STAT 342 Lab 3 Questions

Ben Liu

May 2025

1 Problem 1

Let $G(t)$ be geometric Brownian motion: $G(t) = G(0)e^{\mu t + \sigma B(t)}$ where $B(t)$ is standard Brownian motion and $B(0) = 0$. We find the value m such that $P(G(t) < m) = 0.05$.

Solution. We have that

$$lnG(t) = ln(G(0)) + \mu t + \sigma B(t) \sim N(ln(G(0)) + \mu t, \sigma^2 t)$$

then

$$\begin{aligned}
P(G(t) < m) &= 0.05 \\
\Rightarrow P(\ln G(t) < \ln m) = \Phi(\frac{\ln m - [\ln(G(0)) + \mu t]}{\sigma\sqrt{t}}) \\
&= 0.05 \\
\Rightarrow \frac{\ln m - [\ln(G(0)) + \mu t]}{\sigma\sqrt{t}} &= -1.645 \\
\ln m - [\ln(G(0)) + \mu t] &= -1.645\sigma\sqrt{t} \\
\Rightarrow \ln m &= \ln(G(0)) + \mu t - 1.645\sigma\sqrt{t} \\
\Rightarrow m &= G(0)e^{\mu t - 1.645\sigma\sqrt{t}}
\end{aligned}$$

2 Problem 2

Suppose the price fluctuations of a stock are well described by absorbed Brownian motion, and a company is considered bankrupt if the share drops to zero. If $A(0) = 5$, what the probability the company is bankrupt at $t = 25$?

Solution. We find

$$P(A(25) \leq 0|A(0) = 5) = 1 - P(A(25) > 0|A(0) = 5) = 1 - G_t(5, 0)$$

$$= 1 - [2\Phi(\frac{-5}{\sqrt{25}}) - 1]$$

$$= 2\Phi(-1)$$

$$= 2(0.15866)$$

$$\approx 0.3173$$

3 Problem 3

In a bicycle race between two competitors, let $Y(t)$ denote the amount of time in seconds by which the racer started in the inside position is ahead when $100t$ percent of the race has been completed and suppose that $\{Y(t), 0 \leq t \leq 1\}$ can be effectively modeled as a Brownian motion process with variance parameter σ^2.

If the inside racer is leading by σ seconds at the midpoint of the race, what is the probability that she is the winner?

We have that $Y(0.5) = \sigma$. We find the probability that

$$P(Y(1) > 0|Y(0.5) = \sigma)$$

We have that $Y(1) - Y(0.5) \sim N(0, \sigma^2/2)$ Therefore,

$$P(Y(1) > 0|Y(0.5) = P(Y(0.5) + [Y(1) - Y(0.5)] > 0))$$

$$= P(\sigma + [Y(1) - Y(0.5)] > 0)$$

$$= P(Y(1) - Y(0.5) > -\sigma)$$

standardizing $Y(1) - Y(0.5)$, we have that

$$P(Z > -\sqrt{2}) = 1 - \Phi(-\sqrt{2}) = \Phi(\sqrt{2})$$

$$\approx 0.921$$

If the inside racer wins the race by a margin of σ seconds, what is the probability that she was ahead at the midpoint?

Solution. We note that $Y(t) = \sigma B(t)$. So we have that

$$P(Y(0.5) > 0|Y(1) = \sigma) = P(\sigma B(0.5) > 0|\sigma B(1) = \sigma)$$

$$= P(B(0.5) > 0|B(1) = 1)$$

Since for $B(s)|B(t)$ for $0 < s < t$ we have that $B(s)|B(t) = b \sim N(\frac{s}{t}b, \frac{s(t-s)}{t})$, then

$$B(0.5)|B(1) = 1 \sim N(\frac{1}{2}, \frac{1}{4})$$

Then

$$P(Z > \frac{0 - \frac{1}{2}}{\sqrt{\frac{1}{4}}})$$

$$= P(Z > -1)$$

$$= 1 - \Phi(Z \leq -1)$$

$$1 - \Phi(-1) \approx 0.8413$$