Free Magma : A Library of Universal Catalan Bijections

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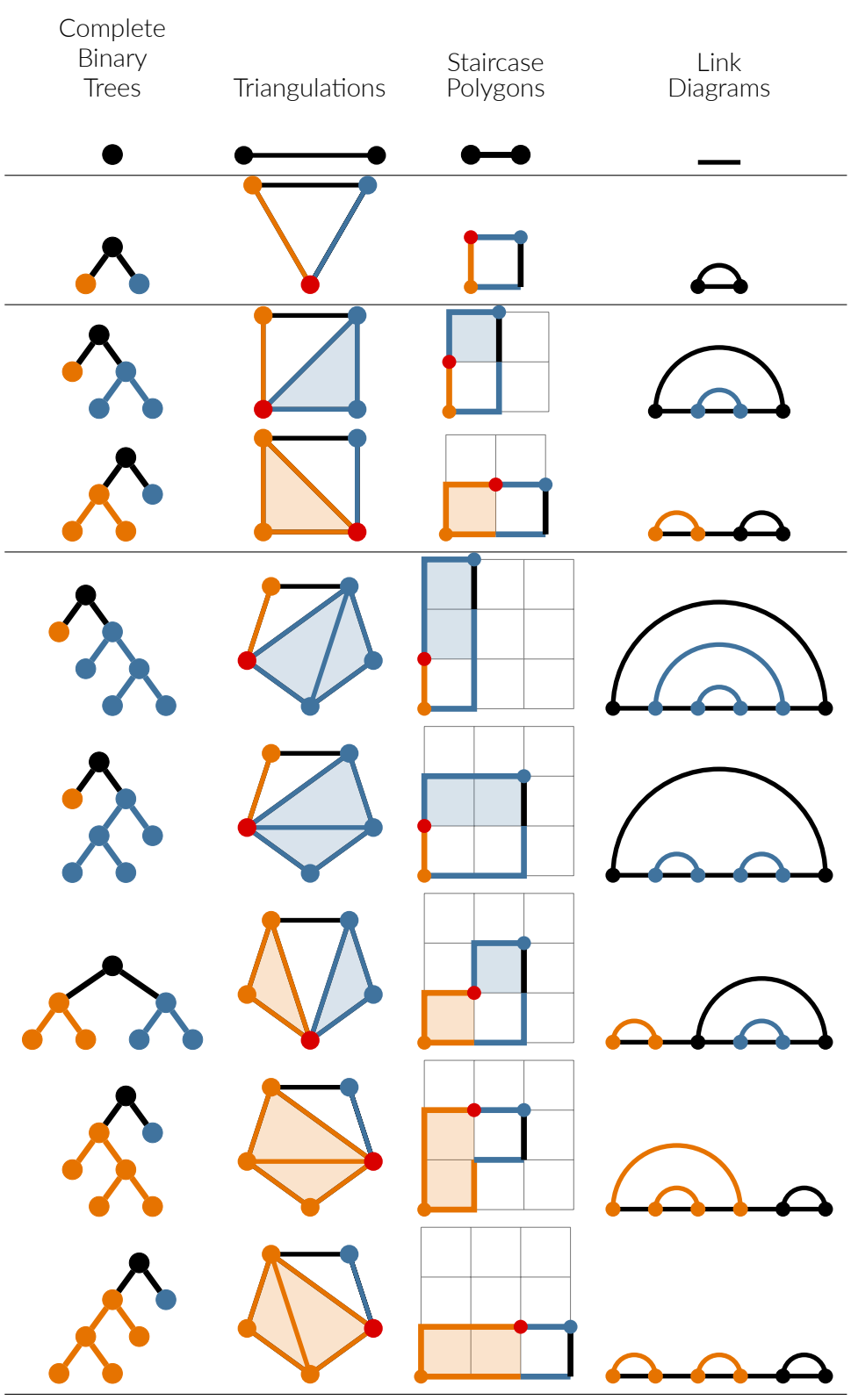
$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862 \ldots$$

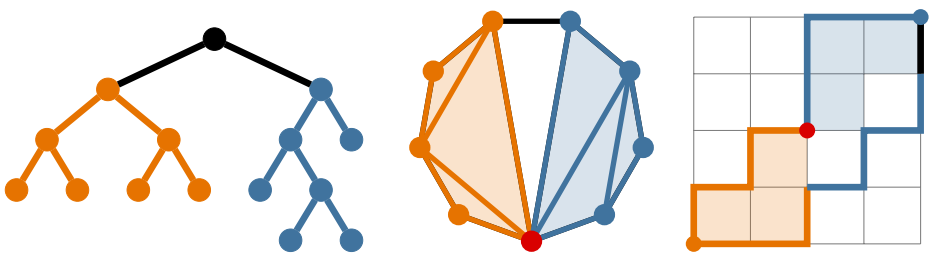
The Catalan Numbers are an integer sequence (OEIS. A000108 [2]) counted by any of 214 combinatorial families described in Stanley's *Catalan Numbers* [3]. We would like to understand properties of these families via bijections to other Catalan families, however, it would be infeasible to design $214 \times 213 = 4582$ direct bijections, and composing bijections lacks canonicity.

Through R. Brak's insight that any Catalan family is a Free Magma on one generator [1], we reduce the problem of finding bijections between pairs of families, to recognising the Free Magma structure within each family (just once for each family!). This means finding a unique way of multiplying and factorising objects.

$$* : \mathtt{C} \times \mathtt{C} \to \mathtt{C} \qquad \Delta : \mathtt{C} \setminus \{\epsilon\} \to \mathtt{C} \times \mathtt{C}$$



Which Link Diagram do these larger objects biject to?



`FreeMagma`

`FreeMagma` is an open-source library of Catalan families written in Python, which provides a universal bijection function. Given two Catalan families, `A`, `B`, we obtain the bijection, `bij::A -> B` recursively:

```
def bij(x):
  if x == A.generator:
    return B.generator
  else:
    first, second := A.factorise(x)
    return B.multiply( bij(first), bij(second) )
```

However, recognising the right multiplication rule is not always a trivial task, hence one purpose of `FreeMagma` is to gather a collection of as many valid multiplication/factorisation rules for Catalan families as possible.

Mapping between representations of geometric Catalan families is no fun if we can't *see* them, so `FreeMagma` provides functions for producing TikZ output (see left) and `ascii` drawings wherever feasible.

Data Structures

A central challenge to this project was deciding on the best way to represent each family's objects in the code, especially as many Catalan families are geometric in nature, with many equivalent embeddings in the plane. Wherever possible, we chose representations that were:

- Intuitive for the user to input
- As canonical as possible
- Easy to manipulate

Caching for Speed

The recursive structure of Catalan families presents a significant danger in exponential blow-up. For instance, the standard algorithm for computing the list objects of size m recursively calls itself $\mathcal{O}(m^2)$ many times! To remedy this, `FreeMagma` caches function calls where-ever possible. When bijecting an object with many occurrences of a sub-object, we only factorise and biject that sub-object once, and reuse that computation for every other occurence.

Acknowledgements

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[1] Richard Brak.
A universal bijection for catalan structures, 2018.

[2] The On-Line Encyclopedia of Integer Sequences.
Catalan numbers - A000108, 2020.

[3] R.P. Stanley.
Catalan Numbers.
Cambridge University Press, 2015.