PMS – Exercise Sheet 8

Exercise 1

Initialisierung

$$A = \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix}$$

$$b = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

$$x_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Erste Iteration

$$d_0 = b - A * x_0 = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

$$r_0 = d_0 = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$$

$$\alpha_0 = \frac{r_0{}^T * r_0}{d_0{}^T * A * d_0} = \frac{\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}^T * \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}}{\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}^T * \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}} = \frac{1}{5}$$

$$x_1 = x_0 + \alpha_0 d_0 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{5} \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} \frac{6}{5} \\ \frac{6}{5} \\ \frac{6}{5} \end{pmatrix}$$

Zweite Iteration

$$r_1 = r_0 - \alpha_0 A * d_0 = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} - \frac{1}{5} \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix} * \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{6}{5} \\ \frac{12}{5} \\ -\frac{6}{5} \end{pmatrix}$$

$$\beta_1 = \frac{{r_1}^T r_1}{{r_0}^T r_0} = \frac{\begin{pmatrix} -\frac{6}{5} \\ \frac{12}{5} \\ -\frac{6}{5} \end{pmatrix}^T * \begin{pmatrix} -\frac{6}{5} \\ \frac{12}{5} \\ -\frac{6}{5} \end{pmatrix}}{\begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}^T * \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}} = \frac{2}{25}$$

$$d_1 = r_1 + \beta_1 d_0 = \begin{pmatrix} -\frac{6}{5} \\ \frac{12}{5} \\ -\frac{6}{5} \end{pmatrix} + \frac{2}{25} \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix} = \begin{pmatrix} -\frac{18}{25} \\ \frac{72}{25} \\ -\frac{18}{25} \end{pmatrix}$$

$$\alpha_1 = \frac{{r_1}^T * r_1}{{d_1}^T * A * d_1} = \frac{\begin{pmatrix} -\frac{6}{5} \\ \frac{12}{5} \\ -\frac{6}{5} \end{pmatrix}^T * \begin{pmatrix} -\frac{6}{5} \\ \frac{12}{5} \\ -\frac{6}{5} \end{pmatrix}}{\begin{pmatrix} -\frac{18}{25} \\ \frac{72}{25} \\ -\frac{18}{25} \end{pmatrix}^T * \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix} * \begin{pmatrix} -\frac{18}{25} \\ \frac{72}{25} \\ -\frac{18}{25} \end{pmatrix}} = \frac{5}{3}$$

$$x_2 = x_1 + \alpha_1 d_1 = \begin{pmatrix} \frac{6}{5} \\ \frac{6}{5} \\ \frac{6}{5} \end{pmatrix} + \frac{5}{3} \begin{pmatrix} -\frac{18}{25} \\ \frac{72}{25} \\ -\frac{18}{25} \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$$

$$r_2 = r_1 - \alpha_1 A * d_1 = \begin{pmatrix} -\frac{6}{5} \\ \frac{12}{5} \\ -\frac{6}{5} \end{pmatrix} - \frac{5}{3} \begin{pmatrix} 1 & 1 & 4 \\ 1 & 1 & 1 \\ 4 & 1 & 1 \end{pmatrix} * \begin{pmatrix} -\frac{18}{25} \\ \frac{72}{25} \\ -\frac{18}{25} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Abbruch, da Residuum $r_2 = 0 \Rightarrow$ keine weitere Änderung an gefundenem Optimum $x_2 = \begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix}$

Exercise 2

a) As each step of the algorithm finds the local minimum of one dimension, for a 3-dimensional systems it will at most take three iterations.

b) In large systems, the conjugate gradient algorithm allows for fast approximation, doing only a few iterations until the solution is "good enough". Gauss elimination does not allow this. TODO: More?

c) A has to be a symmetric, positive-definite matrix. All symmetric matrices are also square.

d)
  - A negative-definite $\Rightarrow$ global Maximum can be found with CG. Replace d_i with $-d_i$ for ascending directions (TODO: Is that correct?)
  - A singular positive indefinite (TODO: What is that? Does positive indefinite even exist?) $\Rightarrow$ No unique Minimum. CG calculates just one possible Minimum
  - A indefinite $\Rightarrow$ Saddlepoint-Problem. CG won't work (or one step-size should be ∞, because one can find a point at which one can walk indefinitely and still get smaller objective function values).
  - A not square $\Rightarrow$ if $A \in \mathbb{R}^{m \times n}$ with $m > n$ (A has more rows than colums) this is a „Lineares Ausgleichsproblem“. This can be solved by decomposing A in a specific way and solving another system of linear equations (which can again be done using CG).