Callum Gray
1559659

Continued Fractions

A continued fraction is a number represented through an iterative series of fractions of the form:

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

Using this form can show deeper patterns in the structure of numbers, and can be used to form rational approximations of irrational numbers, often with small amounts of computation.

Alternative Notation

The more fractions that form a number, the more cumbersome they are to display. This problem leads to another notation for writing continued fractions; a list [a_0; a_1, a_2, … a_n] with the elements in this particular list corresponding to the picture above. Note that only the first digit is followed by a semicolon, as it gives the integer part of the continued fraction.

Evaluating a continued fraction

To get the number to its simple form of a single numerator and denominator, evaluate the fraction at the very bottom, and work backwards, using the result of that fraction to evaluate the next one.

Forming a continued fraction

Take the number 123/49.
This can be written as 2.510204… or 2 + 0.510204…
Taking the reciprocal of 0.510204… we get 1.96, which can be written as 1 + 0.96.
The reciprocal of 0.96 is 1.041666666 which is 1 + 0.041666666.
The reciprocal of 0.041666666 is simply 24, which terminates the continued fraction.

Fraction	**Integer Part**	**Non-Integer Part**		**Reciprocal of Non-Integer Part**		**Continued Fraction**
123/49	2	0.510204	25/49	1.96	49/25	[2 …]
49/25	1	0.96	24/25	1.041666	25/24	[2; 1 …]
25/24	1	0.041666	1/24	24	24	[2; 1, 1 …]
24	24	-	-	-	-	[2; 1, 1, 24]

All rational numbers will eventually terminate through this process and give a continued fraction which can be evaluated. However, irrational numbers will never terminate through this process, and instead give infinite continued fractions. These are incredibly useful for giving approximations to these numbers, with more iterations giving more accuracy. Some notable infinite continued fractions are shown below:

π: [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 3 …]

e: [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10 …]

$\sqrt{2}$: [1; 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2, 2 …]

φ (Golden Ratio): [1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, …]

Rate of Approximations

Some approximations require more computation that others to converge to the desired result. Having larger values for denominators will cause a faster rate of convergence. Famous mathematician Ramanujan worked closely with infinite fractions and found remarkably accurate approximations. Take π^4:

[97; 2, 2, 3, 1, 16539 …]

Notice that the 6th element is abnormally large, which allows a continued fraction terminating with 16539 to give a very accurate π^4. Taking the 4th root of this, a close approximation to π is given:

$$\left(\frac{2143}{22}\right)^{\frac{1}{4}}$$

Now looking at φ we have

[1; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 …]

which has the smallest possible values for each denominator, meaning it takes many iterations to get close to the real value. For this reason, the golden ratio is often referred to as the “most irrational” fraction. The table below shows the speed at which pi and phi approach their true values, with each bold section representing correct digits.

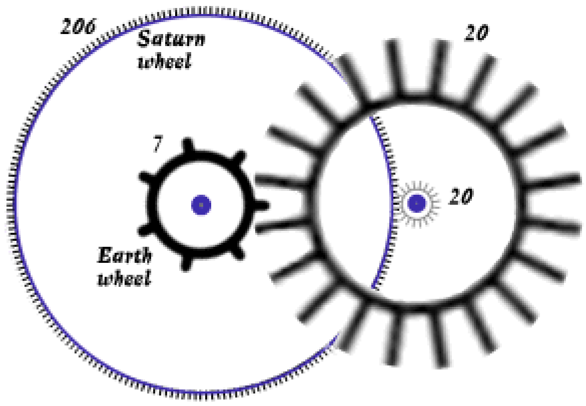
Iterations:	π	Φ
1	**3.**00000000000000	**1.**00000000000000
2	**3.14**285714285714	2.00000000000000
3	**3.1415**0943396226	**1.**50000000000000
4	**3.141592**92035398	**1.6**6666666666666
5	**3.141592653**01190	**1.6**0000000000000

An Interesting Application of Continued Fractions

Prominent mathematician and physicist of the 1600’s, Christiaan Huygens, was working on a mechanical model of solar system and desired the ratio of teeth on the model’s gears to produce an accurate scaled version of planetary orbits.

Let’s take Saturn as an example. In Huygens’ day, it was believed that Saturn takes roughly 29.46 years to orbit the sun (modern day approximations give about 29.43). One gear, E, was set to emulate Earth’s orbit, and one gear, S, set to emulate Saturn’s orbit. To function correctly, the ratio S/E is needed to be as close as possible to this number, and so a continued fraction of it was calculated.

After a few iterations, 206/7 is given, so the amount of teeth given to E was 7, and the teeth given to S was 206, as shown in the diagram below.



References:
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https://www.youtube.com/watch?v=CaasbfdJdJg&ab_channel=Mathologer
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Image Sources:

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