "central value" and the second term, 0.1 cm, the "error" or "uncertainty".

If pressed, we could get a little bit better precision from the picture. However, in a real situation, the precision of 0.1 cm for measurements done with the centimeter ruler is as good as you can get. In this course we will report error in measurements to be ± the last readable digit.

5.3 How to report error

Number of Significant Digits

As a general rule you should round the error to just one significant digit for your final answer and keep two significant digits for intermediate results. So for a time measurement *T*, we would write

$$T = 3.6 \pm 0.2 \text{ sec}$$

Now, what about the number of significant digits for the central value of T itself? Suppose the calculator gives us the value T = 3.57361382 sec instead of 3.6 sec. It is ridiculous to write our final answer for T as

$$T = 3.57361382 \pm 0.2 \text{ sec}$$

Since our error already affects the number 5 in the tenth place of 3.57361382, we should round our value to that decimal place. So the correct answer is

$$T = 3.6 \pm 0.2 \text{ sec}$$

This principle is true in general. We should round our central value to the rightmost decimal place at which our error applies. Thus if we have two decimal places in our error we should round our central value to the hundredths decimal place.

$$T = 3.57 \pm 0.18 \text{ sec}$$

5.3 Maintaining Significant Digits in Calculations

Once the number of significant figures various values have been determined, the issue then becomes dealing with significant figures when these values are used in calculations. When combining values with different degrees of precision, the precision of the final answer can be no greater than the least precise measurement. However, it is a good idea to keep one more digit than is significant during the calculation to reduce rounding errors. In the end, however, the answer must be expressed with the proper number of significant figures.

Addition and Subtraction

When adding and subtracting, round the final result to have the same precision (same number of decimal places) as the least precise initial value, regardless of the significant figures of any one term. For example, 98.112 + 2.3 = 100.412 but this value must be rounded to 100.4 (the precision of the least precise term).

Multiplication, Division, and Roots

When multiplying, dividing, or taking roots, the result should have the same number of significant figures as the least precise number in the calculation. For example, (3.69)x(2.3059) = 8.5088, which should be rounded to 8.51 (three significant figures like 3.69).

Logarithms and Antilogithms

When calculating the logarithm of a number, retain in the mantissa (the number to the right of the decimal point in the logarithm) the same number of significant figures as there are in the number whose logarithm is being found. For example,

$$\log(3.000\times10^{4}) = 4.477121, \text{ which should be rounded to } 4.477$$

$$\log(3\times10^{4}) = 4.477121, \text{ but this value should be rounded to } 4$$

When calculating the antilogarithm of a number, the resulting value should have the same number of significant figures as the mantissa in the logarithm. For example,

$$\text{antilog}(0.301) = 1.9998, \text{ which should be rounded to } 2.00,$$

$$\text{antilog}(0.30) = 1.9998, \text{ which should be rounded to } 2.0$$

Multiple Mathematical Operations

If a calculation involves a combination of mathematical operations, perform the calculation using more figures than will be significant to arrive at a value. Then, go back and look at the individual steps of the calculation and determine how many significant figure could carry through to the final result based on the above conversions. For Example,

$$X = ((5.254+0.0016)/34.6) - 2.231*10^{-3}$$

Calculate the value of X using more digits than will be significant. In this case X = 0.1496649538

Then, go back and look at each piece of the equation to determine the significant figures.

> 5.254 + 0.0016 = 5.256 (since the sum is limited to the thousandths place by 5.254)
>
> 5.256 / 34.6 = 0.152 (since the quotient is limited to 3 significant figures by 34.6)
>
> 0.152 - 0.002231 = 0.149 (since the difference is limited to the thousandths place by 0.152)

The value initially obtained for X (0.1496649538) should be rounded to have 3 significant digits

Therefore, the final answer is 0.150 or 1.50×10^{-1}.

The Rules of Rounding

When a value contains too many significant figures, it must be rounded off. There are two methods that are commonly used to minimize the error introduced into a value to rounding.

Method 1: This method involves *underestimating* the value when rounding the five digits 0, 1, 2, 3, and 4, and *overestimating* the value when rounding the five digits 5, 6, 7, 8, and 9. With this approach, if the value of the digit(s) to the right of the last significant figures smaller than 5, drop this digit and leave the remaining number unchanged. Thus, 2.794 becomes 2.79. If the value of the digit(s) to the right of the last significant digit is 5 or larger, drop this digit and add 1 to the preceding digit. Thus, 2.795 becomes 2.80.

Method 2: This method takes into account that zero doesn't really require rounding and when rounding 5, this value is exactly centered between the underestimated value if it is rounded down and the overestimated value if it is rounded up. Therefore, five should be rounded up half of the time and down half of the time. Since it would difficult to keep track of this when performing numerous measurements or calculations, 5 is rounded down when the preceding significant digit is even and 5 is rounded up when the preceding significant digit is odd. Values less than 5 are rounded down and values greater than 5 are rounded up. For example, 2.785 would be rounded down to 2.78 and 2.775 would be rounded up to 2.78.